Mechanics of Materials

Chapter 6

Deflection of Beams

6.1 *Introduction*

- Because the design of beams is frequently governed by rigidity rather than strength. For example, building codes specify limits on **deflections** as well as stresses. Excessive deflection of a beam not only is visually disturbing but also may cause damage to other parts of the building. For this reason, building codes limit the maximum deflection of a beam to about 1/360 th of its spans.

- A number of analytical methods are available for determining the deflections of beams. Their common basis is the differential equation that **relates** the deflection **to** the bending moment. The solution of this equation is complicated because the bending moment is usually a discontinuous function, so that the equations must be integrated in a piecewise fashion.

Consider **two such methods** in this text:

- **Method of double integration** The primary advantage of the double- integration method is that it produces the equation for the deflection everywhere along the beams.

- **Moment-area method** The moment- area method is a semigraphical procedure that utilizes the properties of the area under the bending moment diagram. It is the quickest way to compute the deflection at a **specific location** if the bending moment diagram has a simple shape.

- The ***method of superposition***, in which the applied loading is represented as a series of simple loads for which deflection formulas are available. Then the desired deflection is computed by adding the contributions of the component loads (principle of superposition).

6.2 *Double- Integration Method*

■ Figure 6.1 (a) illustrates the bending deformation of a beam, the displacements and slopes are very **small** if the stresses are below the elastic limit. The deformed axis of the beam is called its *elastic curve*. Derive the differential equation for the elastic curve and describe a method for its solution.

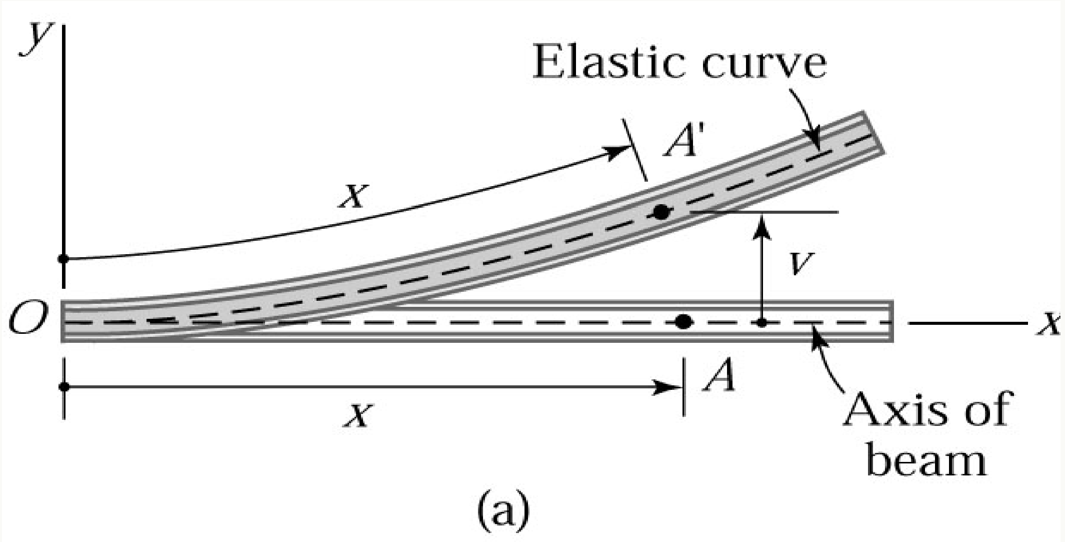


Figure 6.1 (a) Deformation of a beam.

a. *Differential equation of the elastic curve*

- As shown, the vertical deflection of *A*, denoted by *v*, is considered to be positive if directed in the positive direction of the *y*-axis-that is, upward in Fig . 6.1 (a). Because the axis of the beam lies on the neutral surface, its length does not change. Therefore, the distance , measured along the elastic curve, is also *x*. It follows that the horizontal deflection of *A* is negligible provided the slope of the elastic curve remains small.

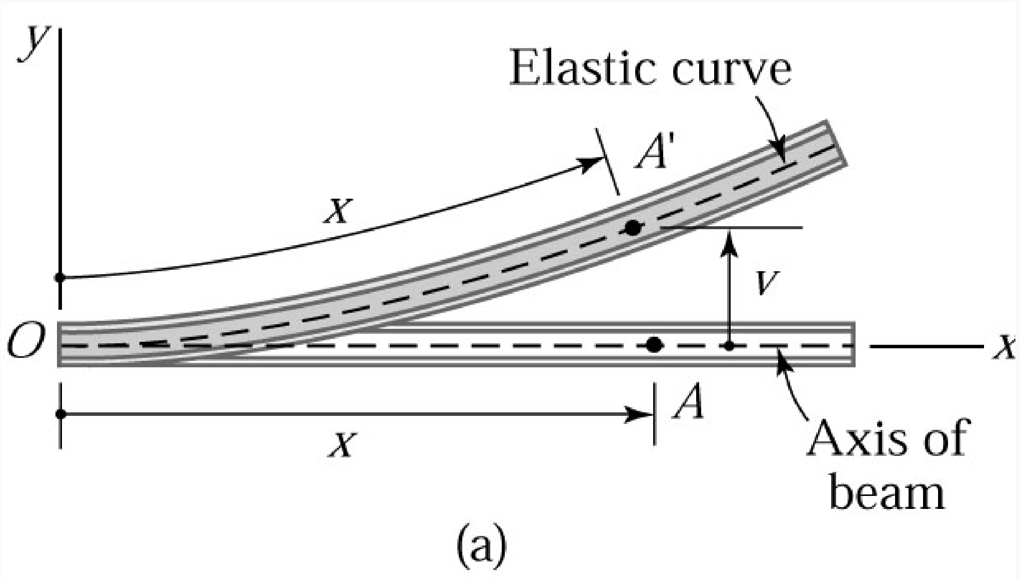


Figure 6.1 (a) Deformation of a beam.

- Consider next the deformation of an infinitesimal segment *AB* of the beam axis, as shown in Fig. 6.1 (b). The elastic curve *A'B'* of the segment has the same length *dx* as the undeformed segment.

- If we let *v* be the deflection of *A*, then the deflection of *B* is *v +dv*, with *dv* being the infinitesimal change in the deflection segment are denoted by θ and $\theta + d\theta$. From the geometry of the figure,

$$\frac{dv}{dx} = \sin\theta \cong \theta \qquad (6.1)$$

- From Fig. 6.1 (b),

$$dx = \rho\, d\theta \qquad (a)$$

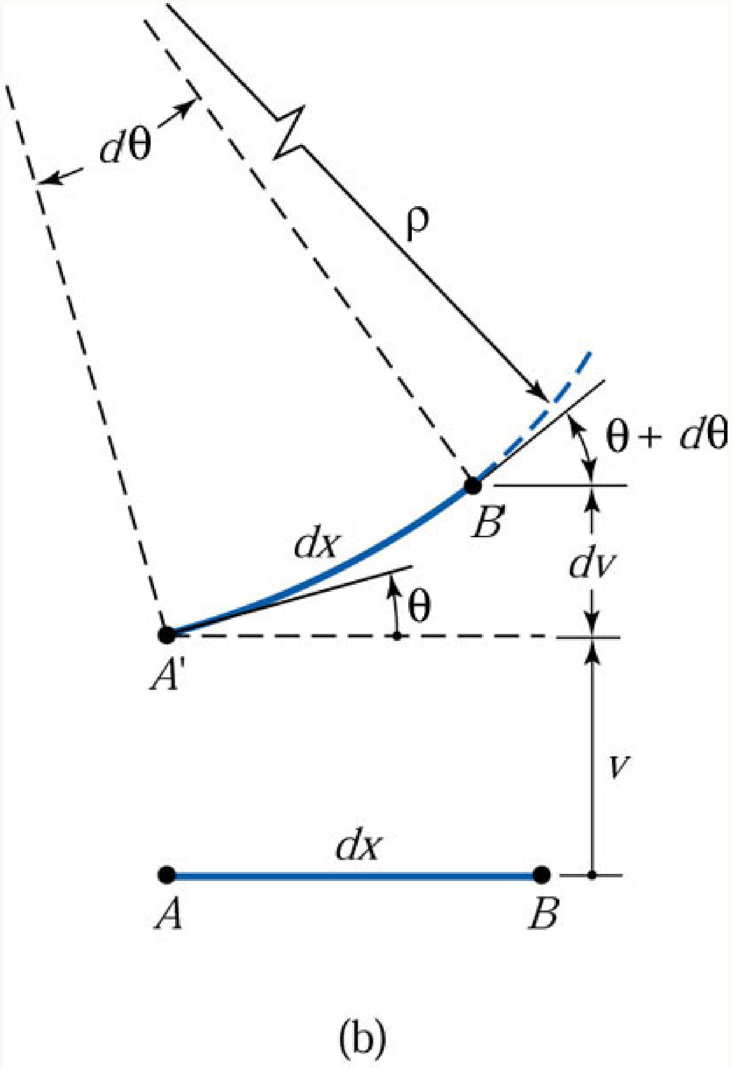


Figure 6.1 (b) Deformation of a differential element of beam axis

The approximation is justified because θ is small. From Fig. 6.1 (b),

$$dx = \rho \, d\theta \qquad \text{(a)}$$

where ρ is the radius of curvature of the deformed segment. Rewriting Eq. (a) as $1/\rho = d\theta/dx$ and substituting θ from Eq. (6.1),

$$\frac{1}{\rho} = \frac{d^2 v}{dx^2} \qquad (6.2)$$

When deriving the flexure formula in Art. 5.2, we obtained the ***moment-curvature relationship***

$$\frac{1}{\rho} = \frac{M}{EI} \qquad \text{(5.2b. repeated)}$$

where M is the bending moment acting on the segment, E is the modulus of elasticity of the beam material, and I represents the modulus of inertia of the cross-sectional area about the neutral (centroidal) axis.

Substitution of Eq.(5.2b) into Eq.(6.2)yields

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \qquad (6.3a)$$

which the ***differential equation of the elastic curve***. The product *EI*, called the *flexural rigidity* of the beam, is usually constant along the beam. It is convenient to write Eq. (6.3a)in the form

$$EI\,v'' = M \qquad (6.3b)$$

Where the prime denotes differentiation with respect to x ; that is, $dv/dx = v'$, $d^2v/dx^2 = v''$, and so on.

b. *Double integration of the differential equation*

If *EI* is constant and *M* is a known function of x, integration of Eq. (6.3b) yields

$$EIv' = \int M dx + C_1 \qquad (6.4)$$

A second integration gives

$$EIv = \iint M dx dx + C_1 x + C_2 \qquad (6.5)$$

where C_1 and C_2 are constants of integration to be determined from the prescribed constraints (for example, the boundary conditions) on the deformation of the beam. Because Eq. (6.5) gives the deflection *v* as a function of x, it is called the *equation of the elastic curve*.

- In Eq. (6.5), the term $\int\int Mdxdx$ gives the shape of the elastic curve. The position of the curve is determined by the constants of integration : C_1 represents a rigid-body rotation about the origin and C_2 is a rigid-body displacement in the *y*-direction. Hence, the computation of the constants is equivalent to adjusting the position of the elastic curve so that it fits properly on the supports.

- If the bending moment of flexural rigidity is not a smooth function of x, a separate differential equation must be written for each beam segment between the discontinuities. This means that if there are *n* such segments, two integrations will produce **2*n*** constants of integration (two per segment). There are also **2*n*** equations available for finding the constants.

- The elastic curve must not contain gaps or kinds. In other words, the slopes and deflections must be continuous at the junctions where the segments meet. Because there are n-1 junctions between the n segments, these ***continuity conditions*** give us $2(n-1)$ equations.
- **Two** additional equations are provided by the ***boundary conditions*** imposed by the supports, so that there are a total of $2(n-1)+2=$ ***2n*** equations.

c. ***Procedure for double integration***

The following procedure assumes that *EI* is constant in each segment of the beam:

- Sketch the elastic curve of the beam, taking into account **the boundary conditions** zero displacement at pin supports as well as zero displacement and zero slope at built-in (cantilever) supports.
- Use the method of sections to determine the bending moment *M* at an arbitrary distance x from the origin. Always show *M* acting in the *positive direction* on the free-body diagram. If the loading has discontinuities, a **separate expression for M** must be obtained for each segment between the discontinuities.

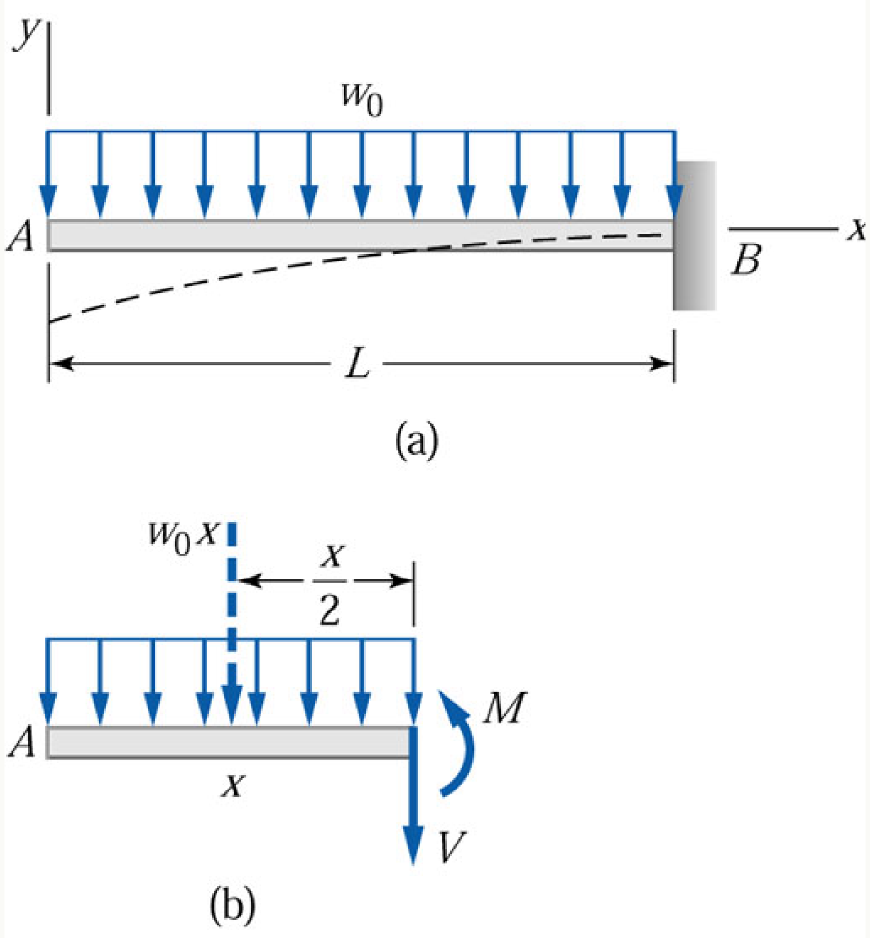
- By integration the expressions for M twice, obtain an expression for $EI\,\upsilon$ in each segment. Do not forget to include the constants of integration.

- Evaluate the constants of integration from the boundary integration and the continuity integration on slope and deflection between segments.

