Chapter 13
ELASTIC PROPERTIES OF MATERIALS

GOALS

When you have mastered the contents of this chapter, you will be able to achieve the following goals:

Definitions

Define each of the following terms, and use it in an operational definition:

- elastic body
- stress
- strain
- elastic limit
- Young's modulus
- bulk modulus
- modulus of rigidity

Hooke's Law

State Hooke's law.

Stress and Strain

Calculate the strain and stress for various types of deformation.

Elasticity Problems

Solve problems involving the elastic coefficients.

PREREQUISITES

Before you begin this chapter, you should have achieved the goals of Chapter 4, ***Forces and Newton's Law***, Chapter 5, ***Energy***, and Chapter 8, ***Fluid Flow***.

Chapter 13
ELASTIC PROPERTIES OF MATERIALS

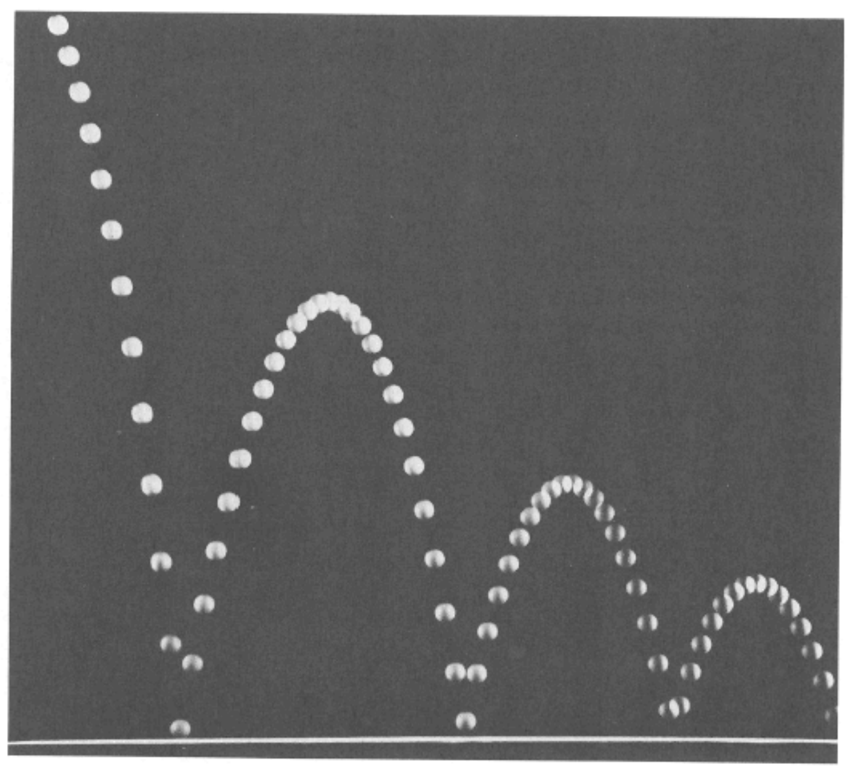
13.1 Introduction

In everyday conversation if someone speaks to you about an elastic body, you probably immediately think of a rubber band. A rubber band yields a great deal to a distorting force, and yet it returns to its original length after the distorting force is removed. Can you think of some biological examples of elastic bodies? In this chapter we will examine the elastic properties of materials.

13.2 Elasticity

Elasticity is a fundamental property of materials. Springs of all kinds are examples of elastic bodies. Let us consider the characteristics of a spring. We find that a spring will respond to distorting force and then return to its original shape after the distorting force is removed. Any material or body can be deformed by an applied force. If it returns to its original shape after the force is removed, it is said to be elastic. Most substances are elastic to some degree. In a technical sense a substance with a high elasticity is one that requires a large force to produce a distortion-for example, a steel sphere.

A multiple flash photograph of a bouncing ball. Many physics principles can be studied in this picture—projectile motion, transformation of energy, changes of momentum, elastic properties of material, among others. How would the picture be altered if the ball and surface that it strikes were perfectly elastic? (Picture from *PSSC Physics*, D.C. Heath and Company, Lexington, Mass., 1965.)



