Which implies that $y(t) = t^2$ solves the DE. (One may easily check that, indeed $y(t) = t^2$ does solve the DE/IVP. $\square$

Exercises

In 1-8, solve the ODE/IVP using the Laplace Transform

1. $y'' + 4y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$
2. $y'' + 4y' + 3y = t^2, \quad y(0) = 1, \quad y'(0) = 0$
3. $y'' - 3y' + 2y = \sin t, \quad y(0) = 0, \quad y'(0) = 0$
4. $y'' - 3y' + 2y = e^t, \quad y(0) = 1, \quad y'(0) = 0$
5. $y'' - 2y = t^2, \quad y(0) = 1, \quad y'(0) = 0$
6. $y'' - 4y = e^{2t}, \quad y(0) = 0, \quad y'(0) = -1$
7. $y'' + 3ty' - y = 6t, \quad y(0) = 0, \quad y'(0) = 0$
8. $y''+ty'-3y = -2t, \; y(0) = 0, \; y'(0) = 1$ (You will need integration by parts or use technology)

5.4 Unit Step Functions and Periodic Functions

In this section we will see that we can use the Laplace transform to solve a new class of problems efficiently. In particular, we will be able to consider discontinuous forcing functions. First, we make a definition.

<u>The Unit Step Function</u>

Let

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

This function is also called a Heaviside function.

Example 5.20 *Plot the graphs of (a)* $u(t)$*, (b)* $u(t-1)$*, (c)* $u(t) - u(t-1)$ *(d)* $(\sin t)\,[u(t) - u(t-1)]$