Frequently only the magnitude of the deflection, called the ***displacement***, is required. We denote the displacement by δ; that is, $\delta = |v|$

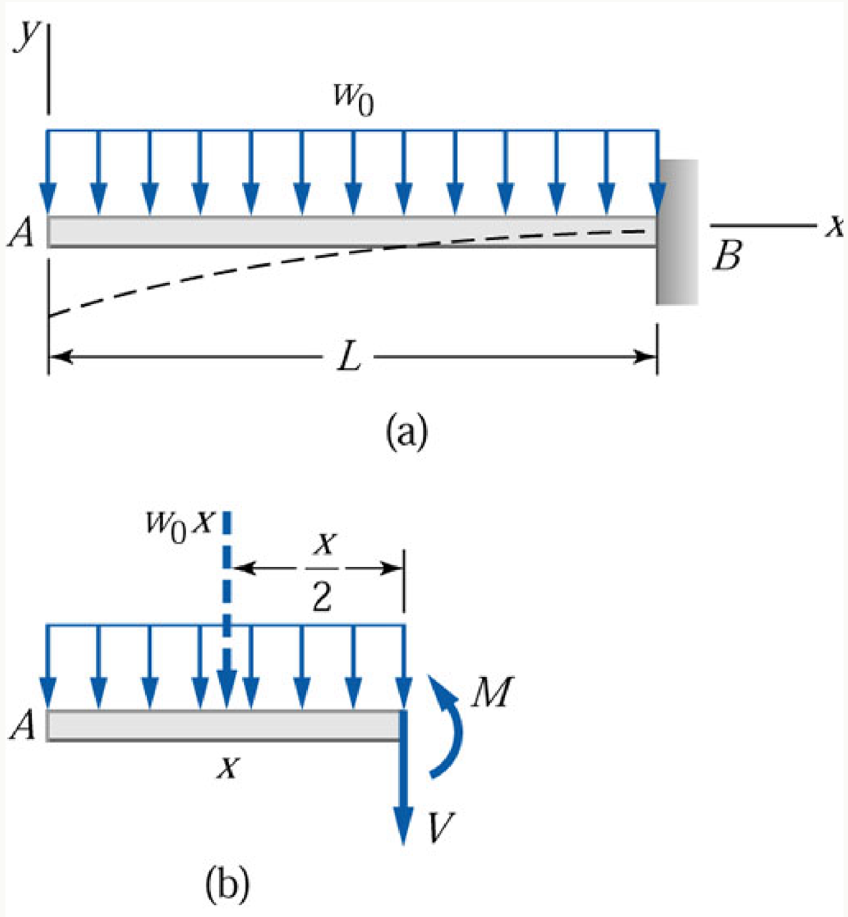
***Sample Problem* 6.1**

The cantilever beam *AB* of length *L* shown in Fig.(a) carries a uniformly distributed load of intensity w_0 , which includes the weight of the beam. (1) Derive the equation of the elastic curve. (2) Compute the maximum displacement if the beam is a W12×35 section using L = 8 ft, w_0 = 400 lb/ft, and $E = 29 \times 10^6$ psi.



Solution

Patr1



The dashed line in Fig. (a) represents the elastic curve of the beam. The bending moment acting at the distance *x* from the left end can be obtained from the free-body diagram in Fig. (b) (note that *V* and *M* are shown acting in their positive directions):

$$M = -w_0 x \frac{x}{2} = -\frac{w_0 x^2}{2}$$

Substituting the expression for *M* into the differential equation EI υ ”= *M*,

$$EI\upsilon'' = -\frac{w_0 x^2}{2}$$

Successive integrations yield

$$EI\upsilon' = -\frac{w_0 x^3}{6} + C_1 \qquad \text{(a)}$$

$$EI\upsilon = -\frac{w_0 x^4}{24} + C_1 x + C_2 \qquad \text{(b)}$$

The constants C_1 and C_2 are obtained from the boundary conditions at the built-in end *B*, which are :

1. $\upsilon'\big|_{x=L} = 0$ (support prevent rotation at B) . Substituting $\upsilon' = 0$ and $x = L$ into Eq. (a),

$$C_1 = \frac{w_0 L^3}{6}$$

2. $\upsilon\big|_{x=L} = 0$ (support prevent deflection at B) . With $\upsilon = 0$ and $x = L$, Eq.(b) becomes

$$0 = \frac{w_0 L^4}{24} + \left(\frac{w_0 L^3}{6}\right) L + C_2$$

$$C_2 = \frac{w_0 L^4}{8}$$

If we substitute C_1 and C_2 into Eq. (b), the equation of the elastic curve is

$$EI\upsilon = \frac{w_0 x^4}{24} + \frac{w_0 L^3}{6} x - \frac{w_0 L^4}{8}$$

$$EI\upsilon = \frac{w_0}{24}\left(-x^4 + 4L^3 x - 3L^4\right)$$

Answer

part 2

From Table B.7 in Appendix B (P521), the properties of a W12×35 shape are I = 285 in.^{4} and S = 45.6 in.^{3} (section modulus). From the result of part 1. the maximum displacement of the beam is (converting feet to inches)

$$\delta_{max} = \left|\upsilon\right|_{x=0} = \frac{woL^4}{8EI} = \frac{(400/12)(8\times 12)^4}{8(29\times 10^6)(625)} = 0.0428 \quad in.$$

Answer

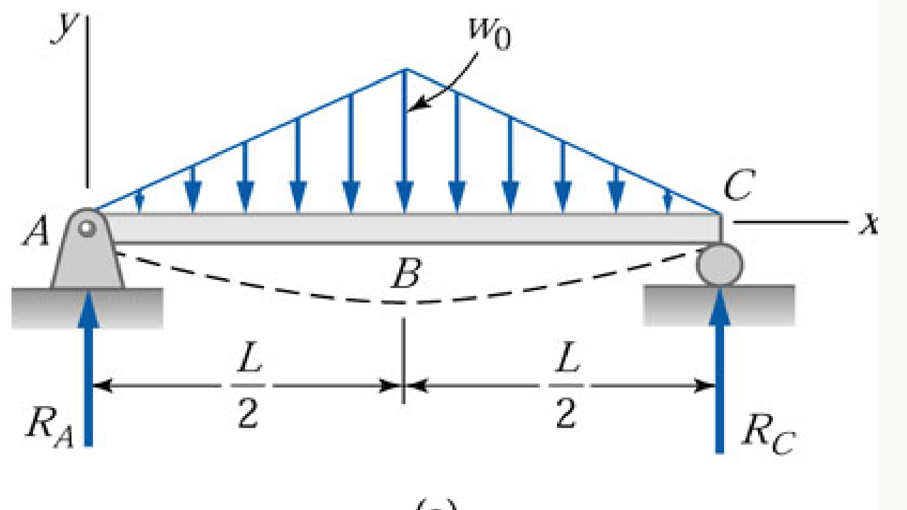
The magnitude of the maximum bending moment, which occurs at B, is $M_{\max} = w_0 L^2/2$. Therefore. the maximum bending stress is

$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{w_0 L^2}{2S} = \frac{(400/12)(8 \times 12)^2}{2(45.6)} = 33700\, psi$$

which close to the proportional limit of 35000 psi (P503) for structural steel. The maximum displacement is very small compared to the length of the beam even when the material is stressed to its proportional limit.

Sample Problem **6.2**

The simple supported beam *ABC* in Fig.(a) carries a distributed load of maximum intensity w_0 over its span of length L. Determine the maximum displacement of the beam.



(a)

Solution

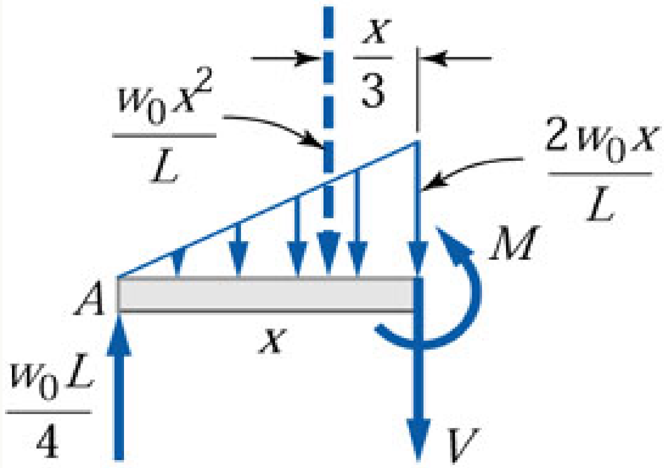
The bending moment and the elastic (the dashed line in Fig. (a)) are symmetric about the midspan. Therefore, we will analyze only the left half of the beam (segment *AB*).

Because of the symmetry, the reactions are

$$R_A = R_C = w_0 L / 4.$$

The bending moment in *AB* can be obtained from the free-body diagram in Fig. (b), yielding

$$M = \frac{w_0 L}{4}x - \frac{w_0 x^2}{L}\left(\frac{x}{3}\right) = \frac{w_0}{12L}\left(3L^2 x - 4x^3\right)$$



(b)

$$EI\upsilon'' = \frac{w_0}{12L}\left(3L^2 x - 4x^3\right)$$

$$EL\upsilon' = \frac{w_0}{12L}\left(\frac{3L^2 x^2}{2} - x^4\right) + C_1 \qquad \text{(a)}$$

$$EI\upsilon = \frac{w_0}{12L}\left(\frac{L^2 x^3}{2} - \frac{x^5}{5}\right) + C_1 x + C_2 \qquad \text{(b)}$$

The two constant can be evaluated from the following two conditions on the the elastic curve of segment *AB*:

1. $\upsilon\,|_{x=0}=0$ (no deflection at *A* due to the simple support).

$$EI\upsilon = \frac{w_0}{12L}\left(\frac{L^2x^3}{2} - \frac{x^5}{5}\right) + C_1x + C_2$$

$$C_2 = 0$$

2. $\upsilon\,'|_{x=L/2} = 0$ (due to symmetry, the slope at midspan is zero),

$$EL\upsilon' = \frac{w_0}{12L}\left(\frac{3L^2x^2}{2} - x^4\right) + C_1$$

$$0 = \frac{w_0}{12L}\left(\frac{3L^4}{8} - \frac{L^4}{16}\right) + C_1$$

$$C_1 = \frac{5w_0L^3}{192}$$

The equation of the elastic curve for segment *AB*:

$$EI\upsilon = \frac{w_0}{12L}\left(\frac{L^2x^3}{2} - \frac{x^5}{5}\right) - \frac{5w_0L^3}{193}x$$

$$EI\upsilon = -\frac{w_0 x}{960L}\left(25L^4 - 40L^2x^2 + 16x^4\right)$$

By symmetry, the maximum displacement occurs at midspan. Evaluation Eq. (c) at $x = L/2$,

$$EI\upsilon\Big|_{x=L/2} = \frac{w_0}{960L}\left(\frac{L}{2}\right)\left[25L - 40L^2\left(\frac{L}{2}\right)^2 + 16\left(\frac{L}{2}\right)^4\right] = -\frac{w_0L^4}{120}$$

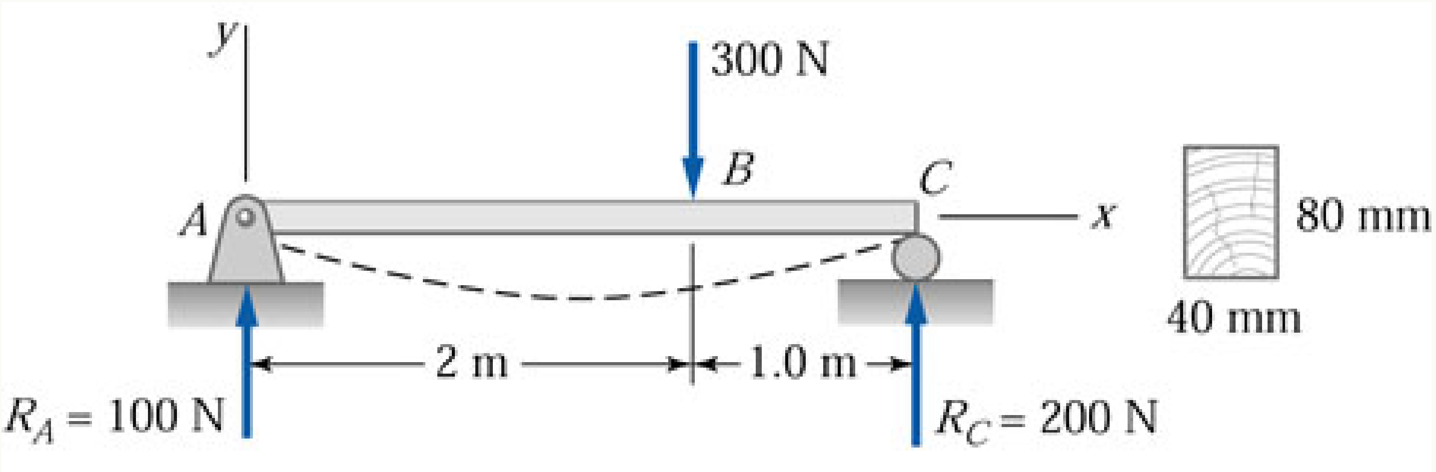
The negative sign indicates that the deflection is downward. The maximum displacement is

$$\delta_{max} = |\upsilon|_{x=l/2} = \frac{w_0L^4}{120EI}\downarrow$$

Answer

Sample Problem 6.3

The simply supported wood beam *ABC* in Fig. (a) has the rectangular cross section shown. The beam supports a concentrated load of 300 N located 2 m from the left support. Determine the maximum displacement and maximum slope angle of the beam. Use E = 12 Gpa for the modulus of elasticity. Neglect the weight of the beam.



(a)

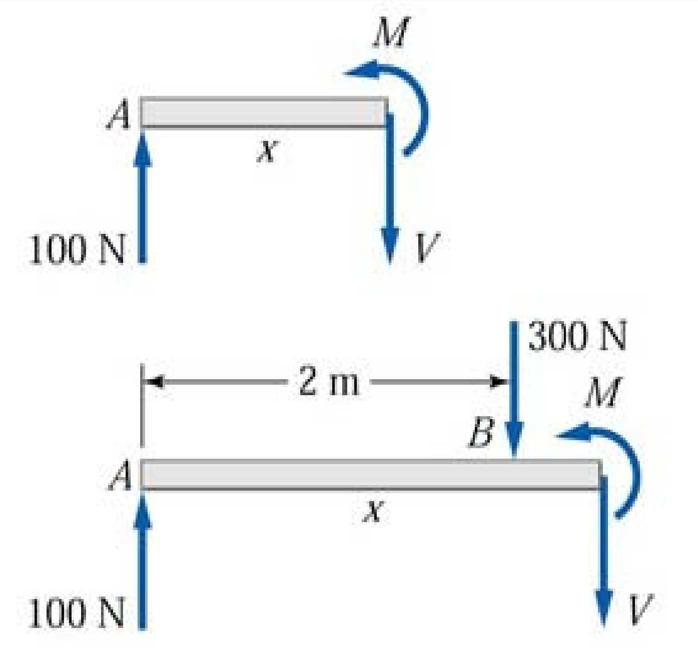
Solution

The moment of inertia of the cross-sectional area is

$$I = \frac{bh^3}{12} = \frac{40(80)^3}{12} = 1.7067 \times 10^6\, mm^4 = 1.7067 \times 10^{-6}\, m^4$$

Therefore, the flexural rigidity of the beam is

$$EI = (12\times10^9)(1.7067\times10^{-6}) = 20.48\times103 \text{ N} \bullet \text{m}^2$$



(b)

Because the loading is discontinuous at *B*. the beam must be divided into two segments: AB and BC. The beading moments in the two segments of the beam can be derived from the free-body diagrams in Fig.(b).

