Chapter 1
Matrices and Systems of Linear Equations

§ 1.1: Introduction to Matrices and Systems of Linear Equations
§ 1.2: Echelon Form and Gauss-Jordan Elimination

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Outline

1. § **1.1 and**§**1.2**

Linear Equations

Definition

A **linear equation** in the n variables $x_1, x_2, \cdots, x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where the **coefficients** $a_1, a_2, \cdots, a_n$ and the **constant term** b are constants.

Example: $3x + 4y + 5z = 12$ is linear. $x^2 + y = 1, \sin y + x = 10$ are not linear.

A **solution** of a linear equation $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$ is a vector $[s_1, s_2, \ldots, s_n]$ whose components satisfy the equation when we substitute $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$.

Example: The linear equation $x_1 - x_2 + 2x_3 = 3$ has $[3, 0, 0], [0, 1, 2]$ and $[6, 1, -1]$.
(In general, $x_1 = 3 + s - 2t, x_2 = s, x_3 = t$.)

Systems of Linear Equations (SLEs)

Definition

A **system of linear equations** is a finite set of linear equations, each with the same variables. A **solution** of system of linear equations is a vector that is *simultaneously* a solution of each equation in the system. The **solution set** of a system of linear equations is the set of *all* solutions of the system.

Example The system

$$2x - 3y = 7$$
$$3x + y = 5$$

has $[2, -1]$ as a solution.

Consistency and type of solutions

Note: A system of linear equations is called **consistent** if it has at least one solution. A system with no solutions is called **inconsistent**.

A system of linear equations with real coefficients has either

1. a unique solution (a consistent system) or
2. infinitely many solutions (a consistent system) or
3. no solutions (an inconsistent system).

Solving a System of Linear Equations

Example: Solve the system

$$x - y - z = 4$$
$$2y + z = 5$$
$$3z = 9$$

Note: To solve this system, we usually use **back substitution**.

$$3z = 9 \rightarrow z = 3$$
$$z = 3, \ 2y + z = 5 \rightarrow 2y = 2 \rightarrow y = 1$$
$$z = 3, \ y = 1, \ x - y - z = 4 \rightarrow x = 8$$

$[8, 1, 3]$ is the unique solution for this SLEs.

Using Augmented Matrix

Assume we have the following SLEs with m equations and n unknowns:

$$a_{1,1}x_1 + a_{1,2}x_2 \ldots a_{1,n}x_n = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 \ldots a_{2,n}x_n = b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{n,2}x_2 \ldots a_{m,n}x_n = b_m$$

We can represent it as follows:

	x_1	x_2	$\ldots$	$\ldots$	x_n	
Eq. 1	$a_{1,1}$	$a_{1,2}$	$\ldots$	$\ldots$	$a_{1,n}$	b_1
Eq. 2	$a_{2,1}$	$a_{2,2}$	$\ldots$	$\ldots$	$a_{2,n}$	b_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$			$\vdots$
Eq. m	$a_{m,1}$	$a_{m,2}$	$\ldots$	$\ldots$	$a_{m,n}$	b_m

Augmented Matrix

The following matrix is called the **augmented matrix** of this SLEs:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \ldots & \ldots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \ldots & \ldots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & & & \vdots \\ a_{m,1} & a_{m,2} & \ldots & \ldots & a_{m,n} & b_m \end{bmatrix}$$

Augmented Matrix can be used to solve SLEs.

Exercises:

Find the augmented matrices of the linear systems:

27. $\begin{array}{l} x - y = 0 \\ 2x + y = 3 \end{array} \implies \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array} \right]$

29. $\begin{array}{l} x + 5y = -1 \\ -x + y = -5 \\ 2x + 4y = 4 \end{array} \implies \left[\begin{array}{cc|c} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array} \right].$

Methods to solve SLEs using Aug. Matrix

- Changing the order of equations

- Changing the order of variables (**We will never use this method**)

- Multiplying an equation by a non-zero constant

- Adding two equations.

We will describe these as operations on the augmented matrix of the given SLEs.

