## 12.21 The Wankel Rotary Engine

STEWART'S CALCULUS: 6 ed. §§6.1, 11.1 and 11.2, 6ET ed. §§6.1, 10.1 and 10.2
BACKGROUND: You need to know how to plot polygons and parametric curves, how to compute the area between two curves and how to use the `piecewise` and `animate` commands. In particular the area under a parametric curve $(x(t), y(t))$ is

$$A = \int_{x_1}^{x_2} y(x)\, dx = \int_{t_1}^{t_2} y(t) \frac{dx}{dt}\, dt$$

Further the length of an arc of a circle with central angle θ is $s = r\theta$.

GOAL: In this project you will study the motion of Felix Wankel's rotary engine[5] of the type once used by Mazda. The basic geometry consists of two circles and an equilateral triangle which just fits inside a curve called an epitrochoid.
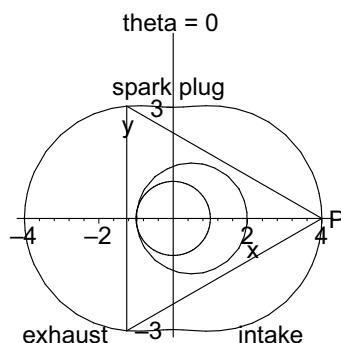

In our simple model, the inner circle has radius 1, center at the origin and does not move. The outer circle has radius $3/2$, is always tangent to the inner circle and is constrained to roll counterclockwise without slippage around the inner circle. The equilateral triangle is concentric with the outer circle and has vertices which are at distance $7/2$ from the center of the triangle. This triangle, called the rotor, rolls with the outer circle. At time $t = 0$ the two circles are tangent to each other at the point $x = -1$ while one vertex of the triangle (called P) is on the positive x-axis at $x = 4$ as shown above. As time progresses, the outer circle rolls around the inner circle and the vertex P traces out a parametric curve $P(\theta) = (x(\theta), y(\theta))$ as a function of the angle θ through which the point of tangency has moved on the inner circle, measured counterclockwise from the negative x-axis. This curve is called an epitrochoid, and is the shape of a cross section of the "cylinder" of a Wankel engine.

In the engine, each of the three spaces between the rotor and the walls of the cylinder is called a "chamber". As the rotor revolves, each chamber passes through the various phases of the Otto cycle: injection, compression, ignition, expansion and exhaust. We will describe these phases for the chamber in front of the vertex P:

[5]For more information, see the Wikipedia entry on the Wankel engine.

From $\theta = -5\pi/2$ to $\theta = -\pi$, the fuel-air mixture is injected through the open intake valve on the lower right side:
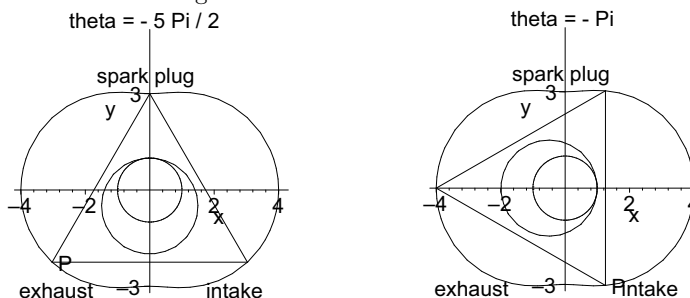
From $\theta = -\pi$ to $\theta = \pi/2$, the fuel-air mixture is compressed.

Just after maximum compression at $\theta = \pi/2$, the spark plug ignites the fuel.
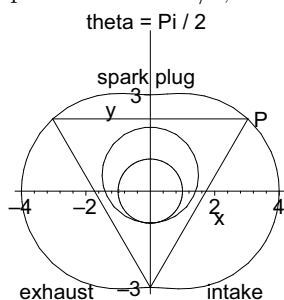
From $\theta = \pi/2$ to $\theta = 2\pi$, the fuel expands providing a power stroke.

From $\theta = 2\pi$ to $\theta = 7\pi/2 \approx -5\pi/2$, the used fuel is compressed and escapes through the open exhaust valve on the lower left side.
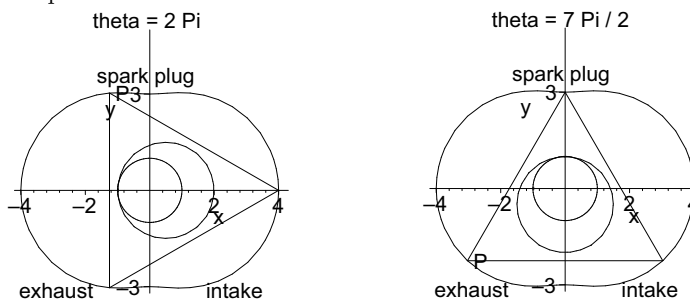

ASSIGNMENT:

1. The inner circle does not move. Write an equation, `eq1`, for the inner circle and use `implicitplot` to plot it.

2. The center of the outer circle (which is also the center of the triangular rotor) changes with the angle to the point of tangency with the inner circle, θ. Write an equation, `eq2`, for the outer circle for general θ. Then use `subs(theta=Pi/3,eq2)` and `implicitplot` to plot it when $\theta = \pi/3$.