In comparing the elasticity of materials there are certain terms we need to define. Suppose that we have a steel wire that is held rigidly at the top end and has a load fastened to the lower end .See Figure 13.1. The wire is then said to be under *stress*. The magnitude of this stress is found by dividing the applied force (the weight in this case) by the cross-sectional area. Thus,

stress = F/A (13.1)

The SI units of stress are newtons/meter². If the load is doubled, the wire will be stretched by an amountΔL. We now introduce another term called *strain*. Strain is a measure of the distortion of an object, and it is defined as the change in a spatial variable divided by the original value of that variable. For example, in Figure 13.1 the variable is length. So,

strain = $\Delta L/L_0$ (13.2)

Strain is a dimensionless number. An elastic coefficient is defined as the stress divided by strain. There are three types of distortions that may be produced. These are change in length, change in volume without change in shape and change of shape without change in volume. The only distortion that a fluid resists is volume change. A liquid has a greater ability to resist a change in volume than a gas. Hence, a liquid has a larger value for an elastic coefficient, called *bulk modulus*, than a gas. Solids may have all three types of distortion.

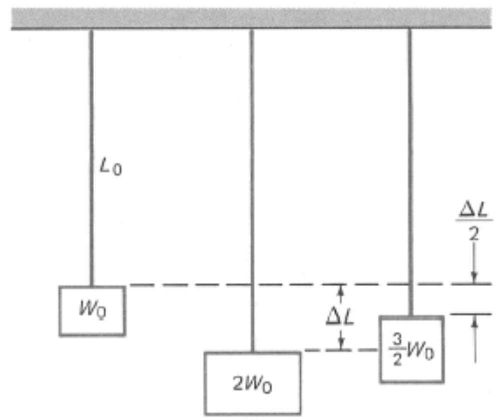


FIGURE 13.1
Strain is proportional to stress. An increase in force of $W_0/2$ produces an elongation of $\Delta L/2$ and an increase in force of W_0 produces an elongation of ΔL.

13.3 Hooke's Law

In 1676 in his study of the effects of tensile forces, Robert Hooke formulated and stated the law that is still used to define elastic properties of a body. He observed that the increase in length of a stretched body is proportional to applied force F as shown in the experiment above Figure 13.1.

$F = kx$ (13.3)

Where x is the length increase (m), and k is a proportionality constant or *spring constant* (N/m). This equation is shown graphically in Figure 13.2. Note that k is the slope of the line.

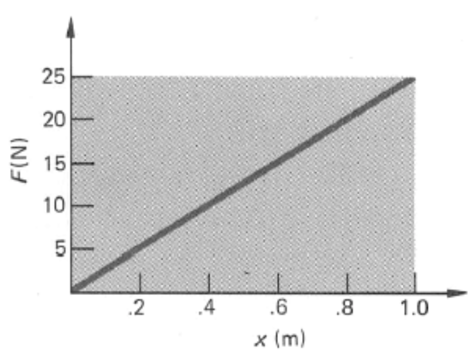


FIGURE 13.2
Hooke's law, $F = kx$, governs the stress-strain relationship within the elastic limits.

The curve shown in Figure 13.2 applies if the body returns to its original size and shape after the distorting force is removed. If the body does not return to its original condition, it is said to have been distorted beyond its *elastic limit* and takes on a permanent change in length. Hooke's law may be stated in a more general form as follows:

within the elastic limits strain is proportional to the stress,

or strain = C x stress. We will now apply this law to the three different types of distortions we mentioned above.

EXAMPLE

Find the stress on a bone (1 cm in radius and 50 cm long) that supports a mass of 100 kg. Find the strain on the bone if it is compressed 0.15 mm by this weight. Find the proportionality constant C for this bone.

$$\text{stress} = F/A = (100 \text{ kg})(9.8 \text{ m/s}^2)/\pi \times (0.01 \text{ m})^2$$

$$= 3.1 \times 10^6 \text{ N/m}^2$$

$$\text{strain} = \Delta L/L_0$$

$$= (0.15 \times 10^{-3} \text{ m})/(0.5 \text{ m})$$

$$= 3.0 \times 10^{-4}$$

Since strain = C x stress, C = strain/stress = 0.96×10^{-10} m^2/N.