The results are

$$M = \begin{cases} 100xN \cdot m & inAB\ (0 \le x \le 2m) \\ 100x - 300(x-2)N \cdot m & inBC\ (2m \le x \le 3m) \end{cases}$$

They must be treated separately during double integration, integrating twice, we get the following computations;

Segment AB

$$EI\upsilon'' = 100xN \cdot m$$

$$EI\upsilon' = 50x^2 + C_1 N \cdot m^2 \qquad \text{(a)}$$

$$EI\upsilon = \frac{50}{3}x^3 + C_1 x + C_2 N \cdot m^3 \qquad \text{(b)}$$

Segment BC

$$EI\upsilon'' = 100x - 300(x-2)N \cdot m$$

$$EI\upsilon' = 50x^2 - 150(x-2)^2 + C_3 N \cdot m^2 \qquad \text{(c)}$$

$$EI\upsilon = \frac{50}{3}x^3 - 50(x-2)^3 + C_3 x + C_4 N \cdot m^3 \qquad \text{(d)}$$

The four constants of integration, C_1 to C_4, can be found the following boundary and continuity conditions:

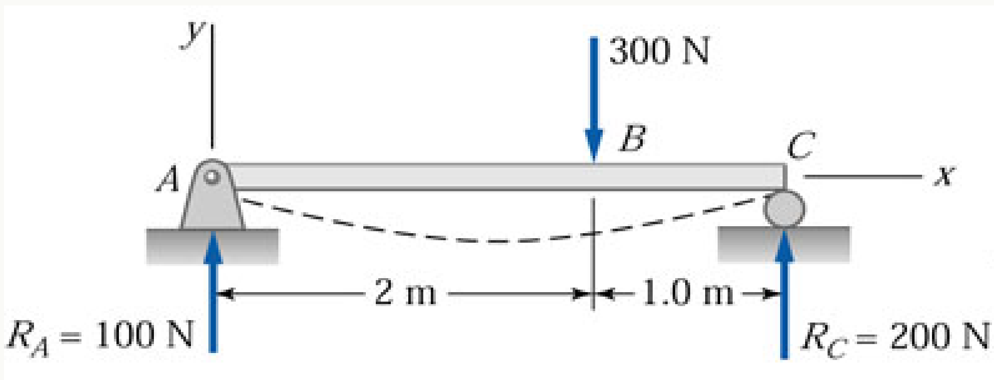
1. $\upsilon\,|_{x=0} = 0$ (no deflection at *A* due to the support).

$$C_2 = 0 \qquad \text{(e)}$$

2. $\upsilon\,|_{x=3\text{m}} = 0$ (no deflection at *C* due to the support).

$$0 = \frac{50}{3}(3)^3 - 50(3-2)^3 + C_3(3) + C_4$$

$$3C_3 + C_4 = -400\ \text{N}\cdot\text{m}^3 \qquad \text{(f)}$$



(a)

3. $\upsilon'|_{x=2m-} = \upsilon'|_{x=2m+}$ (the slope at *B* is continuous).

$$50(2)^2 + C_1 = 50(2)^2 + C_4$$

$$C_1 = C_3 \qquad \text{(g)}$$

4. $\upsilon|_{x=2m-} = \upsilon|_{x=2m+}$ (the slope at *B* is continuous).

$$\frac{50}{3}(2)^3 + C_1(2) + C_2 = \frac{50}{3}(2)^3 + C_3(2) + C_4 \qquad \text{(h)}$$

$$2C_1 + C_2 = 2C_3 + C_4$$

The solution of Eqs.(c)-(h) is

$$C_1 = C_3 = \frac{400}{3} N \cdot m^2 \qquad C_2 = C_4 = 0$$

Substituting the values of the constants and *EI* into Eqs.(a)-(d), we obtain the following results:

Segment AB

$$\upsilon' = \frac{50x^2 - (400/3)}{20.48\times10^3} = (2.441x^2 - 6.510)\times10^{-3}$$

$$\upsilon = \frac{(50/3)x^2 - (400/3)x}{20.48\times10^3} = (0.8138x^3 - 6.510x)\times10^{-3}\,m$$

Segment BC

$$\upsilon' = \frac{50x^2 - 150(x-2)^2 - (400/3)}{20.48\times10^3}$$

$$= \left[2.441x^2 - 7.324(x-2)^2 - 6.150\right]\times10^{-3}$$

$$\upsilon = \frac{(50/3)x^3 - 50(x-2)^3 - (400/3)x}{20.48\times10^3}$$

$$= \left[0.81.38x^3 - 2.44(x-2)^3 - 6.150\,x\right]\times10^{-3}\,m$$

The maximum displacement occurs where the slope of the elastic curve is zero.This point is in the longer of the two segments, Setting υ '= 0 in the segment AB

$$2.441x^2 - 6.510=0 \qquad x = 1.6331 \text{ m},$$

The corresponding deflection is

$$\upsilon|_{x=1.6331\text{m}} = [0.8138(1.6133)^3-6.510(1.6133)]\cdot 10^{-3}$$

$$= -7.09 * 10^{-3} \text{ m} = -7.09 \text{ mm}$$

The negative sign indicates that the deflection is downward, as expected. Thus, the maximum displacement is

$$\delta_{\max} = |\upsilon|_{x=|.633|m} = 7.09mm \downarrow \qquad \textit{Answer}$$

By inspection of the elastic curve in Fig.(a), the largest slope occurs at *C*.

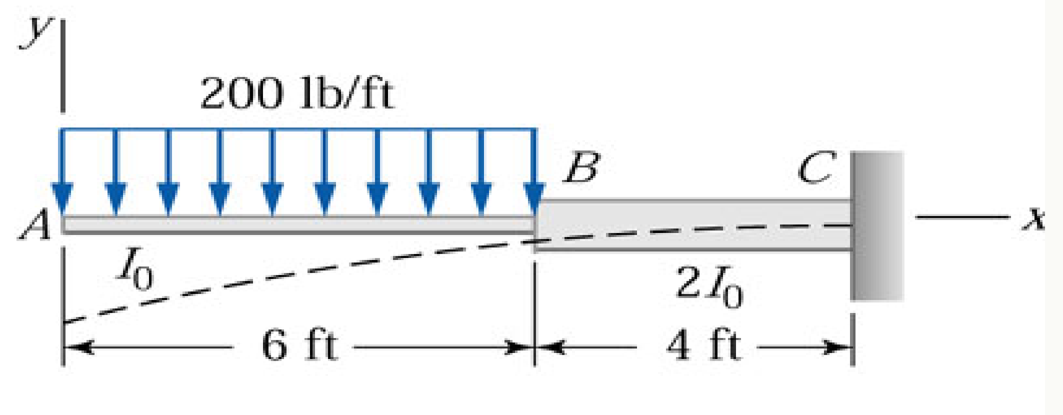
$$\upsilon'|_{x=3m} = [2.44(3)^2 - 7.324(3-2)^2 - 6.150] \times 10^{-3} = 8.50 \times 10^{-3}$$

According to the sign conventions for slopes, the positive value for υ' means that the beam rotates counterclockwise at *C*. Therefore, the maximum slope angle of the beam is

$$\theta_{\max} = |\upsilon'|_{x=3m} = 8.50 \times 10^{-3} rad = 0.487^0 \quad \circlearrowleft \qquad Answer$$

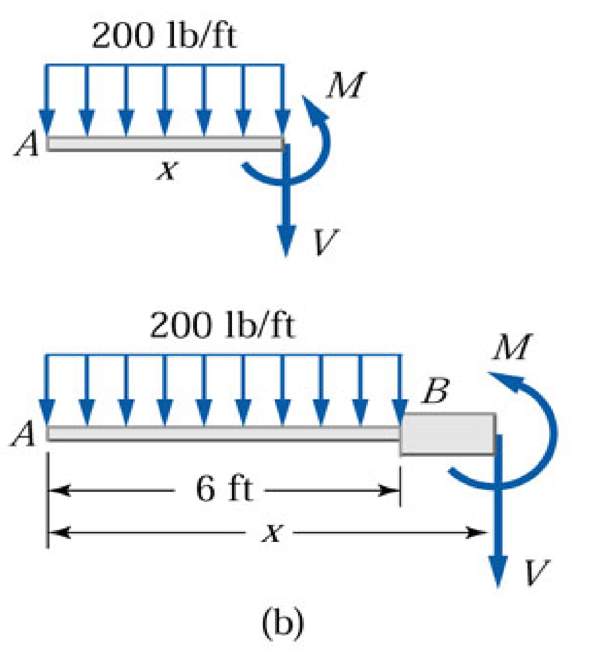
***Sample Problem* 6.4**

The cantilever beam *ABC in* Fig.(a) consists of two segments with different moment of inertia : I_0 for segment *AB* and $2I_0$ for segment *BC*. Segment AB carries a uniformly distributed load of intensity 200 lb/ft. Using $E = 10 \times 10^6$ psi and $I_0 = 40$ in.^{4} , determine the maximum displacement of the beam.



(a)

Solution



The dashed line in Fig.(a) represents the elastic curve of the beam. The bending moments in the two segments, obtain from the free-body diagram in Fig.(b), are

$M = -100\,x^2$ lb·ft in AB ($0 \leqq x \leqq 6$ft)

$M = -1200(x-3)$ lb·ft in BC ($6\text{ft} \leqq x \leqq 10$ft)

Substituting the expressions for M into Eq.(6.3b) and integrating twice yield the following result:

Segment AB **(*I = I₀*)**

$$EI\upsilon'' = -100x^2 lb\cdot ft$$

$$EI_0\upsilon' = -\frac{100}{3}x^3 + C_1 lb\cdot ft^2 \qquad \text{(a)}$$

$$EI_0\upsilon = \frac{25}{3}x^4 + C_1 x + C_2 lb\cdot ft^3 \qquad \text{(b)}$$

Segment BC(I =2 I_0)

$$E(2I_0)\upsilon'' = -1200(x-3)\,lb\cdot ft \quad \text{or} \quad EI_0\upsilon'' = -600(x-3)\,lb\cdot ft$$

$$EI_0\upsilon' = -300(x-3)^2 + C_3\,lb\cdot ft^2 \qquad \text{(c)}$$

$$EI_0\upsilon = -100(x-3)^3 + C_3x + C_4\,lb\cdot ft^3 \qquad \text{(d)}$$

The conditions for evaluating the four constants of integration

1. $\upsilon'|_{x = 10\text{ ft}} = 0$ (no rotation at C due to the built-in support).

$0 = -300(10-3)^2 + C_3$, $\qquad C_3 = 14.70\times10^3$ lb · ft

2. $\upsilon|_{x = 10\text{ ft}} = 0$ (no deflection at C due to the built-in support).

$$0 = -100\,(10-3)^3 + (14.70\times10^3)\,(10) + C_4$$

$$C_4 = -112.7\times103\text{lb}\cdot\text{ft}^3$$

3. $\upsilon'|_{x = 6\text{ ft-}} = \upsilon'|_{x = 6\text{ ft+}}$ (the slope at *B* is continuous).

$$-\frac{100}{3}\left(6^3\right)+C_1=-300(6-3)^2+\left(14.7\times10^3\right) \qquad C_1=19.20\times10 lb\cdot ft^2$$

4. $\upsilon|_{x = 6\text{ft-}} = \upsilon|_{x = 6\text{ ft +}}$ (the displacement at *B* is continuous).

$$-\frac{25}{3}(6)^4+\left(19.20\times10^3\right)(6)+C_2=-100(6-3)^3+\left(14.70\times10^3\right)(6)-\left(112.7\times10^3\right)$$

$$C_2=-131.6\times10^3 lb\cdot ft^3$$

The maximum deflection of the beam occurs at *A*, at x =0.

$$EI_0\,\upsilon\,|_{x=0} = -131.6\times10^3 \text{ lb}\cdot\text{ft}^3 = -227\times106 \text{ lb}\cdot\text{in.}^3$$

The negative sign indicates that the deflection of *A* is downward.

The maximum displacement

$$\delta_{\max}=|\upsilon|_{x=0}=\frac{227.4\times10^6}{EI_0}=\frac{227.4\times10^6}{\left(10\times10^6\right)(40)}=0.569 in.\downarrow \qquad Answer$$

6.4 *Moment- Area Method*

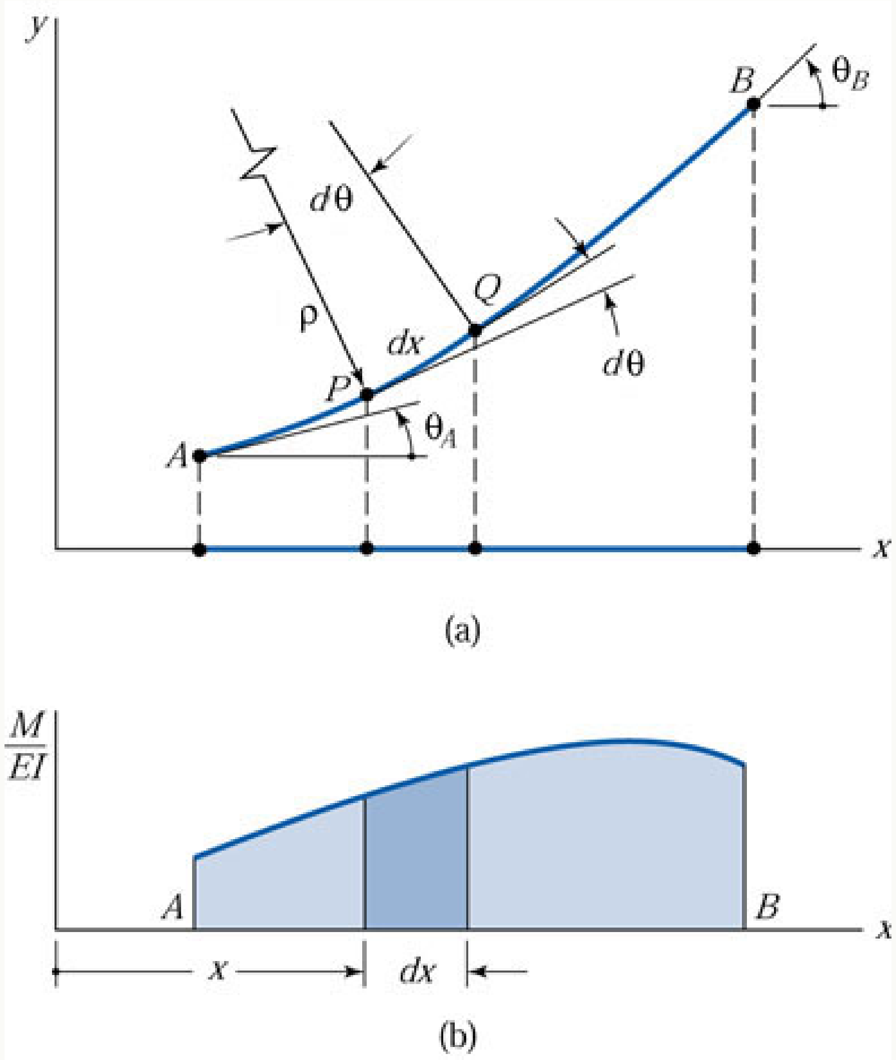
- The moment-area method is useful for determining the slope or deflection of a beam at a specified location. It is a semigraphical method in which the integration of the bending moment is carried out indirectly, using the geometric properties of the area under the bending moment diagram.

- As in the method of double integration, we assume that the deformation is within the elastic range, resulting in small slopes and small displacements.

a. *Moment- Area theorems*

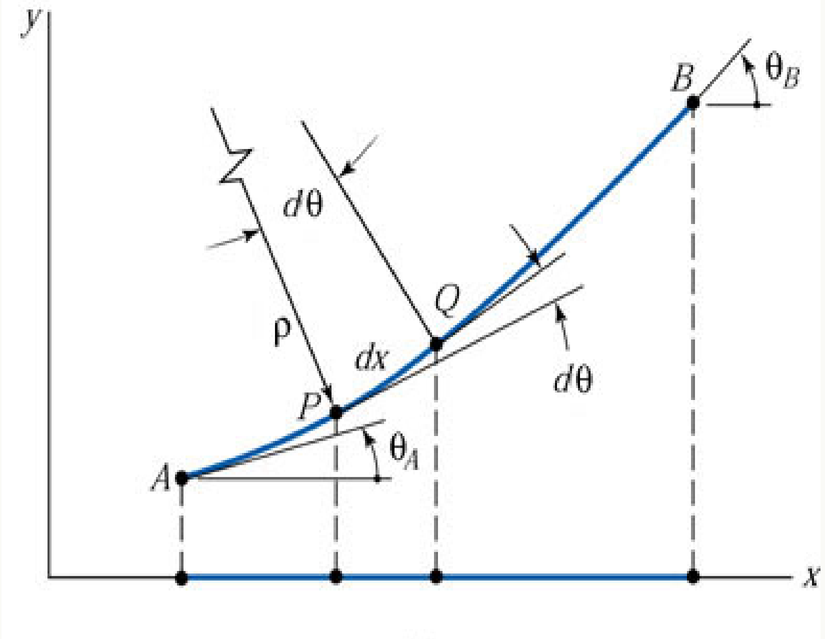
First Moment- *Area Theorem*

Figure 6.4(a) shows the elastic curve AB of an initially straight beam segment. As discussed in the derivation of the flexure formula in Art. 5.2 two cross sections of the beam at P and *Q*, separated by the distance *dx*, rotate through the angle $d\theta$ relative to each other.