Augmented Matrix

There are two important matrices associated with a linear system.
The **coefficient matrix** contains the coefficients of the variables, and
the **augmented matrix** is the coefficient matrix augmented by an
extra column containing the constant terms:

Assume we have the following SLEs with m equations and n
unknowns:

$$a_{1,1}x_1 + a_{1,2}x_2 \ldots a_{1,n}x_n = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 \ldots a_{2,n}x_n = b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{n,2}x_2 \ldots a_{m,n}x_n = b_m$$

Coefficient and Augmented Matrices

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & & \\ a_{m,1} & a_{m,2} & \dots & \dots & a_{m,n} \end{bmatrix}$$

Table : Coefficient Matrix

$$\left[\begin{array}{ccccc|c} a_{1,1} & a_{1,2} & \dots & \dots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \dots & \dots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & & & \vdots \\ a_{m,1} & a_{m,2} & \dots & \dots & a_{m,n} & b_m \end{array}\right]$$

Table : Augmented Matrix

Remark 1**:** Note that if we denote the coefficient matrix of a linear system by A and the column vector of the constant terms by $\mathbf{b}$, then the form of the augmented matrix is $[A \mid \mathbf{b}]$.

Example

Find the coefficient and augmented matrices A and $[A|\mathbf{b}]$ for the system

$$\begin{array}{rrrrr} 2x & +y & -z & = & 3 \\ x & & +5z & = & 1 \\ -x & +3y & -2z & = & 0 \end{array}$$

The coefficient matrix is $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 5 \\ -1 & 3 & -2 \end{bmatrix}$.

The augmented matrix is $[A|\mathbf{b}] = \left[\begin{array}{rrr|r} 2 & 1 & -1 & 3 \\ 1 & 0 & 5 & 1 \\ -1 & 3 & -2 & 0 \end{array}\right]$

Row Echelon Form

Definition

A matrix is in **row echelon form** if it satisfies the following properties:

(1) All rows consisting entirely of zeros are at the bottom.

(2) In a nonzero row, the first nonzero entry (called the **leading entry**) is in a column to the left of any leading entries below it.

Remark 1**:** Note that the textbook's definition of row echelon form requires the leading terms to be 1. You are welcome to use both versions.

Example The following matrices are in row echelon form:

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Example

Assuming that each of the matrices in the previous example is an augmented matrix, write out the corresponding systems of linear equations and solve them. (Here, we will study the last matrix, and the rest will be left as an exercise)

Remark 1**:** If we are asked to study a coefficient matrix A as the augmented matrix $[A|\mathbf{b}]$, then we treat $\mathbf{b}$ as the zero matrix $\mathbf{0}$.

If $[A|\mathbf{0}] = \begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & c \\ \hline 0 & 2 & 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & -1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \end{array} \end{array}$, then we have

$$2x_2 + x_4 - x_5 + 3x_6 = 0 \Rightarrow 2x_2 + x_4 = 0$$
$$-x_3 + x_4 + 2x_5 + 2x_6 = 0 \Rightarrow -x_3 + x_4 = 0$$
$$4x_5 = 0 \Rightarrow x_5 = 0$$
$$5x_6 = 0 \Rightarrow x_6 = 0$$

Set $x_1 = s, x_4 = t$. Then we have the solution set
$\{x_1 = s, x_2 = -t/2, x_3 = t = x_4, x_5 = x_6 = 0\}$ as our solution set.

1.15

Elementary Row Operations

Definition

The following **elementary row operations** can be performed on a matrix:

(1) Interchange two rows.

(2) Multiply a row by a nonzero constant.

(3) Add a multiple of a row to another row.

Example

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 4 \\ 2 & 3 \\ 5 & 6 \end{bmatrix}$$

Table : Operation (1)

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \xrightarrow{R_1 := 2R_1} \begin{bmatrix} 4 & 6 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Table : Operation (2)

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \xrightarrow{R_2 := R_2 + 2R_1} \begin{bmatrix} 2 & 3 \\ 7 & 10 \\ 5 & 6 \end{bmatrix}$$

Table : Operation (3)

Example

Reduce the following matrix to row echelon form:

$$A = \begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 4 & 2 & 4 & 3 & 3 \\ 1 & 3 & 3 & -1 & 0 & 0 \end{bmatrix}$$

Remark 1**:** Note that the entry chosen to become a leading entry is called a **pivot**, and this phase of the process is called **pivoting**.

$$A \xrightarrow[R_2:=R_2+R_1]{R_4:=R_4-R_1} \begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 4 & 2 & 4 & 3 & 3 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \xrightarrow[R_3:=R_3-2R_2]{} \begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{bmatrix}$$

Row Equivalence

Definition

Matrices *A* and *B* are **row equivalent** if there is a sequence of elementary row operations that converts *A* into *B*.