13.4 Young's Modulus

Let us consider a stretched wire. Suppose we have a wire that has a length L and radius r, and suppose a load F is applied to the taut wire to produce an elongation of ΔL. For this case, see Figure 13.3.

$$\text{stress} = F/\pi r^2 = F/A$$

$$\text{strain} = \Delta L/L \quad (13.4)$$

From Hooke's law, stress is proportional to strain, or

$$F/A = Y\Delta L/L$$

where Y is the elastic constant of proportionality for a distortion in length and is called *Young's modulus*. Solving for Y, we get

$$Y = \text{stress}/\text{strain} = (F/A)/(\Delta L/L) = (FL)/(A\Delta L) \quad (13.5)$$

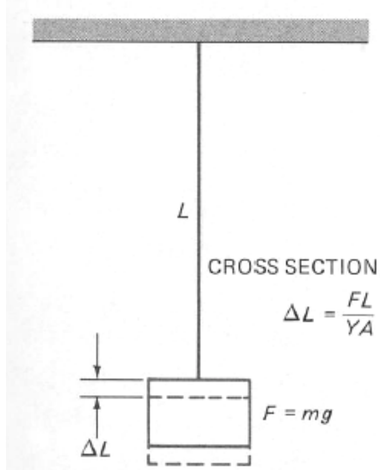


FIGURE 13.3
Steel wire subjected to stretching weight *mg*.

The units of Young's modulus are force/unit area, or N/m^2. See Table 13.1 for the value of Young's modulus for a number of solids.

Note that Y has same units as stress. Because neither liquid nor gases will support a linear strain, they have no measured values for Young's modulus.

TABLE 13.1
Elastic Coefficients of Various Materials

Material	Young's Modulus (10^{10} N/m²)	Modulus of Rigidity (10^{10} N/m²)	Bulk Modulus (10^{10} N/m²)
Aluminum	7.0	2.5	7.0
Bone	0.9–1.3	1.0	–
tensile	1.6	–	1.0
compressive	0.94	–	1.0
Copper	12.0	4.0	13.0
Eggshell	0.006	–	–
Ethanol	–	–	0.09
Granite	5.0	–	4.8
Iron	19.0	6.0	12.0
Mercury	–	–	2.5
Steel	20.0	8.0	16.0
Water	–	–	0.22
Wood (oak)	1.0	1.0	–

EXAMPLES

1. A steel bar 6.00 m long and with rectangular cross section of 5.00 cm x 2.50 cm supports a mass of 2000 kg. How much is the bar stretched?

   From Table 13.1 we find Y for steel is 20.0 x 10^{10} N/m². Solving for ΔL, we get

   $\Delta L = FL/YA$ = (2000) (9.80) (6.00)/(20.0 x 10^{10}) (.050 x .025)

   = 4.70 x 10^{-4} m = 0.47 mm

2. Your leg bones (cross-sectional area about 9.50 cm²) experience a force of approximately 855 N when you walk. Find the fractional amount your leg bones are compressed by walking. Using Y = 1010 N/m² for bone we get:

   $\Delta L/L$ = 8.55 x 10^2 N/(9.5 x 10^{-4} m²) (10^{10} N/m²) = 9 x 10^{-5}

3. Studies show that for strains less than 0.5 percent bones are elastic. Using values from Table 13.1 we will calculate the elastic limit force for compression and stretch of a humerus 20 cm long and 3 cm² in cross-sectional area.