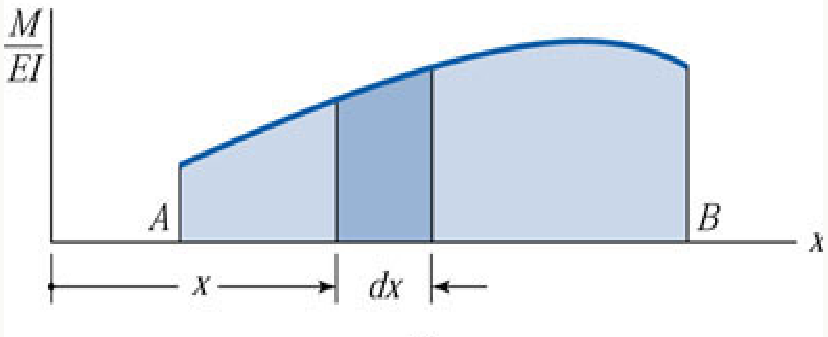


- Because cross sections are assumed to remain perpendicular to the axis of the beam $d\theta$ is also the difference in the slope of the elastic curve between P and Q, as shown in Fig. 6.4 (a).
- From the geometry of the figure, we see that $dx = \rho\, d\theta$,where ρ is the radius of curvature of the elastic curve of the deformed element. Therefore, $d\theta = dx/\rho$, which upon using the moment-curvature relationship

$$\frac{1}{\rho} = \frac{M}{EI} \qquad \text{(5.2b. repeated)}$$



(a)

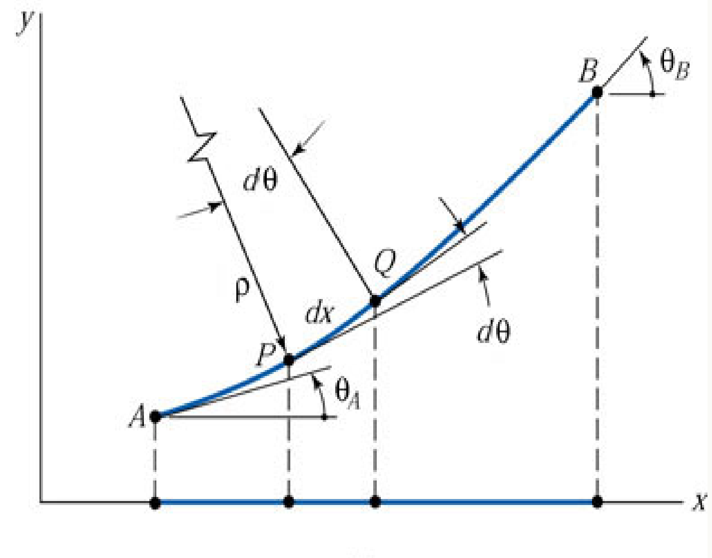


(b)

becomes

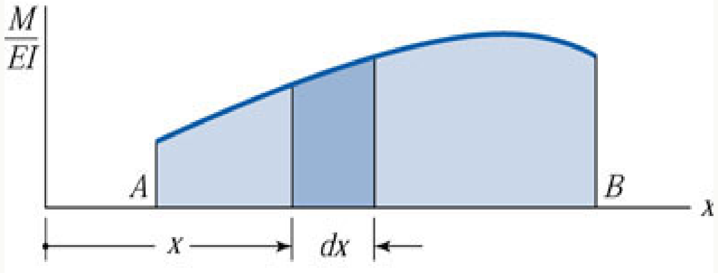
$$d\theta = \frac{M}{EI}dx \quad \text{(a)} \qquad \int_A^B d\theta = \int_A^B \frac{M}{EI}dx \quad \text{(b)}$$

The left side of Eq. (b) is $\theta_B - \theta_A$ which is the change in the slope between *A* and *B*. The right-hand side represents the area under the *M*/(*EI*) diagram between *A* and *B*.



(a)

If we introduce the notation $\theta_{B/A} = \theta_B - \theta_A$, Eq. (b) can be expressed in the form



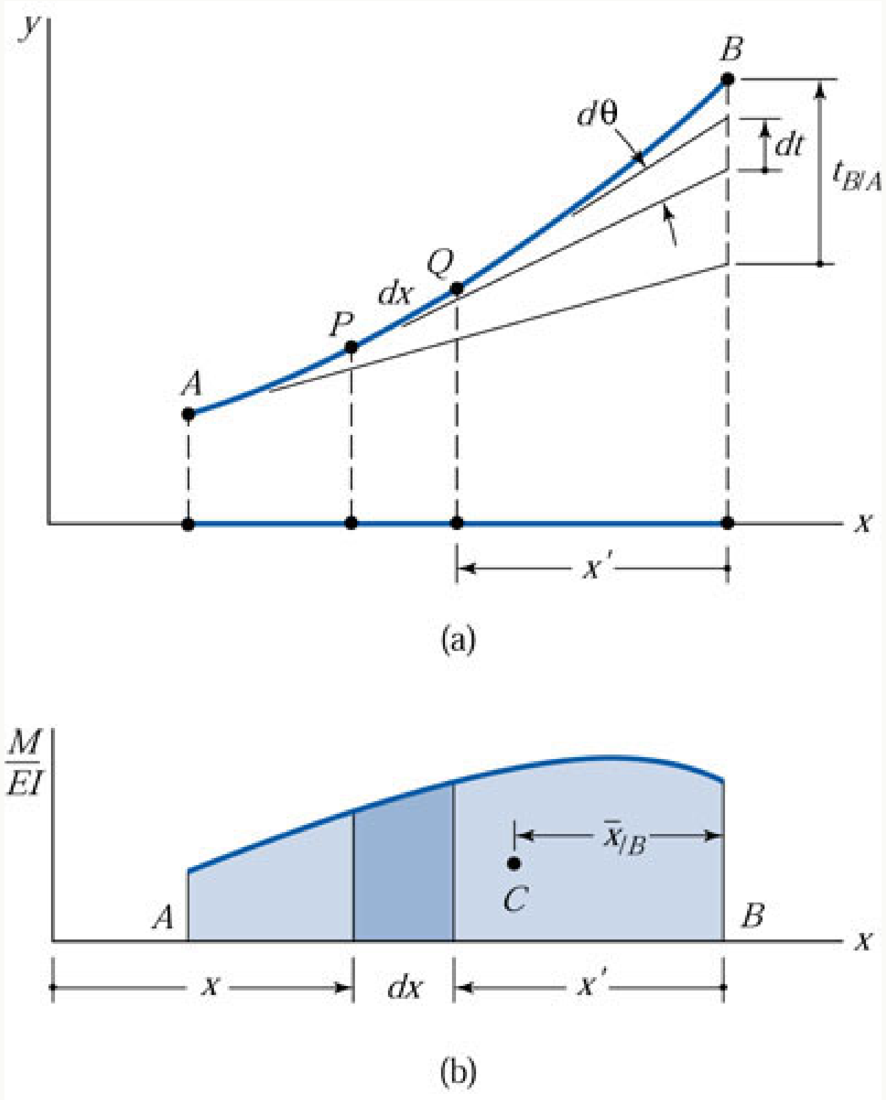
(b)

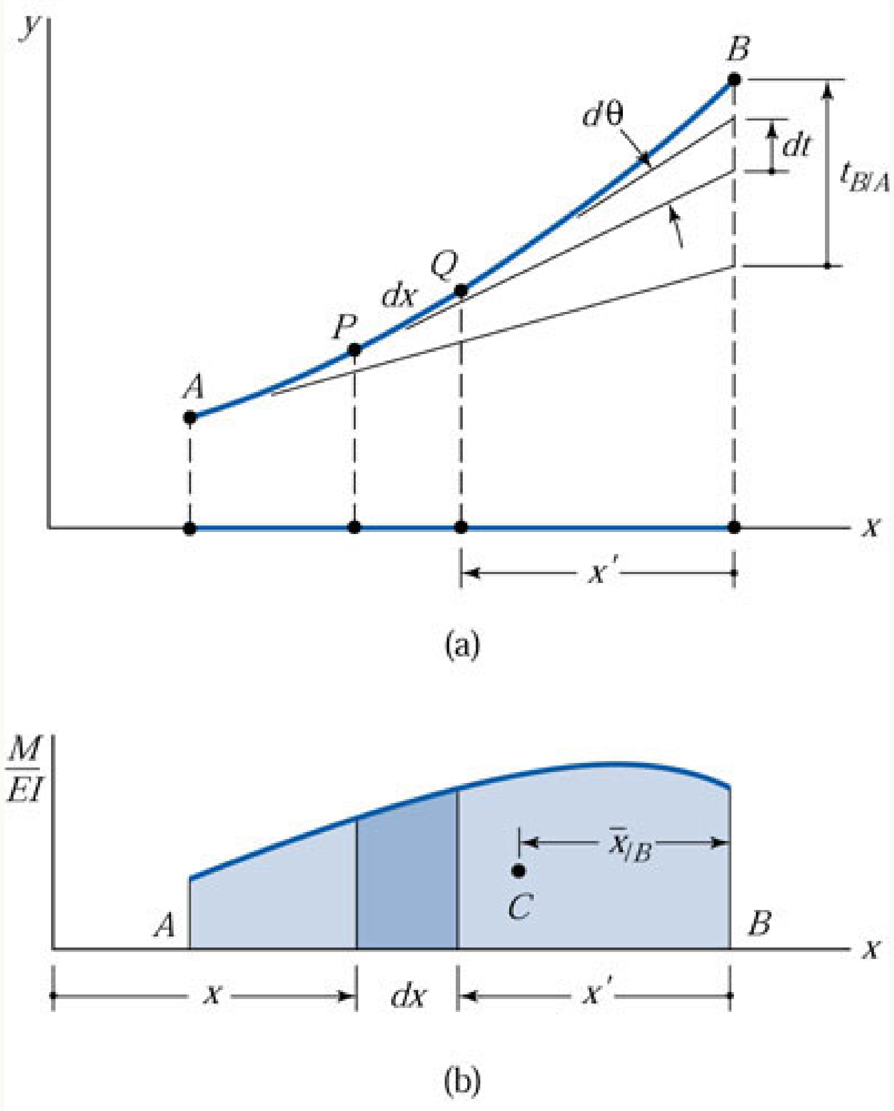
$$\theta_{B/A} = area \quad of \quad \frac{M}{EI} \quad diagram \quad \Big|_A^B \qquad (6.8)$$

which is ***the first moment-area theorem***.

Second Moment-Area Theorem

Referring to the elastic curve AB in Fig.6.5(a), we let $t_{B/A}$ be the vertical distance of point *B* from the tangent to the elastic curve at *A*. This distance is called **the tangential deviation of *B* with respect to *A***. To calculate the tangential deviation, we first determine the contributions *dt* of the infinitesimal element PQ and then use $t_{B/A} = \int_A^B dt$ to add the contributions of all the elements between *A* and *B*.





As shown in the figure, *dt* is the vertical distance at B between the tangents drawn to the elastic curve at P and Q. Recalling that the slopes are very small, we obtain from geometry

$$dt = x' \, d\theta$$

where *x'* is the horizontal distance of the element from *B*.

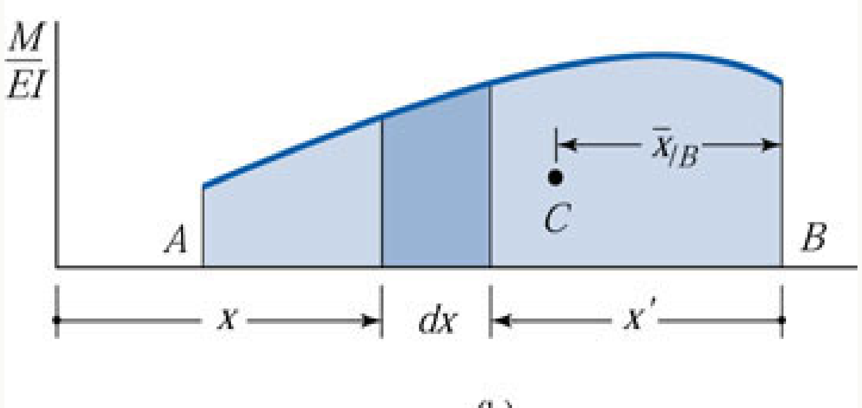
Figure 6.5 (a) Elastic curve of a beam segment; (b) bending moment diagram for the segment.

The tangential deviation is

$$t_{B/A} = \int_A^B dt = \int_A^B x' d\theta$$

Substituting $d\theta$ from Eq. (a)

$$t_{B/A} = \int_A^B \frac{M}{EI} x' dx \qquad \text{(c)}$$



(b)

The right-hand side of Eq.(c) represents the first moment of the shaded area of the *M*/(EI) diagram in Fig. 6.5 (b) about point B. Denoting the distance between B and the centroid *C* of this area by $\bar{x}_{/B}$ (read /*B* as "relative"to B"), we can write Eq.(c) as

$$t_{B/A} = area \quad of \quad \frac{M}{EI} \quad diagram \ \Big|_A^B \cdot \bar{x}_{/B} \qquad (6.9)$$

This is **the second moment-area theorem**. Note that the first moment of area represented by the right-hand side of Eq. (6.9), is always taken about the point at which the deviation is being computed.

- Do not confuse $t_{B/A}$(the tangential deviation of *B* with respect to *A*) with $t_{B/A}$(the tangential deviation of *A* with respect to *B*). In general, these two distance are not equal, as illustrated in Fig.6.6.

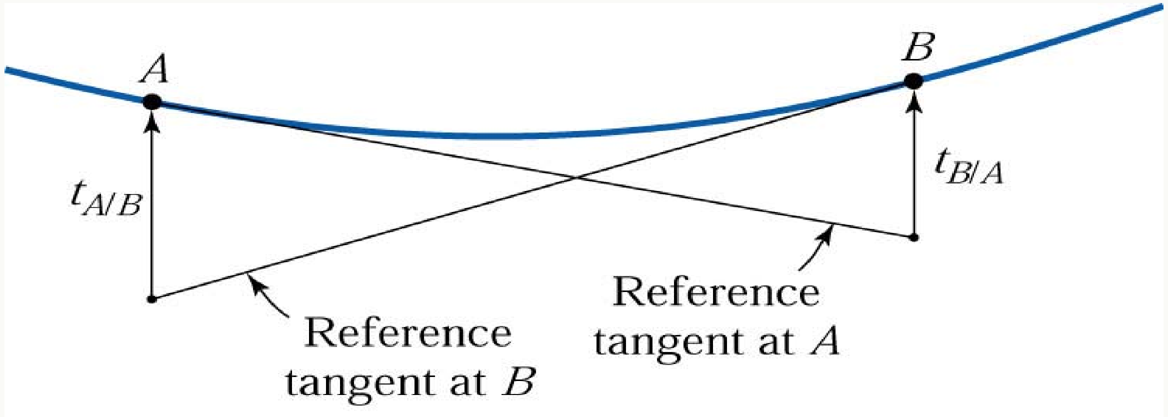
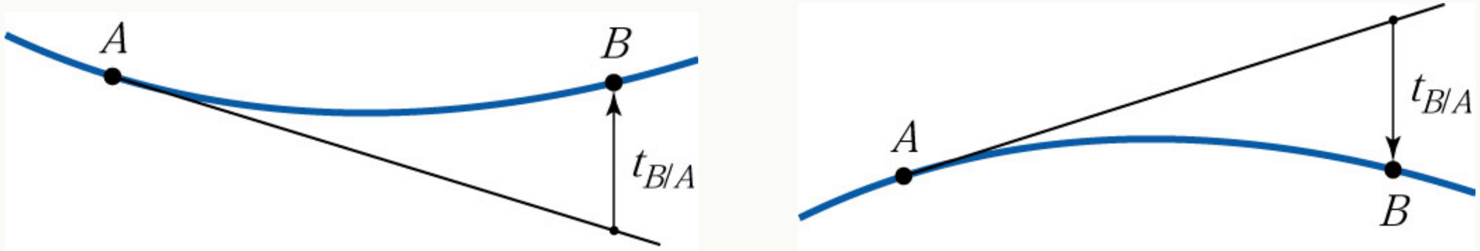


Figure 6.6 Tangential Deviations of the elastic curve.