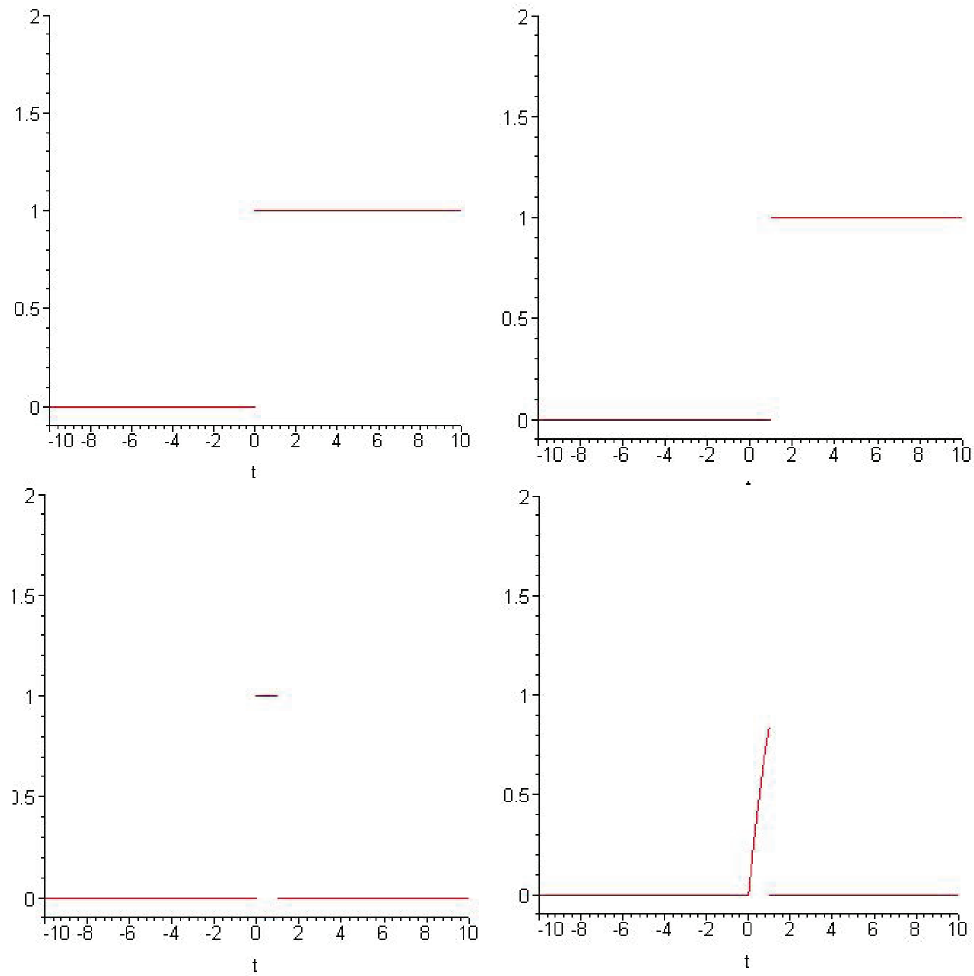

Figure 5.1: **Plots of (a)-(d) in Exercise 5.20**

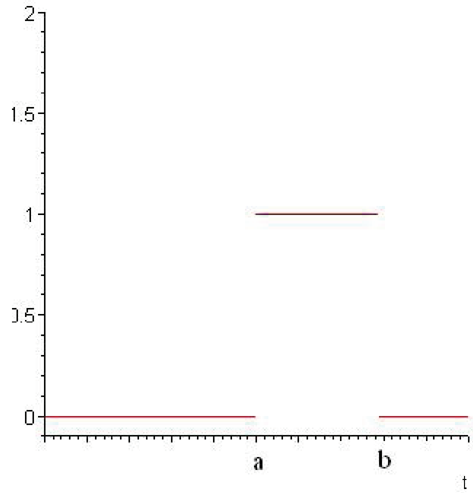

Figure 5.2: **Plot of** $u(t-a)-u(t-b)$**, which is 1 on** (a,b)

Solution: □

Note that the general plot of $u(t-a)-u(t-b)$, where $a<b$ is shown in the plot below:

□

We can use unit step functions to write any case-defined, up to the points where the discontinuity points of the unit step functions.

Example 5.21 *Express*

$$f(t)=\begin{cases} 0 & t<1 \\ t^2 & 1<t<2 \\ -5 & 2<t<3 \\ \sin t & t>3 \end{cases}$$

in terms of unit step functions.

Solution: We may rewrite this function as

$$f(t) = t^2[u(t-1) - u(t-2)] - 5[u(t-2) - u(t-3)] + (\sin t)u(t-3)$$

Note that this can be further simplified as

$$f(t) = t^2u(t-1) - (5+t^2)u(t-2) + (\sin t + 5)u(t-3)$$

□

Below, we describe how to express a case defined function using unit step functions.

Expressing a Case-Defined Function

The function

$$f(t) = \begin{cases} f_1(t) & t_0 < t < t_1 \\ f_2(t) & t_1 < t < t_2 \\ \vdots & \vdots \\ f_n(t) & t_{n-1} < t < t_n \end{cases}$$

can be rewritten as

$$f(t) = f_1(t)[u(t-t_0) - u(t-t_1)] + f_2(t)[u(t-t_1) - u(t-t_2)] + ...$$

$$+f_n(t)[u(t-t_{n-1}) - u(t-t_n)]$$

or

$$f(t) = \sum_{j=1}^{n} f_j(t)[u(t-t_{j-1}) - u(t-t_j)]$$

Note that if

$$f(t) = \begin{cases} f_1(t) & t_0 < t < t_1 \\ f_2(t) & t_1 < t < t_2 \\ \vdots & \vdots \\ f_n(t) & t_{n-1} < t \end{cases}$$

then we would express $f(t)$ as

$$f(t) = f_1(t)[u(t-t_0) - u(t-t_1)] + f_2(t)[u(t-t_1) - u(t-t_2)] + ...$$

$$+f_{n-1}(t)[u(t-t_{n-2})-u(t-t_{n-1})]+f_n(t)u(t-t_{n-1})$$

Laplace Transforms of Step Functions

Laplace Transform of $u(t-a)$
For $a \geq 0$, $$\mathcal{L}[u(t-a)](s)=\frac{e^{-as}}{s}, \quad s>0$$