   compression: $F_c = YA\ \Delta L_c/L$

   = 9.4 x 10^9 N/m² + 3.0 x 10^{-4} m2 x 5.0 x 10^{-3}

   F_c = 14000/N

   stretch: $F_s = YA\ \Delta L_s/L$

   = 16 x 10^9 N/m² x 3.0 x 10^{-4} m2 x 5.0 x 10^{-3}

   F_s = 2.4 x 10^4 N

SIMPLE EXPERIMENTS

You may be interested in some of the characteristics of a rubber band as an elastic body. You can carry out some simple experiments and record your results. Try stretching a rubber band and observe its change in temperature. You will find that its temperature increases on stretching and decreases upon relaxation. Also have a stretched rubber band in vertical position support a fixed load. Heat the band, and observe how the load moves. You will observe that it rises, indicating a contraction of the rubber band. The above results are exactly opposite to what you would observe for a metal wire. Another experiment you can carry out is one in which you can measure stretch as a function of load. As you increase the load and then decrease the load, you will observe that a rubber band does not follow Hooke's law and that it does not return to the original position after the load is removed. This failure to return to the original position is called an *elastic lag* or *hysteresis*.

13.5 Bulk Modulus

Suppose that a specimen such as a sphere is placed in a liquid upon which the pressure can be increased by a force applied to the piston (see Figure 13.4). The change in volume of the sphere is a function of the stress applied. The

stress = F/A = pressure applied

strain = $\Delta V/V$ (13.6)

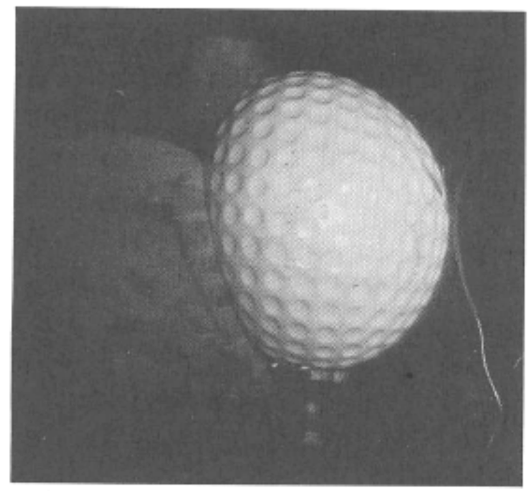
From Hooke's law

stress = B x strain

where B is the constant of proportionality and is called the *bulk modulus*. Then,

$$B=(F/A)/(\Delta V/V)=P/(\Delta V/V)=PV\ \Delta V \qquad (13.7)$$

Deformation of volume can occur in gases, liquids, and solids. Bulk moduli of liquids and solids are high and of the same order of magnitude. Gases are easiest to compress and hence have the lowest bulk modulus. We often speak of the compressibility of a material which is the reciprocal ($1/B$) of its bulk modulus. The values of bulk moduli of some materials are given in Table 13.1.



An impulsive force is applied to an elastic sphere. A deformation is produced on impact, but later the ball will return to its spherical shape. Note cracks in the paint. Analyze the transformation in momentum and energy during the time the club head and the ball are in contact. (Harold E. Edgerton, MIT, Cambridge, Mass.)

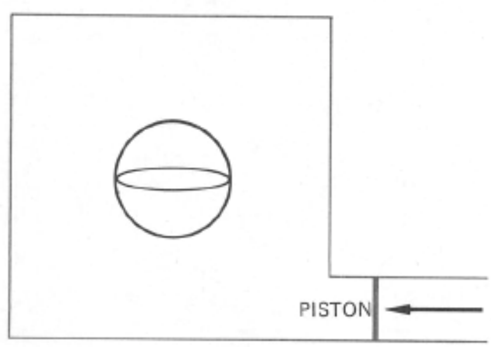


FIGURE 13.4
Uniform pressure is applied to a body to determine its fractional change in volume as a function of applied pressure.

EXAMPLE

Find the pressure necessary to change a volume of water by 1.0 percent. Express the pressure in terms of atmospheric pressure units 1 bar = 10^5 N/m^2.