Sign Conventions

The following rules of sign illustrated in Fig 6.7, apply to the two moment-area theorems:

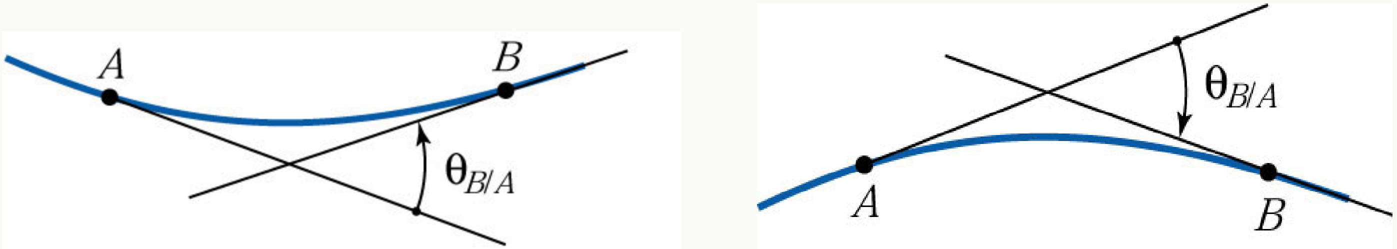
- The tangential deviation $t_{B/A}$ is positive if *B* lies above the tangent line drawn to the elastic curve at *A*, and negative if *B* lies above the tangent line.



(a) Positive deviation; *B* located above reference tangent

(b) Negative deviation; *B* located below reference tangent

- Positive $\theta_{B/A}$ has a counterclockwise direction, whereas negative $\theta_{B/A}$ has a clockwise direction.



(c) Positive change of slope is counterclockwise from left tangent

(d) Negative change of slope is clockwise from left tangent

Figure 6.7 (a through d) SignConventions for tangential deviation and change of slope.

b. Bending moment diagrams by parts

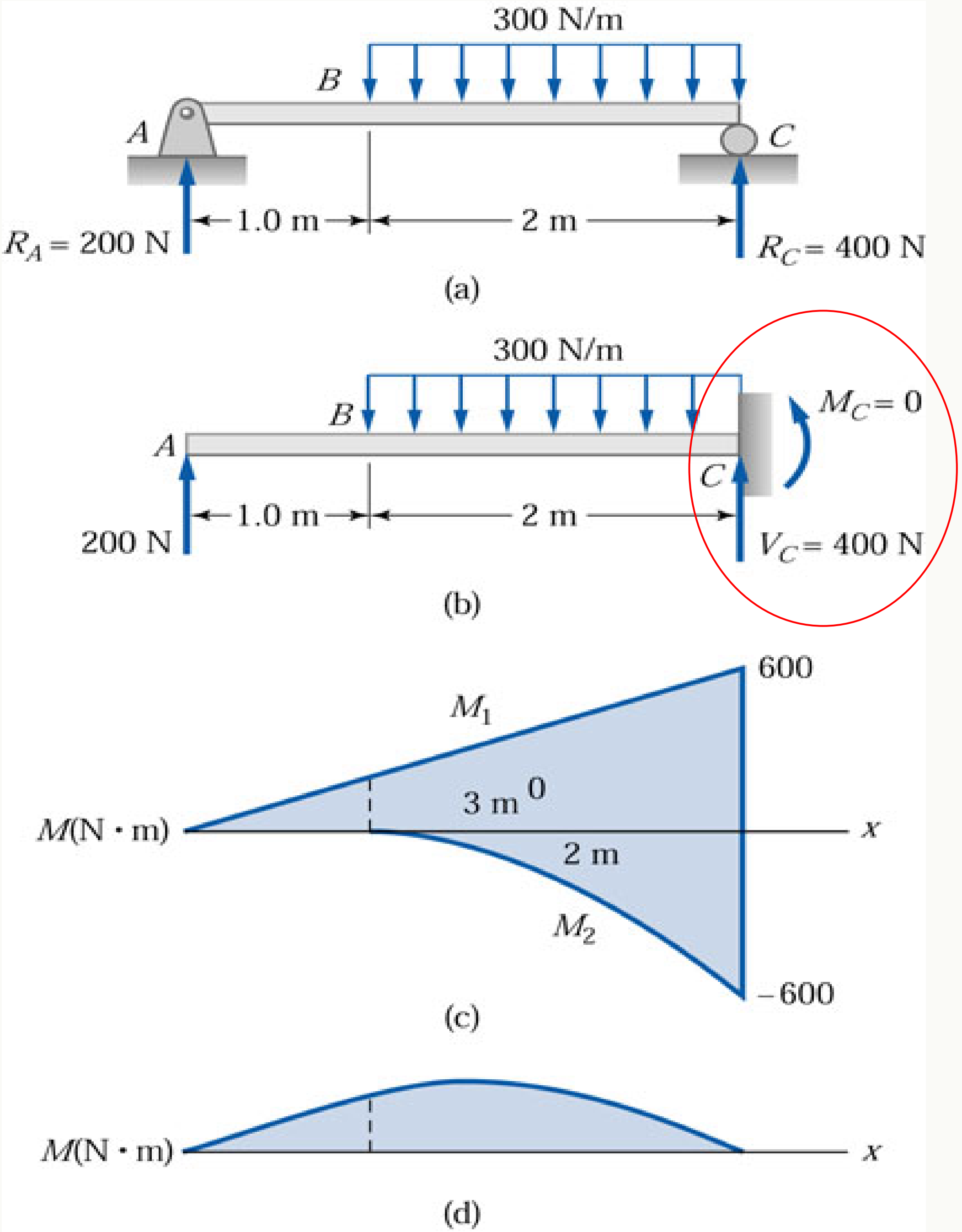
- Application of the moment-area theorems is practical only if the area under the bending moment diagrams and its first moment can be calculated without difficulty.
- The key to simplifying the computation is to divide the bending moment diagram into simple geometric shapes (rectangles, triangles, and parabolas) that have known areas and centroidal coordinates.
- Sometimes the conventional bending moment diagrams lends itself to such division, but often it is preferable to draw the ***bending moment diagram by parts***, with each part of the diagrams representing the effect of one load. (Consider different EI)

- Construction of the bending moment diagram by parts for *simply supported* beams proceeds as follows;

- Calculate the simply support reactions and consider them to be applied loads.

- Introduce a **fixed support** as a convenient location. A simply support of the original beam is usually a good choice, but sometimes another point is more convenient. The beam is now cantilevered from this support.

- Draw a bending moment diagram for each load (including the support reactions of the original beam). If all the diagrams can be fitted on a single plot, do so, drawing the positive moment above the *x*-axis and the negative moment below the *x*-axis.



The moment M_1 due to R_A is positive, whereas the distributed load results in a negative moment M_2. The conventional bending moment diagram M, shown in Fig.6.8(d), obtained by superimposing M_1 and M_2- that is , $M=M_1+M_2$.

Figure 6.8 (a) Simply supported beam; (b)equivalent beam with fixed support at *C*; (c) bending moment diagram by parts; (d) conventional ending moment.

Therefore, the bending moment diagram in Fig. 6.9 (b) now contains three parts.

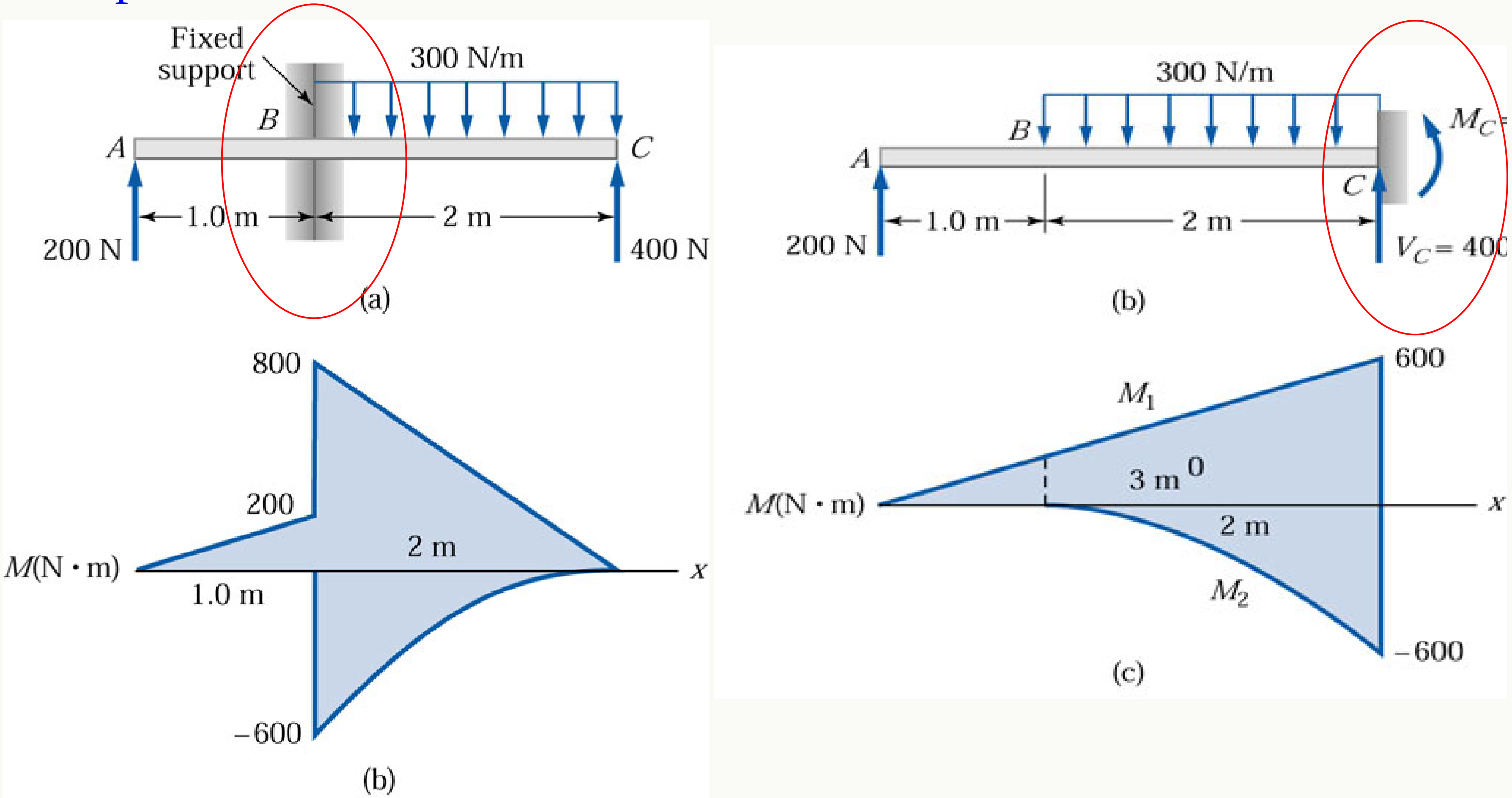


Figure 6.9 (a) Beam with fixed support at *B* that is equivalent to the simply supported beam in Fig.6.8;(b) bending Moment diagram by parts.

- When we construct the bending moment diagram by parts, each part is invariably of the form $M = kx^n$, where n is a nonnegative integer that represents the degree of the moment equation. Table 6.1 shows the properties of areas under the *M*-diagram for n = 0, 1, 2 and 3, This table is useful in computations repuired by the moment-area method.
- **Table 6.1** (see book) *Properties of Area Bounded by* $M = k \cdot x^n$

C. *Application of the moment-area method*

Cantilever Bems Because the support at *A* is fixed, the tangent drawn to the elastic curve at *A* is horizontal. Therefore, $t_{B/A}$ (the tangential deviation of *B* with respect to *A*) has the same magnitude as the displacement of *B*.In other words, , where

$$t_{B/A} = area \quad of \quad \frac{M}{EI} diagram \ \Big]_A^B \cdot \bar{x} /_B$$

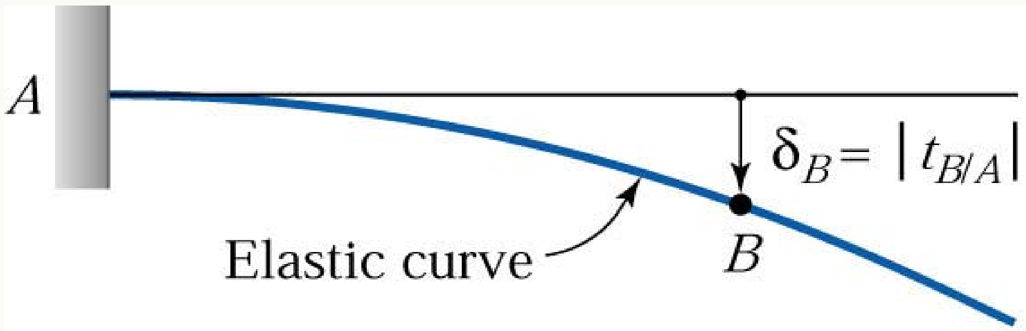


Figure 6.10 The displacement Equals the magnitude of the tangential deviation for point *B*on the cantilever beam.

Simply Supported Beams The problem is to compute the displacement δ_B of a point B located a distance *x* from *A*. Because the point δ_B at which a tangent to the elastic curve is usually unknown. If a tangent is draw to the elastic curve at *A*, the tangential deviation $t_{B/A}$ is evidently not the displacement δ_B. From the figure $\delta_B = \theta_A \cdot x - t_{B/A}$, we must compute the slope angle θ_A as well as $t_{B/A}$.

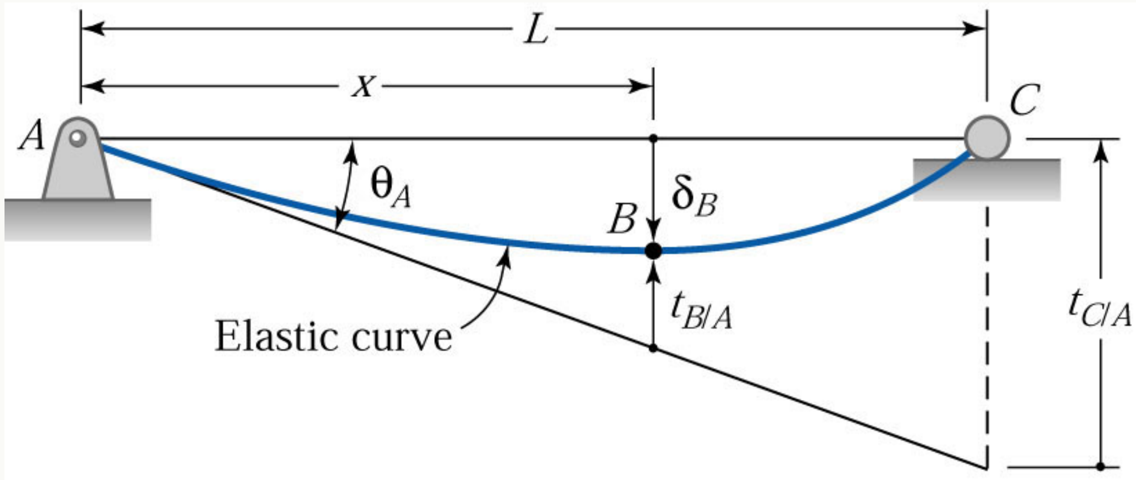


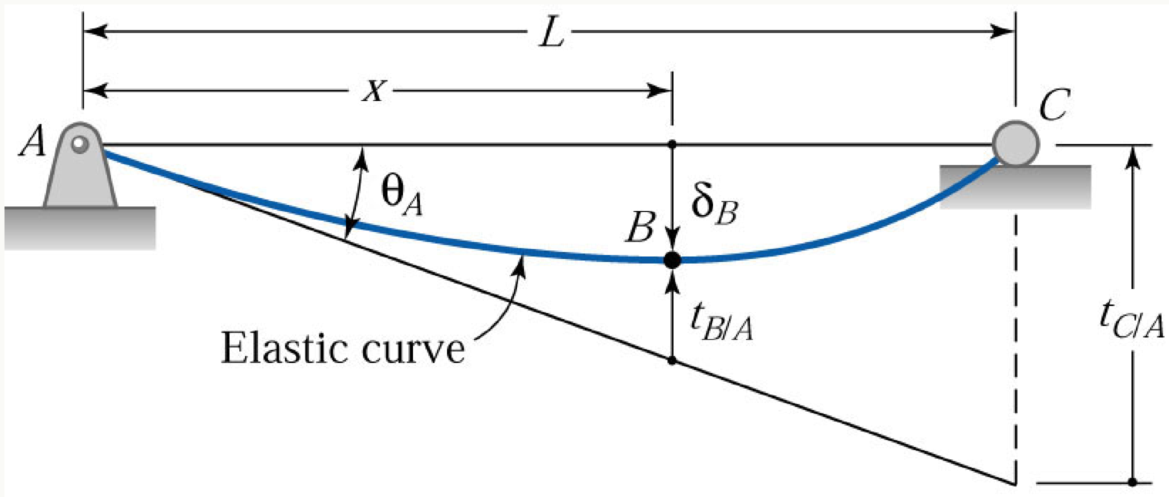
Figure 6.11 Procedure forcalculating δ_B, the displacement of point *B* on the simply supported beam.