More generally,

Laplace Transform of $u(t-a)f(t-a)$ (Pre-Shift Theorem)
For $a \geq 0$, $$\mathcal{L}[u(t-a)f(t-a)](s)=e^{-as}\mathcal{L}[f(t)](s)$$

Proof: By definition

$$\mathcal{L}[u(t-a)f(t-a)]=\int_0^\infty e^{-st}u(t-a)f(t-a)\;dt$$

Since $u(t-a)=0$ for $t<a$, and $u(t-a)=1$ for $t>a$, this integral becomes

$$\int_a^\infty e^{-st}f(t-a)\;dt.$$

Let $w=t-a$ and $dw=dt$. Then this integral becomes

$$\int_0^\infty e^{-s(w+a)}f(w)\;dw$$

or

$$e^{-sa}\int_0^\infty e^{-sw}f(w)\;dw=e^{-sa}\mathcal{L}[f(w)](s)=e^{-sa}\mathcal{L}[f(t)](s)$$

□

We will call this Theorem the Pre-Shift Theorem, since it requires us to rewrite the variable t to $t-a$ in order to use the result as the next examples illustrate.

Example 5.22 *Find*

$$\mathcal{L}[u(t-7)t^2]$$

Solution: We need to rewrite t^2 in terms of $t-7$. So

$$t^2 = ((t-7)+7)^2 = (t-7)^2 + 14(t-7) + 49.$$

Substituting:

$$\mathcal{L}[u(t-7)t^2] = \mathcal{L}[u(t-7)(t-7)^2] + 14\mathcal{L}[u(t-7)(t-7)] + 49\mathcal{L}[u(t-7)]$$
$$= e^{-7s}\left(\frac{2}{s^3} + \frac{14}{s^2} + \frac{49}{s}\right)$$

□

Example 5.23 *Find*

$$\mathcal{L}[u(t-4)\sin 2t]$$

Solution: We need to rewrite $\sin t$ in terms of $t-4$ using a trigonometric identity. So

$$\sin 2t = \sin(2[t-4]+8) = \sin 2(t-4)\cos 8 + \cos 2(t-4)\sin 8$$

Substituting:

$$\mathcal{L}[u(t-4)\sin 2t] = \mathcal{L}\Big[\sin[2(t-4)]\cos 8 + \cos[2(t-4)]\sin 8\Big]$$

$$= e^{-4s}\left(\cos 8\left(\frac{2}{s^2+4}\right) + \sin 8\left(\frac{s}{s^2+4}\right)\right)$$

□

Inverse Laplace Transforms involving e^{-as} (Backward Pre-Shift Theorem)

For $a \geq 0$,

$$\mathcal{L}^{-1}[e^{-as}F(s)] = u(t-a)f(t-a),$$

where $F(s) = \mathcal{L}[f(t)](s)$.

Example 5.24 *Find*

$$\mathcal{L}^{-1}\left[e^{-4s}\frac{1}{s^4}\right]$$

Solution: We know for $F(s) = \frac{6}{s^4}$ that $f(t) = t^3$. So

$$\mathcal{L}^{-1}\left[e^{-4s}\frac{1}{s^4}\right] = \frac{1}{6}\mathcal{L}^{-1}\left[e^{-4s}\frac{6}{s^4}\right] = \frac{1}{6}u(t-4)(t-4)^3$$

□

We now solve a differential equation arising from a spring mass system with discontinuous forcing.