3. Explain why, if the point of tangency of the two circles has moved coun-
terclockwise around the small circle through θ radians, then the vertex P
which was originally on the x-axis will be at

$$x_1 = \frac{1}{2}\cos(\theta) + \frac{7}{2}\cos\left(\frac{\theta}{3}\right) \qquad \text{and} \qquad y_1 = \frac{1}{2}\sin(\theta) + \frac{7}{2}\sin\left(\frac{\theta}{3}\right).$$

HINTS: The main two parts of the derivation are showing (i) that the
center C of the triangle rotates through an angle θ and (ii) that the vertex
P rotates through an angle $\theta/3$ around the center C. Below is a diagram
which shows the inner and outer circles after the point of tangency has
rotated by θ radians.
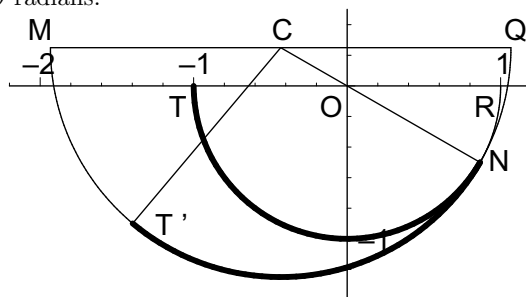
In the diagram, the center of the (small) stationary circle is at the origin,
O. The center of the (large) rotor circle (and the triangle) is at C. The
two circles are currently tangent at N. The points T and T' are the
points on the inner and outer circles which used to be at the initial point
of tangency and the thickened arcs represent the parts of the two circles
that have been in contact with each other. Notice that N is diametrically
opposed to C and that T' is diametrically opposed to the vertex P of the
triangle (not shown). In other words, N, O and C are colinear and P, C
and T' are colinear. So you need to show $\angle ROC = \theta$ and $\angle MCT' = \theta/3$.
Given that $\angle TON = \theta$, identify each of the following and justify your
answer: (When you answer (b) and (d), you will need to use the arclength
formula $s = r\theta$ where θ is in radians, $r = 1$ for the small circle and $r = 3/2$
for the large circle.)

  (a) the angle $\angle ROC$ (which completes part i)

  (b) the arclength $\overset{\frown}{TN}$

  (c) the arclength $\overset{\frown}{T'N}$

  (d) the angle $\angle T'CN$

  (e) the angle $\angle NOR$

  (f) the angle $\angle NCQ$

  (g) the angle $\angle MCT'$ (which completes part ii)

  (h) the coordinates of the point C.

  (i) the coordinates of the point P.

4. Use the previous result to do a parametric plot of the (epitrochoid) cylinder wall along which the vertex P of the triangular rotor travels.

5. The position of the vertex P of the rotor is given in #3 above. The other two vertices of the triangular rotor are $2\pi/3$ apart. Write down the positions of the other two rotors as (x_2, y_2) and (x_3, y_3) for general θ. Then use `subs` and `plot` to plot the triangle when $\theta = \pi/3$.

6. Combine the plots of both circles, the triangle and the epitrochoid for $\theta = \pi/3$ using `display`.

7. Write a Maple procedure `F(theta)` which plots both circles, the triangle and the epitrochoid at the same time as a function of θ. (See Section **??**.)

8. Make the outer circle and the triangle roll around the inner circle by animating your plot. (See `?animate`.) To see it move, you need to right click in the plot and select ANIMATE > PLAY or click in the plot and then click on the PLAY button which is a triangle pointing to the right on the toolbar.

9. Write a Maple procedure with argument θ to compute the area between a side of the rotor and the epitrochoid. Observe that for many angles the top or bottom curve will be split into two parts: an edge of the rotor and a piece of the epitrochoid. Use Maple's `piecewise` command to make sure that the correct rules are applied at the correct angles. Look at your animation to determine the angles at which the changes occur.

10. Plot the graph of the area between the rotor and the wall over the interval $(0, 3\pi)$. From the plot, what are the values of θ (expressed as multiples of π) which give the minimum and maximum area? Using your area function, what are the minimum and maximum area?

11. The difference between the maximum and minimum area is known as the *displacement* of the engine. The ratio of the maximum area to the minimum area is called the *compression ratio*. Compute the displacement and compression ratio for your Wankel engine. (Higher compression ratios can be achieved by "rounding" the rotor in an appropriate manner, but we will not deal with that here.)