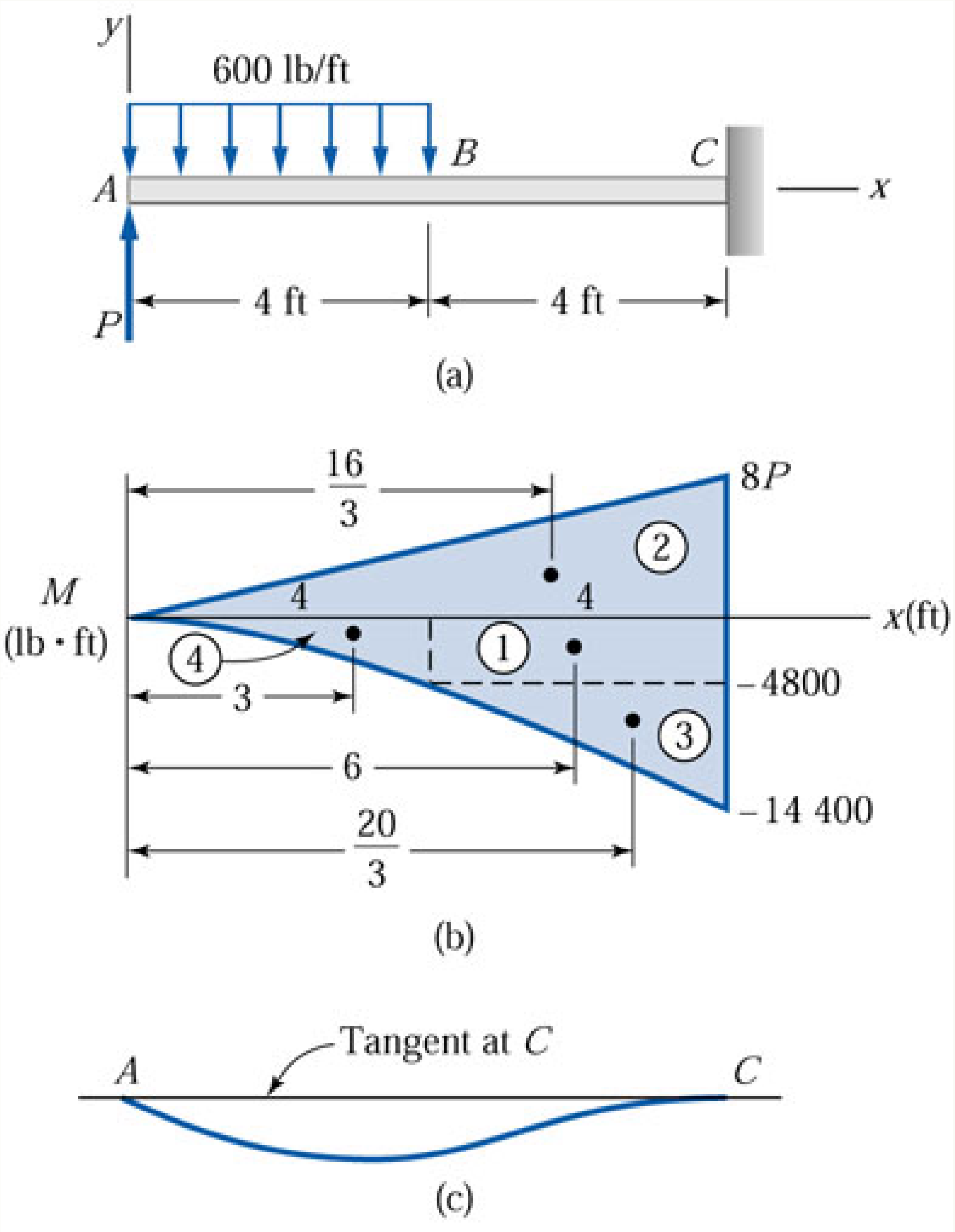
The procedure for computing δ_B consists of the following steps:

- Compute $t_{C/A}$ from

$$t_{C/A} = area \quad of \quad \frac{M}{EI} \quad diagram \quad \Big]_A^C \cdot \bar{x} / c$$

- Determine θ_A from the geometric relationship $\theta_A = \frac{t_{C/A}}{L}$
- Compute $t_{B/A}$ using $t_{B/A} = area \quad of \quad \frac{M}{EI} \quad diagram \quad \Big]_A^B \cdot \bar{x} /_B$
- Calculate δ_B from $\delta_B = \theta_A x - t_{B/A}$

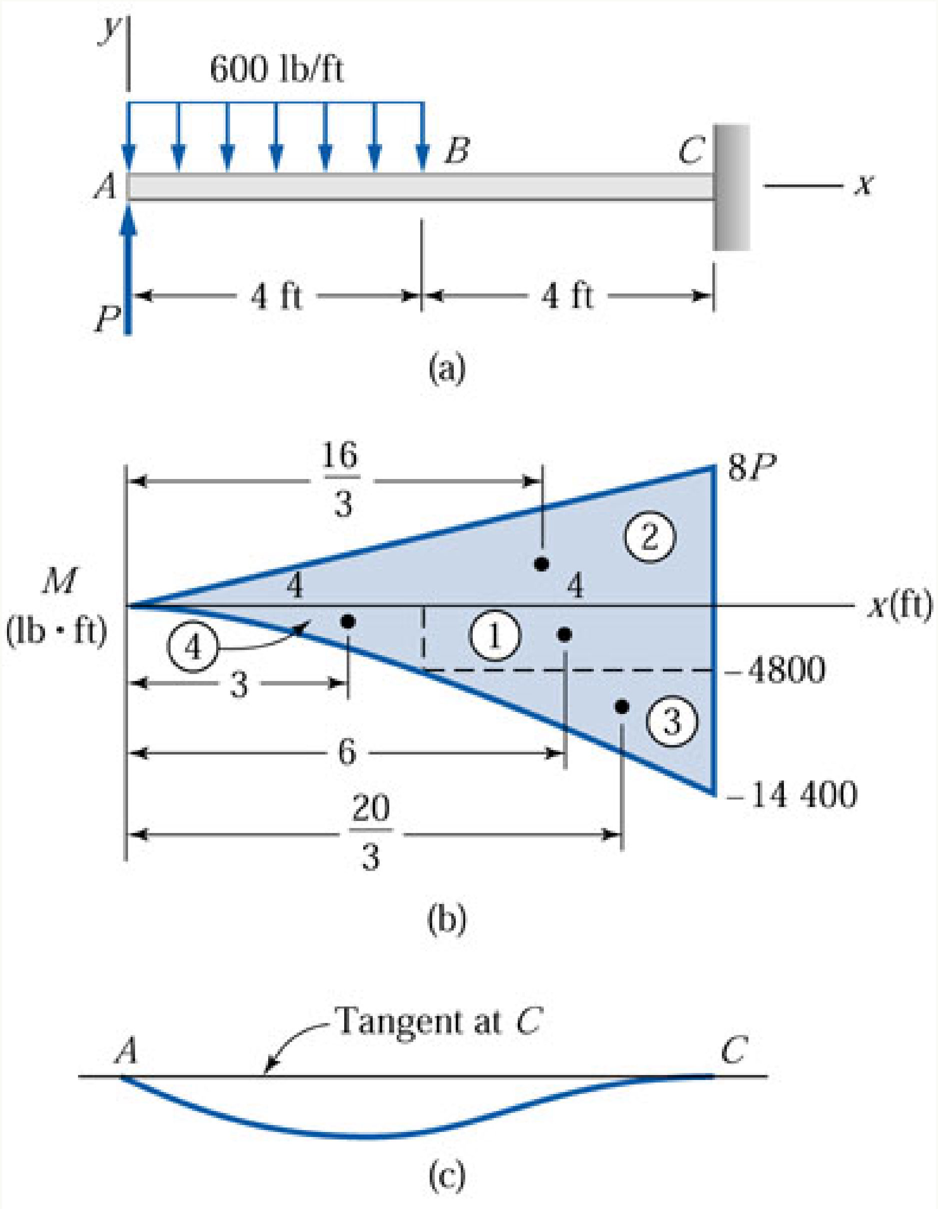




Figures (a) to (c)

Sample Problem 6.8

A 600-lb/ft uniformly distributed load a applied to the left half of the cantilever beam *ABC* in Fig. (a). Determine the magnitude of force P that must the applied as shown so that the displacement at *A* is zero .

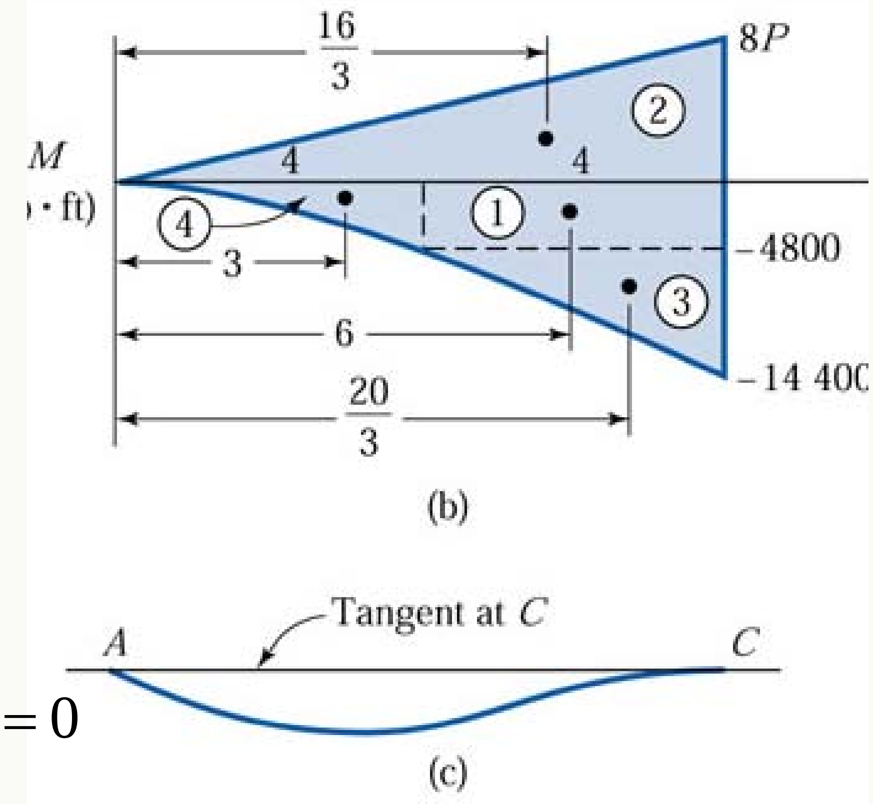


Figures (a) to (c)

Solution

The bending moment diagram, drawn by parts, is shown in Fig.(b). The upper portion is the moment caused by P; the lower part is due to the distributed load. The area under the diagram can be divided into the four simple shapes: the rectangle 1. the triangles2. and 3.and the parabola 4.

The sketch of the elastic curve in Fig.(c) is drawn so that it satisfies the boundary conditions ($\delta_c = \theta_c = 0$) and the requirement the $\delta_A = 0$. Because the slope of the elastic curve at C is zero, the $t_{A/C}$ is zero. Therefore, from the second moment-area theorem. we obtain



$$t_{A/c} = area \;\; of \;\; \frac{M}{EI} \;\; diagram \;\Big]_C^A \cdot \bar{x} /_A = 0$$

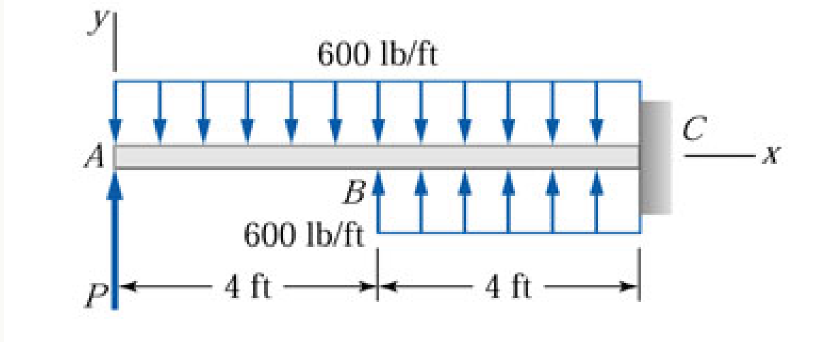
Using the four subareas shown in Fig. (b) to compute the first moment of the bending moment diagram about A, (the constant EI cancels)

$$\frac{1}{2}(8\times 8P)\left(\frac{16}{3}\right) - \frac{1}{3}(4\times 4800)(3) - (4\times 4800)(6) - \frac{1}{2}(4\times 9600)\left(\frac{20}{3}\right) = 0$$

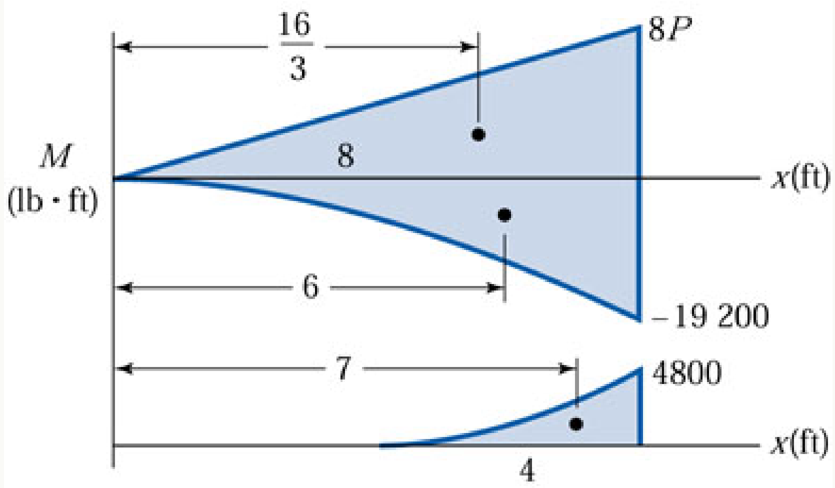
which yields P = 1537.5 lb *Answer*

Alternative Solution

There are other ways of drawing the bending moment diagram by parts. We could, for example, replace the distributed loading with the equivalent loading shown in Fig. (d). The resulting bending moment diagram by parts in Fig. (e) has only three parts: two parabolas and a triangle. Setting the first moment of the bending diagram



(d)