Example 5.25 *Solve*

$$y'' + y = 10[u(t-\pi) - u(t-2\pi)], \quad y(0) = 0, \quad y'(0) = 1$$

and plot its graph from $0 \le t \le 3\pi$. *Explain the behavior if this were a spring-mass system and find amplitude of the steady state.*

Solution: Taking the Laplace transform of both sides and writing $\mathcal{L}[y(t)]$ as $Y(s)$, we obtain:

$$s^2Y(s) - 1 + Y(s) = 10[e^{-\pi s} - e^{-2\pi s}]\frac{1}{s}$$

so

$$Y(s) = 10[e^{-\pi s} - e^{-2\pi s}]\frac{1}{s(s^2+1)} + \frac{1}{s^2+1}.$$

After partial fractions

$$Y(s) = 10[e^{-\pi s} - e^{-2\pi s}]\left(\frac{A}{s} + \frac{Bs+C}{s^2+1}\right) + \frac{1}{s^2+1}.$$

we see that $A = 1, B = -1, C = 0$ So

$$Y(s) = 10[e^{-\pi s} - e^{-2\pi s}]\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{1}{s^2+1}.$$

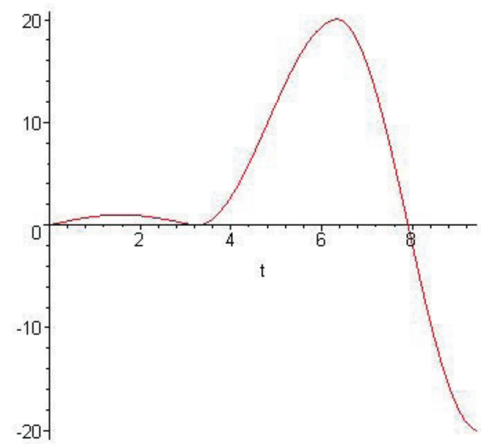

Figure 5.3: **Plot of solution with discontinuous forcing**

Therefore

$$y(t) = 10u(t-\pi)[1-\cos(t-\pi)] - 10u(t-2\pi)[1-\cos(t-2\pi)] + \sin(t).$$

Since for $t > 2\pi$, both $u(t-\pi)$ and $u(t-2\pi)$ equal 1, this will reduce to

$$y(t) = -10\cos(t-\pi) + 10\cos(t-2\pi) + \sin t$$

which can be rewritten using trigonometric identities as:

$$-10[\cos t\cos(-\pi) + \sin t\sin\pi] + 10\cos t + \sin t$$

$= 20\cos t + \sin t$ so the amplitude is $\sqrt{20^2+1} = \sqrt{401} \approx 20$.

We plot this solution.

This example illustrates the effect forcing a particular solution of a spring mass system with a force of $10N$ from $t = \pi$ to $t = 2\pi$ seconds. Notice that around $t = \pi$ the displacement increases to about 20, it is at this time that the forcing is stopped and the spring mass system continues to oscillate at this new amplitude. □

5.4.1 Periodic Functions

Definition 5.26 *A function is* ***periodic*** *if for some* $T > 0$, $f(t+T) = f(t)$ *for all* t. *The smallest such positive value of* T *is called the* ***period*** *of* $f(t)$.

One way to define a periodic function is simply to specify its values on $[0, T]$ and then extend it. We define the **windowed version of a function** $f(t)$ to be

$$f_T(t) = \begin{cases} f(t) & 0 < t < T \\ 0 & else \end{cases}$$

or

$$f_T(t) = f(t)\,[u(t) - u(t-T)]$$

Then we can write:

$$f(t) = \sum_{k=-\infty}^{\infty} f_T(t-kT) = \sum_{k=-\infty}^{\infty} f(t-kT)\,[u(t-kT) - u(t-(k+1)T)]\,,$$

but note that this function is not actually defined at the values of $t = 0, \pm, \pm 2T, ...$, since the unit step functions are not defined there. Note that if we only only care about $f(t)$ when $t > 0$, then

$$f(t) = \sum_{k=0}^{\infty} f_T(t-kT) = \sum_{k=0}^{\infty} f(t-kT)\,[u(t-kT) - u(t-(k+1)T)]\,.$$

<u>Extending a Piece of a Function to a T-Periodic Function</u>

Let $f(t)$ be a function defined for all t. The periodic extension of $f(t)$ via $f_T(t)$ is the function with period T given by

$$\widetilde{f}(t) = \sum_{k=0}^{\infty} f(t-kT)\,[u(t-kT) - u(t-(k+1)T)]\,.$$

Note that this function is actually undefined for: $t = 0, T, 2T, 3T...$ This can be rewritten as:

$$\widetilde{f}(t) = f(t) + \sum_{k=1}^{\infty} [f(t-kT) - f(t-(k-1)T)]\,u(t-kT).$$