$P = B\Delta V/V = 2.2 \times 10^9$ N/m^2 x 1.0×10^{-2}

$= 2.2 \times 10^7$ N/m^2 $= 2.2 \times 10^2$ bars = 220 bars

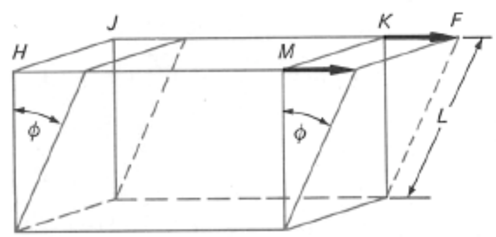
13.6 Modulus of Rigidity

The third type of deformation is one of shape but with a constant volume. As an example consider your book resting on a table as you apply a horizontal force to the top cover. You will deform your book in the direction of the applied force. That is, each page tends to slide over the page below it. This may be represented by Figure 13.5. The force F is applied to the plane of $HJKM$ the stress of the force is given by (F/A), where A = area of $HJKM$. Strain ϕ is expressed in radians and given by $\Delta L/L = \phi$. Then by Hooke's law $F/A = \eta\phi$ where η is the constant of proportionality called the shear modulus or the modulus of rigidity, hence

$$\eta = (F/A) / \phi \qquad (13.8)$$

Note that the radian measure of ϕ (rad) is a ratio of lengths and is therefore unitless. The moduli of rigidity of some materials are given in Table 13.1.

FIGURE 13.5
Applied force F produces shear distortion measured by movement of $HJKL$ plane through a distance $L\phi$. The rigidity modulus η is given by $\eta = \frac{F/A}{\phi}$.



EXAMPLE

Find the force necessary to produce a shear break of a bone with 3 cm^2 cross section if the break strain is 6° = 0.10 rad.

$F = \eta\phi A = 10 \times 10^9$ N/m^2 x 10^{-1} x 3×10^{-4} m^2

$F = 2 \times 10^5$ N.

The elastic properties of a material depend upon its molecular structure. Other properties that are closely related to its elastic characteristics and depend also upon molecular structure are ductility, malleability, and hardness.

ENRICHMENT

13.7 Energy of A Hooke's Law System

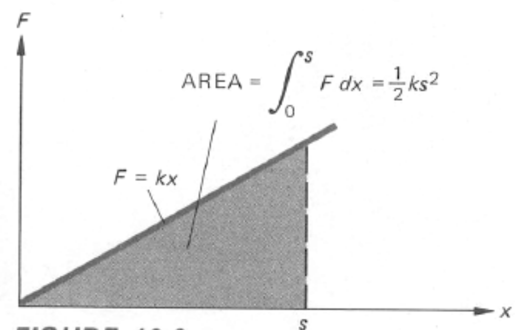


FIGURE 13.6
Work done in compressing or stretching a spring is the area under the F−x graph.

Energy is stored in a compressed or stretched spring. The work that is done against the elastic forces in deforming a body is a measure of elastic potential energy stored in the deformed body. Consider the case of a spring that obeys Hooke's law, $F = kx$. The work dW done in stretching a spring a distance dx is given by $dW = kx\,dx$.

$$W = \int_0^s kx\,dx = (1/2)\,ks^2 \qquad (13.9)$$

see Figure 13.6 in which s is the total stretch, so

$$PE = (1/2)\,ks^2 \qquad (13.10)$$

Similarly, for energy stored in shear we find, where $F = \eta A\phi$ and $ds = L\,d\phi$:

$$W = \int_0^{\phi_f} \eta A\phi \times L\,d\phi = (1/2)\eta AL\phi_f^2 \qquad (13.11)$$

We can derive the expression for the coefficient of rigidity for a cylinder as follows: Consider a twisting force applied perpendicular to the radius at the top of the cylinder of length l. If this force produces a twist of the top surface of ϕ (rad), then the strain can be written as $r\phi/l$, at a radius r within the cylinder at its surface. From the general form of Hooke's law we can write the stress at this top surface as

stress = η strain where η is the modulus of rigidity

Thus

stress = $\eta r\phi/l$

and the torque per unit area at radius r is given by

stress x $r = (\eta r\phi/l)\,r$

If we integrate this torque over the entire area we get

torque = $\int_0^R 2\pi r\,dr \times r\eta r\phi/l = 2\pi\eta\phi R^4/(4l) = \pi\eta R^4\phi/(2l)$

Solving for η:

$\eta = (2l \times torque)/(\pi R^4\,\phi)$ where R = radius of cylinder

SUMMARY

Use these questions to evaluate how well you have achieved the goals of this chapter. The answers to these questions are given at the end of this summary with the number of the section where you can find related content material.