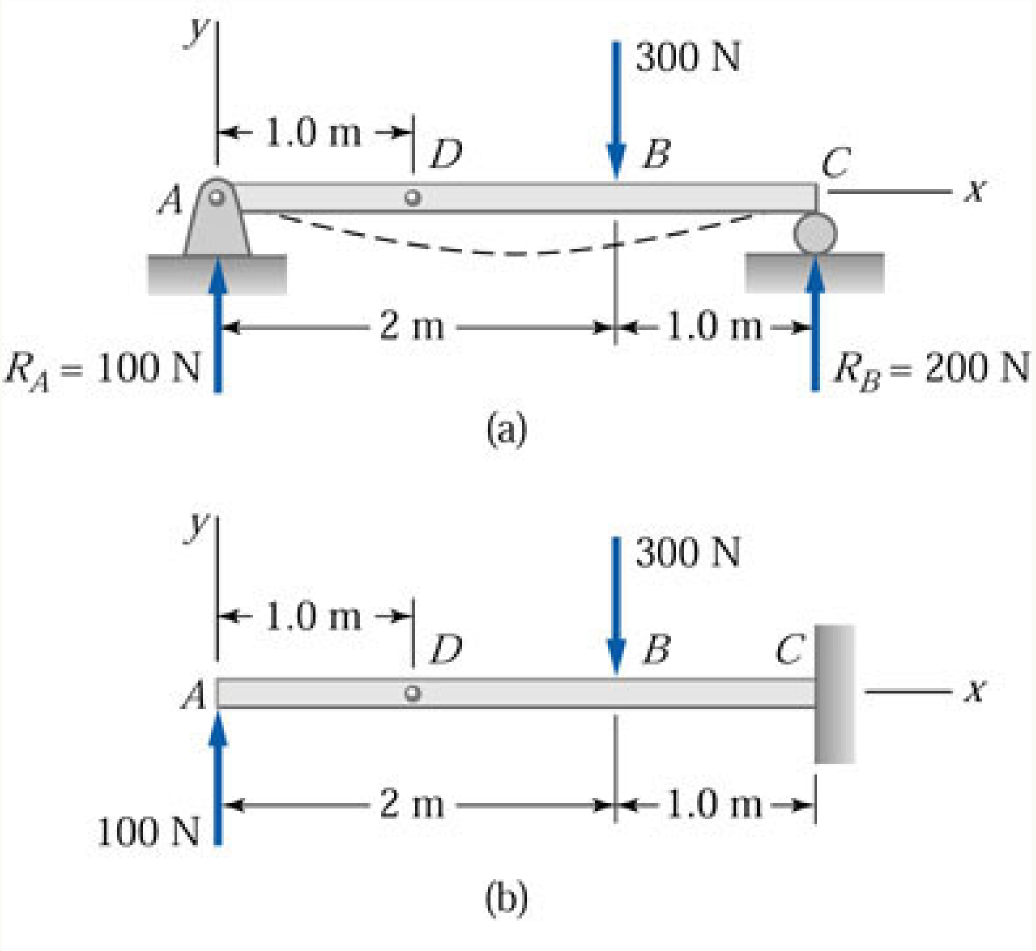
(e)

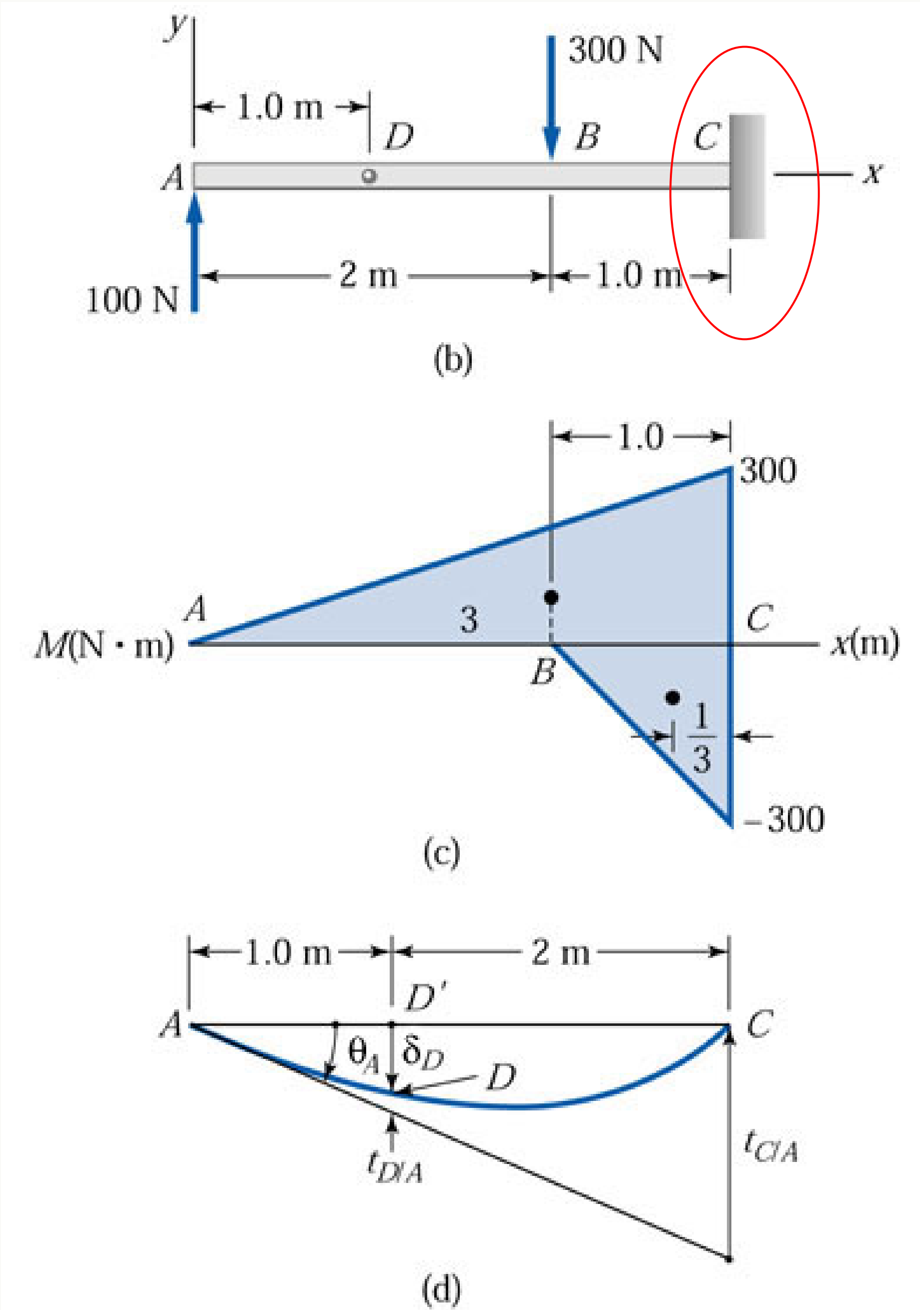
$$\frac{1}{2}(8 \times 8P)\left(\frac{16}{3}\right) - \frac{1}{3}(8 \times 19200)(6) + \frac{1}{3}(4 \times 4800)(7) = 0$$

giving us as before, P = 1537.5 lb *Answer*

Sample Problem 6.9

The simply supported beam in Fig.(a) supports a concentrated load of 300 N as shown Using EI = 20.48×10^3 N • m^2, determine (1) the slope of the elastic curve at *A*, θ_A . and (2) the displacement at *D*, δ_D .





Solution

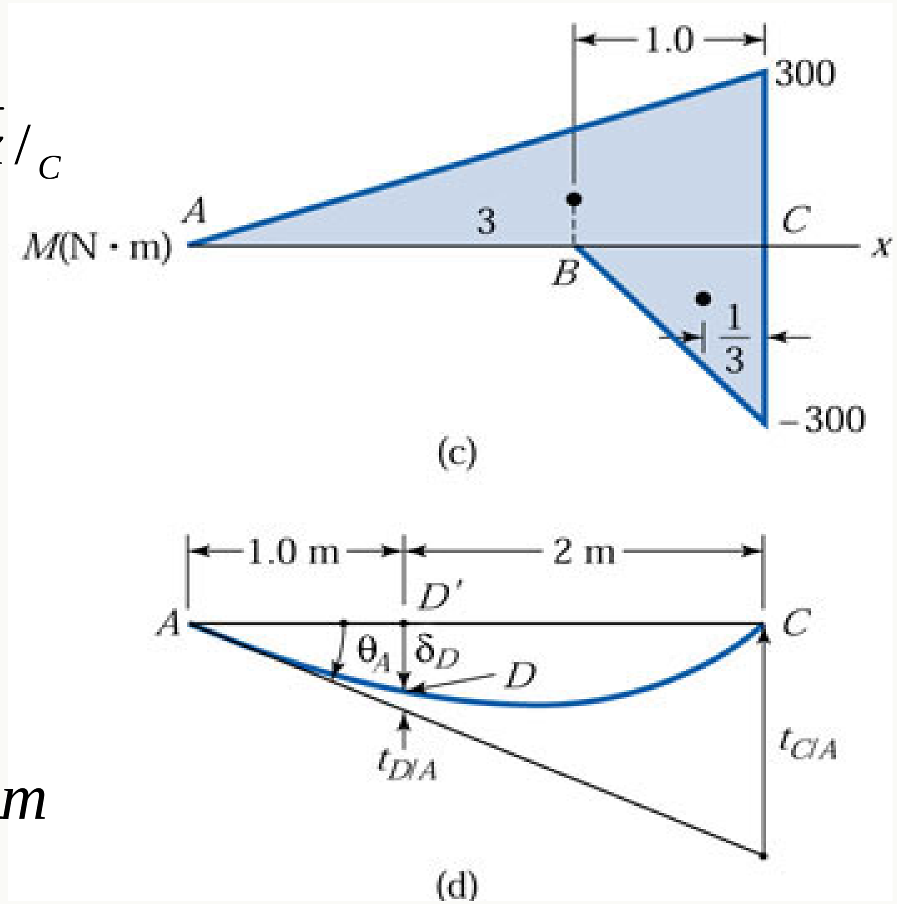
Introduce a fixed support at *C* and consider the reaction at *A* to be an applied load, as shown in Fig. (b). The resulting bending moment diagram is shown in Fig.(c). The sketch of the elastic curve of the original beam in Fig. (d) identifies the slope angle θ_A and the displacement δ_D, where are to be found, together with the tangential deviations $t_{C/A}$ and $t_{D/A}$.

Part1

The tangential deviation $t_{C/A}$:

$$t_{C/A} = area \quad of \quad \frac{M}{EI} \quad diagram \;]_C^A \cdot \bar{x}/_C$$

Note that $t_{C/A}$ is position which means that C is above the reference tangent at A.



$$t_{C/A} = \frac{1}{20.48\times10^3}\left[\frac{1}{2}(3\times300)(1.0)\cdot\right.$$

$$\left. -\frac{1}{2}(1.0\times300)\left(\frac{1}{3}\right)\right] = 0.019531m$$

Assuming small slopes, we obtain from geometry of Fig.(d)

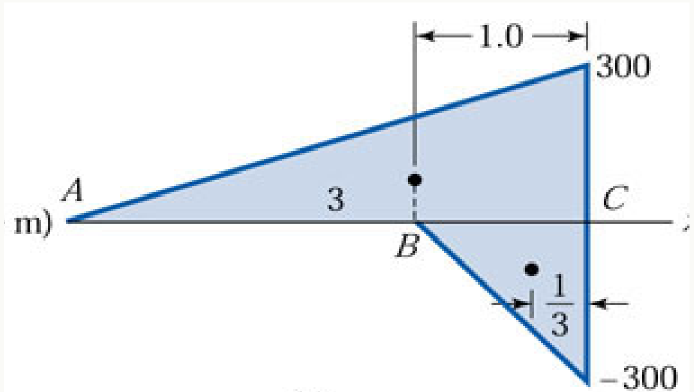
$$\theta_A = \frac{t_{C/A}}{AC} = \frac{0.019531}{3} = 6.510\times10^{-3}\,rad = 0.373^\circ$$

Answer

Part 2

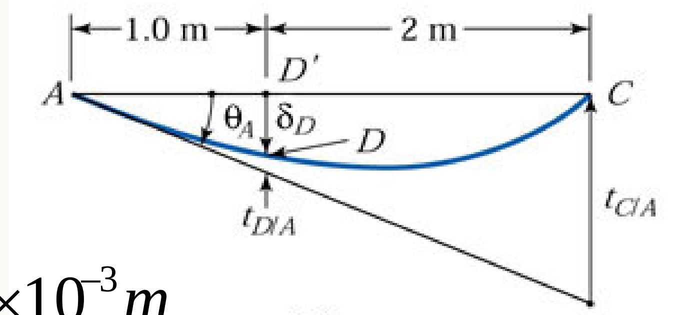
The tangential deviation of D relative to A is

$$t_{D/A} = area of\ \frac{M}{EI}\ diagram\,_A^D \cdot \bar{x}/_D$$



(c)

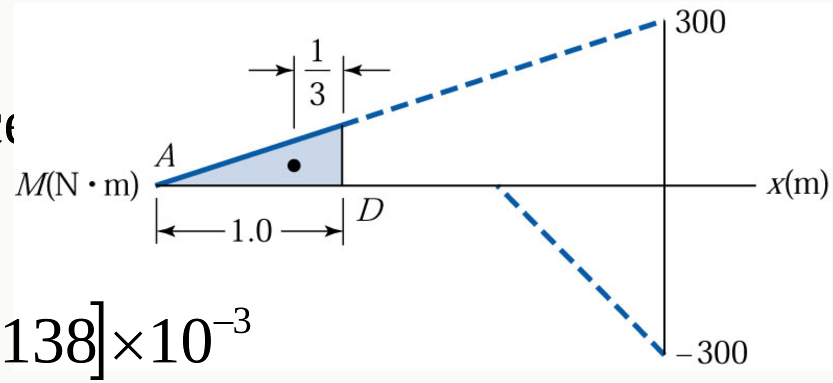
Referring to Fig. (e) to obtain the first moment about D of the bending moment diagram between A and D.



(d)

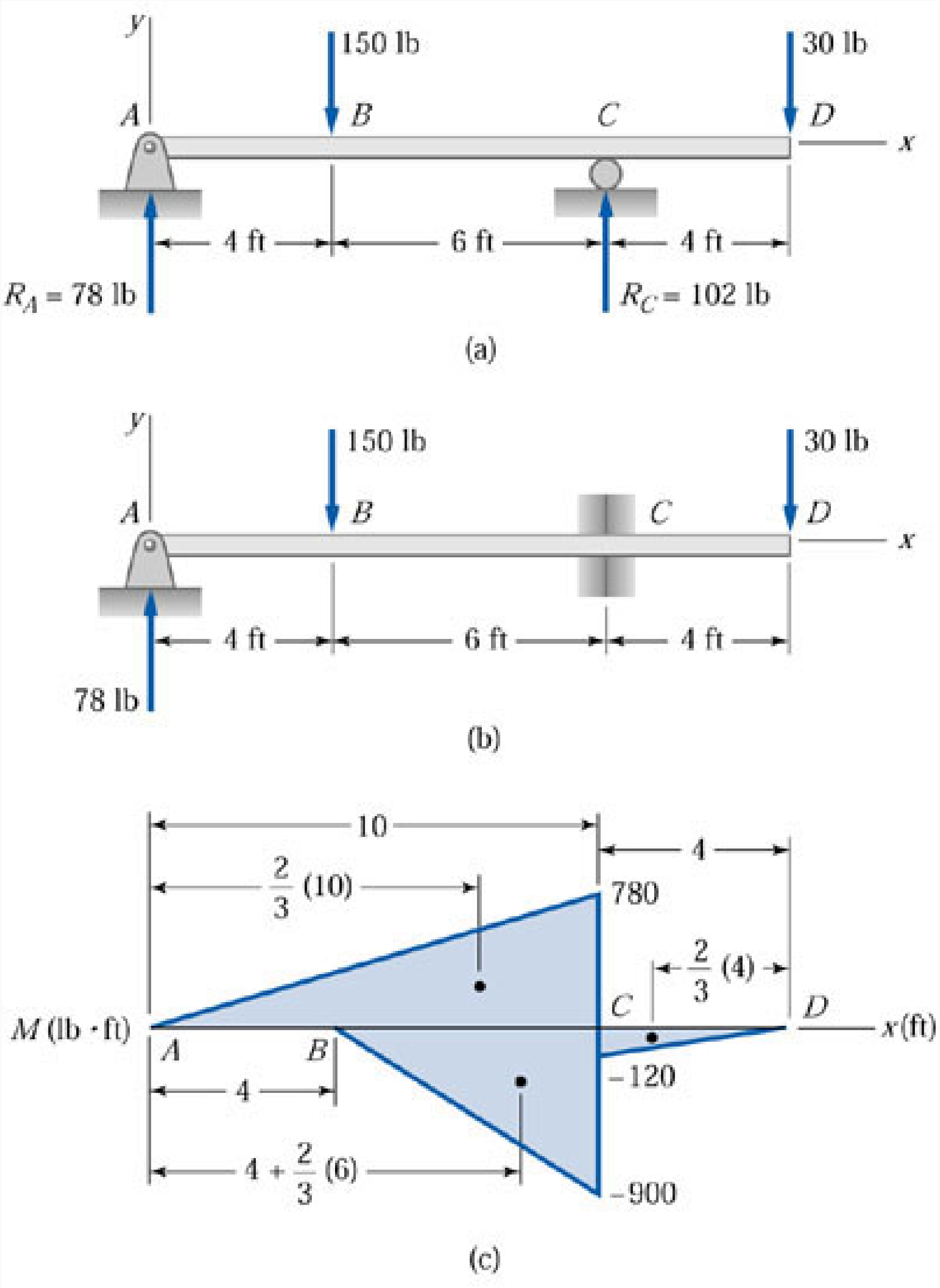
$$t_{D/A} = \frac{1}{20.48\times10^{3}}\left[\frac{1}{2}(1.0\times100)\left(\frac{1}{3}\right)\right] = 0.8138\times10^{-3}\,m$$