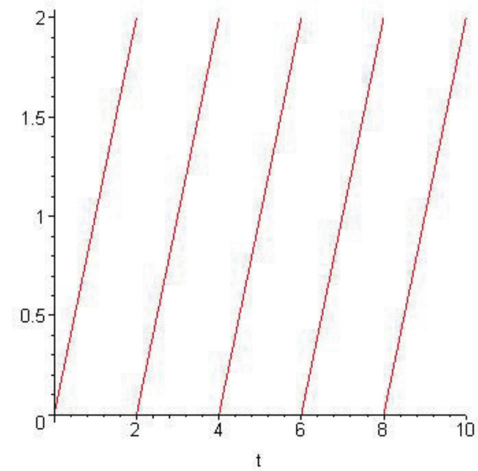

Figure 5.4: **Plot of periodic function generated by** $f(t) = t$ **on** $(0, 2)$

If we only care about this function on a finite interval, we do not need all the terms in this infinite sum.

Example 5.27 *Suppose that* $f(t) = t$ *and we want to create* $f_T(t)$ *for* $T = 2$ *and extend it to a periodic function* $\widetilde{f}(t)$*. Plot the graph of* $\widetilde{f}(t)$ *on* $[0, 10]$ *and express* $\widetilde{f}(t)$ *in terms of unite step functions on* $[0, 10]$.

Solution: Effectively, we are taking $f(t) = t$ on the interval $(0, 2)$ repeating it, so its graph on $[0, 10]$ is in Figure 5.4.1.

Note that for $t > 0$,

$$\widetilde{f}(t) = \sum_{k=0}^{\infty} (t - 2k) \left[u(t - 2k) - u(t - 2(k + 1)) \right].$$

Note that this is (after expanding)

$$\widetilde{f}(t) = t - 2u(t - 2) - 2u(t - 4) - 2u(t - 6) - ...$$

$$= t - 2\sum_{k=1}^{\infty} u(t-2k)$$

□

Example 5.28 *Solve*

$$y'' + y = \widetilde{f}(t), y(0) = 0,\ y'(0) = 0$$

where $\widetilde{f}(t)$ *is as in Example 5.27.*

Solution: Since

$$\widetilde{f}(t) = t - 2\sum_{k=1}^{\infty} u(t-2k)$$

we take the Laplace transform of both sides to obtain:

$$(s^2+1)Y(s) = \frac{1}{s^2} - 2\sum_{k=1}^{\infty}\frac{e^{-2ks}}{s}$$

$$Y(s) = \frac{1}{s^2(s^2+1)} - 2\sum_{k=1}^{\infty}\frac{e^{-2ks}}{s(s^2+1)}$$

$$Y(s) = \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) - 2\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\sum_{k=1}^{\infty} e^{-2ks}$$

so

$$y(t) = t - \sin(t) - 2\sum_{k=1}^{\infty}(1-\cos(t-2k))u(t-2k).$$

A plot of the solution for $t = 0$ to $t = 44$ is shown.

□

The following is also helpful for a periodic function with windowed version $f_T(t)$.

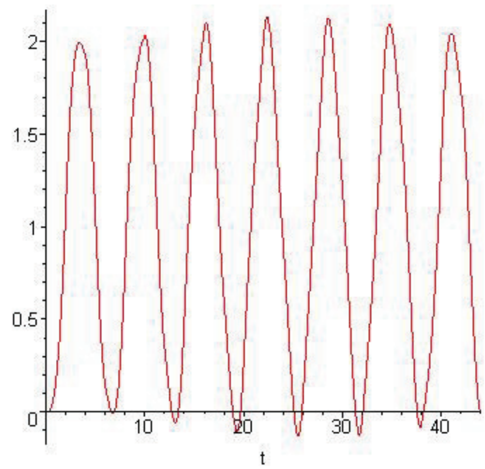

Figure 5.5: **Solution of IVP in Example 5.28**

<u>Laplace Transform of Periodic Functions</u>

For a periodic function $\widetilde{f}(t)$ with associated windowed version $f_T(t)$ we have

$$\mathcal{L}[\widetilde{f}(t)] = \frac{1}{1 - e^{-Ts}} F_T(s) = F_T(s) \sum_{k=0}^{\infty} e^{-kTs},$$