Definitions

1. Any object can be deformed by applying a force to the object. An object is elastic if ________ after the force is removed. However, if the object has been deformed beyond its________, it will remain in a deformed state.
2. The stress on an object is defined as the ratio of the________ to the_________.
3. The strain of an object is defined as the ratio of the________ to the_________.
4. The generalized statement of Hooke's law states that________ is proportional to________ of the system.
5. Young's modulus is a constant with units of _________, which characterizes the response of a solid to _________ and which has a magnitude equal to ________.
6. The bulk modulus of a material is a constant with units of ________, which characterizes the response of a material to _________.
7. The shear modulus, also called the________, is a constant with units of _________, which characterize the response of a material to_________ and which has a magnitude equal to _________.
8. A steel rod 2 m long and 0.5 cm^2 in area stretches 0.24 cm under a tension 12,000 N.
   a. What is the stress of the rod?
   b. What is the strain of the rod?
9. What is the compressibility of water?

Elasticity Problems

10. What is Young's modulus for the steel rod in question 8 above?
11. If the volume of an iron sphere is normally 100 cm^3 and the sphere is subjected to a uniform pressure of 10^8 N/m^2, what is its change in volume?

Answers

1. It returns to its original state, elastic limit (Section 13.3)
2. Applied force, cross-sectional area (Section 13.2)
3. Change in a spatial variable, original value of that variable (Section 13.2)
4. Strain, stress (Section 13.3)
5. N/m^2, forces applied to change its length, $FL/A\Delta L$ (Section 13.4)
6. N/m^2, forces applied to change its volume without changing its shape, $FV/A\Delta L$ (Section 13.5)
7. Modulus of rigidity, N/m^2 force applied to change its shape without changing its volume, $(F/A)\phi$ (Section 13.6)
8. a. 2.4×10^8 N/m^2
   b. 12×10^{-4} (Section 13.4)
9. 4.5×10^{-10} m^2/N (Section 13.5)
10. 20×10^{10} N/m^2 (Section 13.4)
11. 6.3×10^{-2} cm^3 (Section 13.5)

ALGORITHMIC PROBLEMS

Listed below are the important equations from this chapter. The problems following the equations will help you learn to translate words into equations and to solve single-concept problems.

Equations

stress = F/A (13.1)

strain = ΔL /L0 or ΔV/ V or ϕ (13.2, 13.6)

F = kx (13.3)

Y = FL / A Δ L (13.5)

B = P / Δ V / V (13.7)

h = (F/A)/ ϕ (13.8)

PE = (1/2) ks2 (13.9)

Problems

1. If the stress produced in stretching a wire is 5.00×10^{6} N/m^2 by an applied force of 10.0 N, what is the cross-sectional area of the wire?
2. If the strain for the wire above is 0.100 percent, what is the length of the wire that will have an elongation of 1.00 mm?
3. In an experiment one finds that a force of 160 N produces a stretch of 8.00 cm in a given spring. What is the spring constant of the spring?
4. What is Young's modulus for the wire described in problems 1 and 2?
5. A 5.00-cm cube of gelatin has its upper surface displaced 1.00 cm by a tangential force 0.500 N. What is shear modulus of this substance?
6. What is the potential energy stored in the spring in problem 3 when it is stretched 8 cm?

Answers

1. 2.00×10^{-6} m^2
2. 1.00 m
3. 2.00×10^{3} N/m
4. 5.0×10^{9} N/m^2
5. 400 N/m^2
6. 6.4 J

EXERCISES

These exercises are designed to help you apply the ideas of a section to physical situations. When appropriate the numerical answer is given in brackets at the end of the exercise.