Theorem

Matrices A and B are row equivalent if and only if they can be reduced to the same row echelon form.

Remark 1**:** In fact, If A, B are row equivalent and B,C are row equivalent, then A, C are row equivalent. The reason is simple: " We can reverse the elementary row operations".

Gaussian Elimination

Gaussian Elimination

(1) Write the augmented matrix of the system of linear equations.

(2) Use elementary row operations to reduce the augmented matrix to row echelon form.

(3) Using back substitution, solve the equivalent system that corresponds to the row-reduced matrix.

Remark 1**:** We have already applied all three steps in different examples.

Example: Gaussian Elimination

Solve the following SLEs using Gaussian Elimination:

$$3x_1 + x_2 - x_3 + 2x_4 = 1$$
$$x_1 - x_2 + x_3 + 2x_4 = 2$$
$$2x_1 + 2x_2 + x_3 + 6x_4 = 3$$

$$[A|\mathbf{b}] \overset{(1)}{=} \begin{bmatrix} 3 & 1 & -1 & 2 & | & 1 \\ 1 & -1 & 1 & 2 & | & 2 \\ 2 & 2 & 1 & 6 & | & 3 \end{bmatrix} \xrightarrow[R_1:=R_1-3R_2]{R_3:=R_3-2R_2} \begin{bmatrix} 0 & 4 & -4 & -4 & | & -5 \\ 1 & -1 & 1 & 2 & | & 2 \\ 0 & 4 & -1 & 2 & | & -1 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & 1 & 2 & | & 2 \\ 0 & 4 & -4 & -4 & | & -5 \\ 0 & 4 & -1 & 2 & | & -1 \end{bmatrix} \xrightarrow[(2)]{R_3:=R_3-R_2} \begin{bmatrix} 1 & -1 & 1 & 2 & | & 2 \\ 0 & 4 & -4 & -4 & | & -5 \\ 0 & 0 & 3 & 6 & | & 4 \end{bmatrix}$$

Remark 1**:** The Gaussian Elimination does not ask us to use the row operations in a specific order. While there is a methodical way to do row operations, it is sometimes faster not to use it. In fact, the methodological way can be summarized as repeating the following two steps:

1. If needed, switch rows to get the next leading term.
2. Otherwise, if R_i has the next leading term then use the operations of the form $R_j := R_j - \alpha R_i$ whenever $j > i$ to cancel out non-zero entries below this leading term.

Example: Cont'd (Step 3 in Gaussian Elimination)

$$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 &= 2 \\ 4x_2 - 4x_3 - 4x_4 &= -5 \\ 3x_3 + 6x_4 &= 4 \end{aligned}$$

Set $x_4 = t$. then

$x_3 = \frac{4}{3} - 2x_4 = \frac{4}{3} - 2t,$

$x_2 = \frac{-5}{4} + x_3 + x_4 = \frac{-5}{4} + \left(\frac{4}{3} - 2t\right) + t = \frac{1}{12} - t,$

$x_1 = 2 + x_2 - x_3 - 2x_4 = 2 + \left(\frac{1}{12} - t\right) - \left(\frac{4}{3} - 2t\right) - 2t = \frac{9}{12} - t$

Solution set is $\{x_1 = \frac{9}{12} - t, \ x_2 = \frac{1}{12} - t, x_3 = \frac{4}{3} - 2t, x_4 = t \}$.

Remark 1: Note that we assigned $x_4 = t$. WHY?

Reduced Row Echelon Form

Definition

A matrix is in **reduced row echelon form** if it satisfies the following properties:

(1) It is in row echelon form.

(2) The leading entry in each nonzero row is a 1 (called a **leading 1**).

(3) Each column containing a leading 1 has zero everywhere else.

Remark 1: Reduced row echelon form can be seen as combining row echelon form and back substitution in Gaussian Eliminiation.

Example:
$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$,
$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

$\begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 & -2 \end{bmatrix}$

Gauss-Jordan Elimination

Definition

(1) Write the augmented matrix of the system of linear equations.

(2) Use elementary row operations to reduce the augmented matrix to reduced row echelon form.

(3) If the resulting system is consistent, solve for the leading variables in terms of any remaining free variables.