Proof: Since for $t > 0$ we have

$$f_T(t) = \widetilde{f}(t)\,[u(t) - u(t-T)]$$

Since $\widetilde{f}$ is $T-$periodc we have

$$f_T(t) = f(t)u(t) - f(t-T)u(t-T).$$

Taking the Laplace transform of both sides yields:

$$F_T(s) = \mathcal{L}[\widetilde{f}(t)] - e^{-sT}\mathcal{L}[\widetilde{f}(t)].$$

Therefore,

$$\mathcal{L}[\widetilde{f}(t)] = \frac{1}{1 - e^{-sT}} F_T(s).$$

Note that we have a the form of the sum of an infinite geometric sequence, namely:

$$\frac{1}{1-e^{-sT}} = 1 + e^{-sT} + e^{-2sT} + ...$$

So

$$\mathcal{L}[\widetilde{f}(t)] = F_T(s)\sum_{k=0}^{\infty} e^{-kTs}.$$

□

Exercises

In 1-5, write the function in terms of unit step functions and take the Laplace Transform

1. $f(t) = \begin{cases} 1 & t < 1 \\ e^t & t > 1 \end{cases}$

2. $$f(t) = \begin{cases} \sin t & t < \pi \\ \cos t & t > \pi \end{cases}$$

3. $$f(t) = \begin{cases} \sin(2t) & t < 2\pi \\ 0 & t > 2\pi \end{cases}$$

4. $$f(t) = \begin{cases} 1 & 0 < t < 2 \\ 2 & 2 < t < 4 \\ 6 & t > 4 \end{cases}$$

5. $$f(t) = \begin{cases} t^2 & 0 < t < 2 \\ 8 - t^2 & 2 < t < 5 \\ e^{-3t} & t > 5 \end{cases}$$

6. Solve $y'' + 2y' + 4y = u(t-2) - u(t-3), y(0) = 0,\ y'(0) = 0.$

7. Solve $y'' + 2y' + 4y = t^2u(t-2) - t^2u(t-3), y(0) = 0,\ y'(0) = 0.$

8. Solve $y'' + 2y' + 4y = e^t[u(t-2) - u(t-3)], y(0) = 0,\ y'(0) = 0.$

9. Graph the function $f(t) = 1 - u(t-1) + u(t-2) - u(t-3) + ...$

10. Solve $y'' + 2y' + 4y = f(t), y(0) = 0,\ y'(0) = 0$, where $f(t)$ is given in the previous problem.

11. Graph the function $f(t) = t - (2t-2)u(t-1) + (2t-4)u(t-2) - (2t-6)u(t-3) + ...$

12. Solve $y'' + 2y' + 4y = f(t), y(0) = 0,\ y'(0) = 0$, where $f(t)$ is given in the previous problem.

13. Consider $f(t) = e^{2t}$ made into a periodic function $\widetilde{f}(t)$ by taking $f_T(t)$ where $T = 1$.

    (a) Plot $\widetilde{f}(t)$ for $0 < t < 4$.

    (b) Find $\mathcal{L}[\widetilde{f}(t)]$

    (c) $y'' + 2y' + 3y = \widetilde{f}(t), y(0) = 0,\ y'(0) = 0,$

14. Use the differentiation theorem to verify that $\mathcal{L}[t\, u(t-a)] = e^{-as}\frac{1}{s^2}$

15. Use appropriate theorems to compute $\mathcal{L}[t \sin t e^t u(t-a)]$