Section 13.4

1. A wire 0.70 mm in diameter and 2.0 m long was stretched 1.6 mm by a load of 20 N. Find Young's modulus for the wire. [6.5×10^{10} N/m^2]
2. A load of 18×10^4 N is placed upon a vertical steel support of 6.0-m height and with a cross-sectional area of 20 cm^2. How much is the support compressed by the load? [2.7×10^{-3} m]

Section 13.5

3. What is the decrease in volume of 2 liters of water if it is subjected to a pressure of 10^{10} N/m^2? Compare this decrease with the decrease in 2 liters of mercury under the same pressure. [9×10^{-3}, ratio H_2O/Hg $\approx$ 11]
4. How much pressure would be required to reduce the volume of a block of aluminum by a factor of one part in one thousand? [7×10^7 N/m^2]

Section 13.6

5. The lower end of a steel wire is 1.0 m long and has a radius of 1.0 mm and is twisted through an angle of 360°. Given $\eta = 2\ L/\pi r^4 \phi$, where l = length, L = torque, r = radius, η = shear modulus, and ϕ = twist in rad, what is the torque required? [0.79 N-m]

PROBLEMS

The following problems may involve more than one physical concept. When appropriate, the numerical answer is given in bracket at the end of the problem.

6. Assume the femur has a diameter of 3.0 cm and a hollow center of 0.8-cm diameter and a length of 50 cm. If it is supporting a load of 600 N, what is the stress in the femur? How much will it be shortened by this load? $Y = 16 \times 10^9$ N/m^2. [9.1×10^5 N/m^2; 2.9 x 10-5m]
7. Given the density of sea water as 1.03 g/cm^3 at the surface, what is its density at a depth where the pressure is 1.00×10^8 N/m^2? [1.08 gm/cm^3]
8. A steel shaft with a radius of 1.00 cm and a length of 2.00 m transmits 50.0 kilowatts of power at 2400 rpm.
    a. What is the torque?
    b. Through what angle in radians is the shaft twisted?
    [199 N-m; 0.317 rad]
9. Find the equation for the energy a bone can absorb within its elastic limit in terms of its Young's modulus, cross-sectional area length and compression. [$(1/2)\ (YA/L)\ (\Delta L)^2$]

10. Two masses are suspended on a copper and on an iron wire (see Figure 13.7). What is the stress in each wire? What is the elongation for each? What is the elastic PE in each wire?

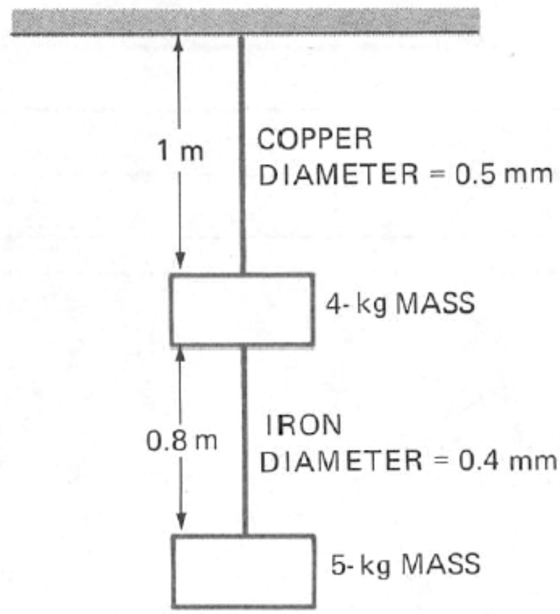


FIGURE 13.7
Problem 10.

[$(F/A)_{Cu} = 4.49 \times 10^8$ N/m^2;
$(F/A)_{Fe} = 3.90 \times 10^8$ N/m^2;
$(\Delta L)_{Cu} = 0.37$ mm;
$(\Delta L)_{Fe} = 0.16$ mm;
$(PE)_{Cu} = 1.65 \times 10^{-1}$ J;
$(PE)_{Fe} = 4.01 \times 10^{-2}$ J]

11. Given Young's modulus for bone = 1.00×10^{10} N/m^2, find the compression experienced by a leg bone 50 cm long subjected to a load of half the weight a 70-kg person. The cross-sectional area of a leg bone is about 5 cm2. [3.4×10^{-5} m]