Example : Find the line of intersection of the planes $3x + 2y + z = -1$ and $2x - y + 4z = 5$.

$$[A|\mathbf{b}] = \begin{bmatrix} 3 & 2 & 1 & | & -1 \\ 2 & -1 & 4 & | & 5 \end{bmatrix} \xrightarrow{R_2 = 3R_2 - 2R_1} \begin{bmatrix} 3 & 2 & 1 & | & -1 \\ 0 & -7 & 10 & | & 17 \end{bmatrix} \xrightarrow[R_1 := R_1/3]{R_2 := -R_2/7}$$

$$\begin{bmatrix} 1 & 2/3 & 1/3 & | & -1/3 \\ 0 & 1 & -10/7 & | & -17/7 \end{bmatrix} \xrightarrow{R_1 := R_1 - 2R_2/3} \begin{bmatrix} 1 & 0 & 67/21 & | & 44/21 \\ 0 & 1 & -10/7 & | & -17/7 \end{bmatrix}$$

Set $z = t$. Then the solution set is $\left\{ x = \frac{44}{21} - \frac{67}{21}t, y = -\frac{17}{7} + \frac{10}{7}t, z = t \right\}$.

Example

Let $\mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Determine

whether the lines $\mathbf{x} = \mathbf{p} + t\mathbf{u}$ and $\mathbf{x} = \mathbf{q} + t\mathbf{v}$ intersect and, if so, find their point of intersection.

$$
\begin{aligned}
\mathbf{p} + s\mathbf{u} &= \mathbf{q} + t\mathbf{v} \\
\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} &= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\
s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \\
s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}
\end{aligned}
$$

$$
[A|\mathbf{b}] = \begin{bmatrix} 1 & 1 & | & 3 \\ 2 & -1 & | & 0 \\ -1 & 0 & | & -1 \end{bmatrix} \xrightarrow[R_3 := R_1 + R_3]{R_2 := R_2 - 2R_1} \begin{bmatrix} 1 & 1 & | & 3 \\ 0 & -3 & | & -6 \\ 0 & 1 & | & 2 \end{bmatrix} \xrightarrow[R_2 := R_2 + 3R_3]{R_1 := R_1 - R_3} \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 0 & | & 0 \\ 0 & 1 & | & 2 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}
$$

This implies $\{s = 1, t = 2\}$ is a solution. Using $s = 1$ on $\mathbf{p} + s\mathbf{u}$, these lines intersect at $(x, y, z) = (0, 4, 0)$.

1.25

Homogeneous Systems

① **Definition** A system of linear equations is called **homogeneous** of the constant term in each equation is zero. Otherwise, it is called **non − homogeneous**. If the [A|**b**] is homogeneous, then **b** = **0**.

② **Theorem 2.3** If [A|**0**] is a homogeneous system of m linear equations with n variables, where $m < n$, then the system has infinitely many solutions.

Proof.

After applying Gauss-Jordan Elimination to [A|**0**], we get [B|**0**] in reduced row echelon form. We make two observations on [B|**0**]:

(1) [B|**0**] **is consistent:** If B has an all-zeros row, then that row corresponds to the equation $0 = 0$ because it is homogenous. So, it has at least one solution.

(2) **It has at least one free variable:** Since B has m equations, it can at most have m non-zero rows. Hence, it has at least $n - m > 0$ free variables.

(1) and (2) together imply that the system has infinitely many solutions. (each free variable introduces a new variable in the solution set) □

What if it is non-homogeneous?

The last row in the following SLEs gives $0 = 1$. This is not possible. So, it has no solution.

$$\begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 1 \end{bmatrix}$$

Remark 1: An SLEs has NO solution (inconsistent), if its augmented matrix $[A|\mathbf{b}]$ is row equivalent to some $[B|\mathbf{c}]$ such that one of the rows of $[B|\mathbf{c}]$ has the form

$$[0 \quad 0 \quad \dots \quad 0 \quad 0 \quad | \quad d \neq 0]$$

Remark 2: It is enough to apply elementary row operations and Gauss-Jordan Elimination to $[A|\mathbf{b}]$. The process will itself tell you whether the system is consistent or inconsistent.

CAUTION

THE QUIZ WILL COVER EVERYTHING IN 1.1 AND 1.2 **EXCLUDING REDUCED ROW ECHELON FORMS** AS WE HAVE NOT COVERED IT YET.