From Fig.(d). we see that the displace
at D is



$$\delta_D = \theta_A \overline{AD'} - t_{D/A} = [60510(1.0) - 0.8138]\times10^{-3}$$

$$= 5.696\times10^{-3}\,m = 5.70\,mm\downarrow \qquad Answer$$



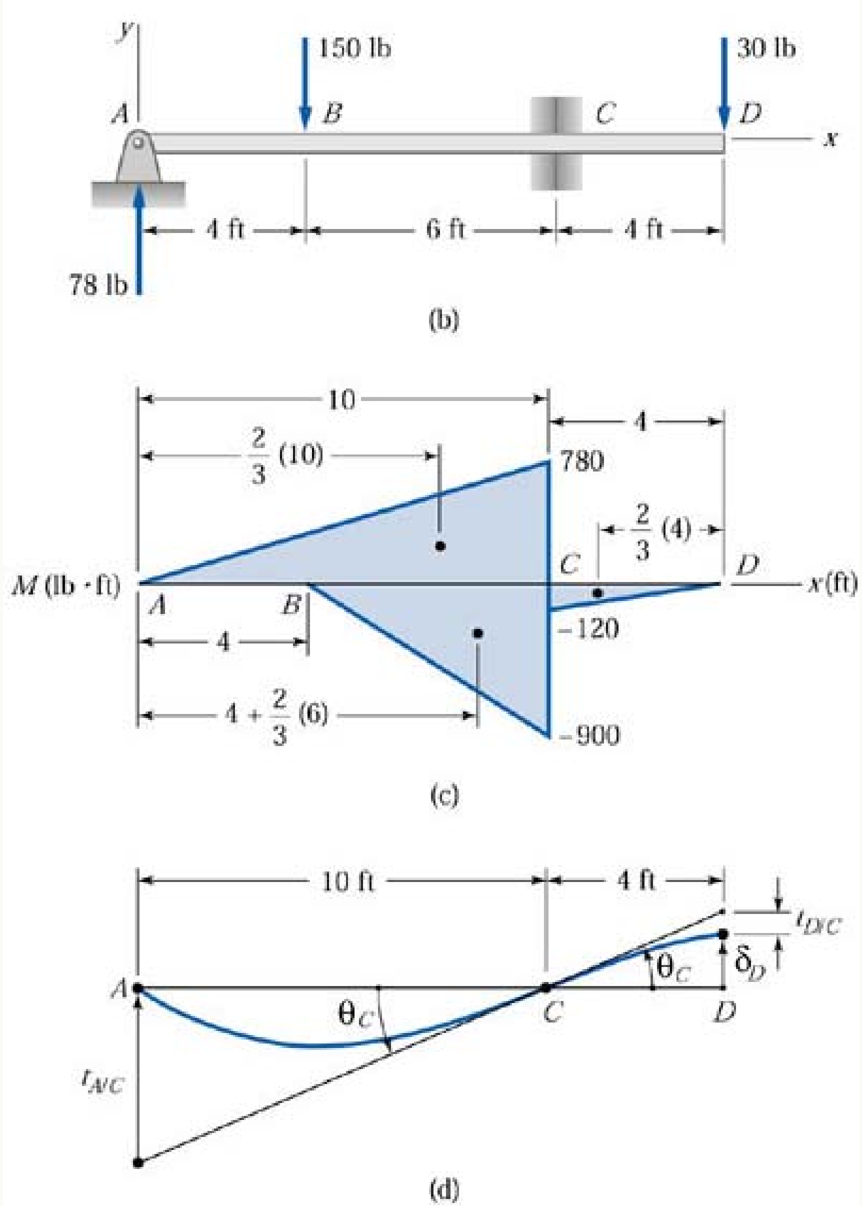
Sample Problem 6.10

Determine the value of EI δ at end D of the overhanging beam in Fig.(a).

Solution

Introduced a built-in support at *C* and shown the reaction at A as an applied load. The result is two beams that are cantilevered from *C*.

The bending moment diagrams by parts for these beams are shown in Fig. (c). The elastic curve of the original beam in Fig. (d) was drawn is assuming that the beam rotates counterclockwise at *C*. The correct direction is determined from the sign of the tangential deviation $t_{A/C}$.

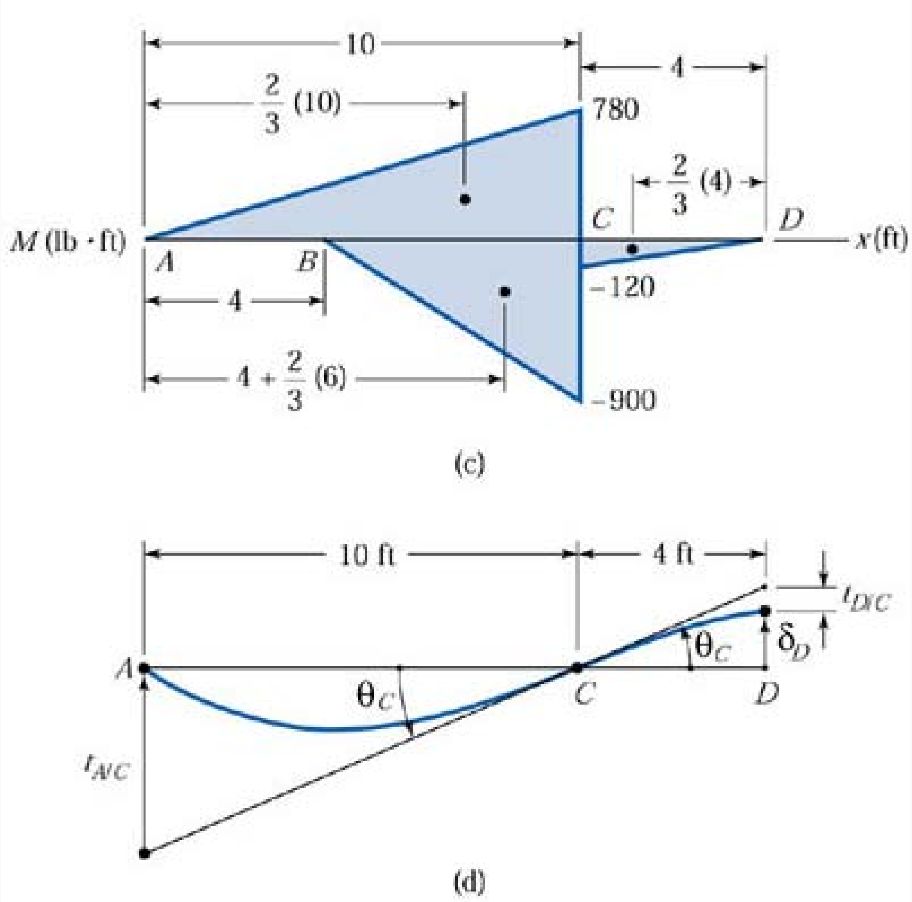


Using the second moment-area theorem.

$$EIt_{A/C} = area\ \ of\ \ M\ \ diagram\Big]_A^C \cdot \bar{x}/_A$$

$$= \frac{1}{2}(10\times780)\left[\frac{2}{3}(10)\right] - \frac{1}{2}(6\times900)\left[4+\frac{2}{3}(6)\right] = 4400\,lb\cdot ft^3$$

The positive value $t_{A/C}$ means that point *A* lies above the reference tangent at *C*, as shown in Fig.(d). thereby verifying our assumption. The slope angle at *C* is (assuming small slopes)



(c)

(d)

$$\theta_C = t_{A/C} / \overline{AC},$$

or

$$EI\theta_C = \frac{4400}{10} = 440 lb \cdot ft^2$$

$$EIt_{D/C} = area \;\; of \;\; M \;\; diagram\Big]_C^D \cdot \bar{x}/_D = -\frac{1}{2}(4\times120)\left[\frac{2}{3}(4)\right] = -640 \;\; lb\cdot ft^3$$

According to Fig.(d), the displacement of *D* is $\delta_D = \theta_C \overline{CD} - |t_{D/C}|.$

$EI\,\delta_D$ = 440(4)-640 = 1120 lb·ft^3 ↑ *Answer*

6.5 *Method of Superposition*

- The method of *superposition*, a popular method for finding slopes and deflections, is based on the ***principle of superposition***:

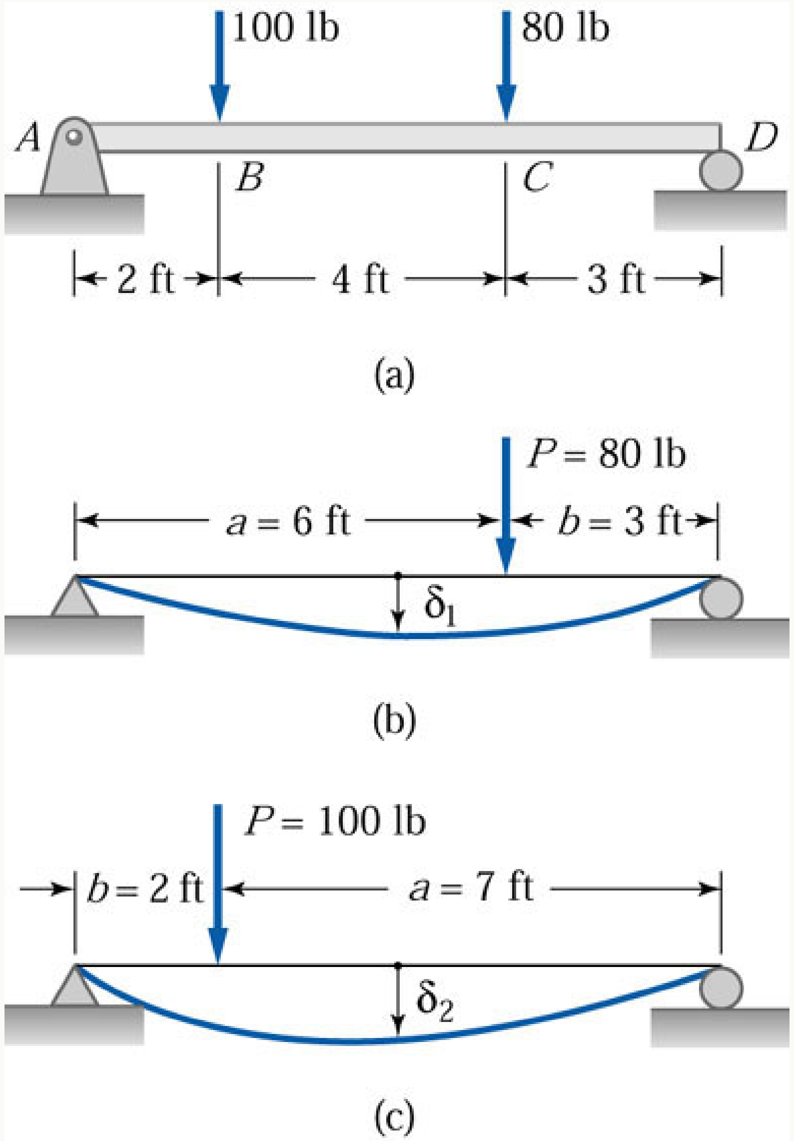
  If the response of a structure is **linear**, then the effect of several loads acting simultaneously *can be obtained by* ***superimposing*** (adding) the effects of the *individual loads*.

- By "linear response" we mean that the relationship between the cause (loading) and the effect (internal forces and deformations) is **linear**, The two repuirements for linear response are (1) the material must obey Hooke's law; and (2) the deformation must be sufficiently small so that their effect on the geometry is neglible.

- The method of superposition permits us to use the known displacements and slopes for simple loads to obtain the deformations for more complicated loadings.

- To use the method effectively requires access to tables that list the formula for slope and deflections for various loading, such as Tables 6.2 and 6.3. More extensive tables can be found in most engineering handbooks.

Sample Problem 6.11



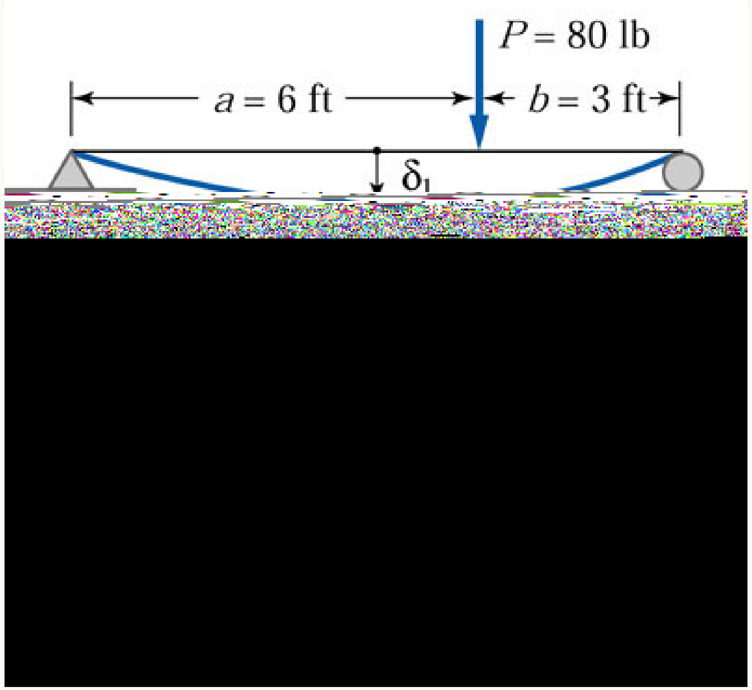
6.11 Figure (a) Through (c)

Compute the midspan value of $EI\,\delta$ for the simply supported beam shown in Fig.(a) that is carrying two concentrated loads.

Solution

The loading on the beam can be considered to be the superposition of the loads shown in Fig. (b) and (c). According to Table 6.3, the displacement at the center of a simply supported beam is given by

$$EI\delta_{center} = \frac{Pb}{48}\left(3L^2 - 4b^2\right)$$



$$EI\delta_1 = \frac{(80)(3)}{48}\left[3(9)^2 - 4(3)^2\right] = 1035 lb \cdot ft^3 \downarrow$$

$$EI\delta_2 = \frac{(100)(2)}{48}\left[3(9)^2 - 4(2)^2\right] = 946 lb \cdot ft^3 \downarrow$$

The midspan deflection of the original beam is obtained by supposition:

$$EI\,\delta = EI\,\delta_1 + EI\,\delta_2 = 1035 + 946 = 1981 \text{ lb} \cdot \text{ft}^3 \quad \textit{Answer}$$

Sample Problem **6.12**

The simply supported beam in Fig.(a) carries a uniformly distributed load over part of its length. Compute the midspan displacement.

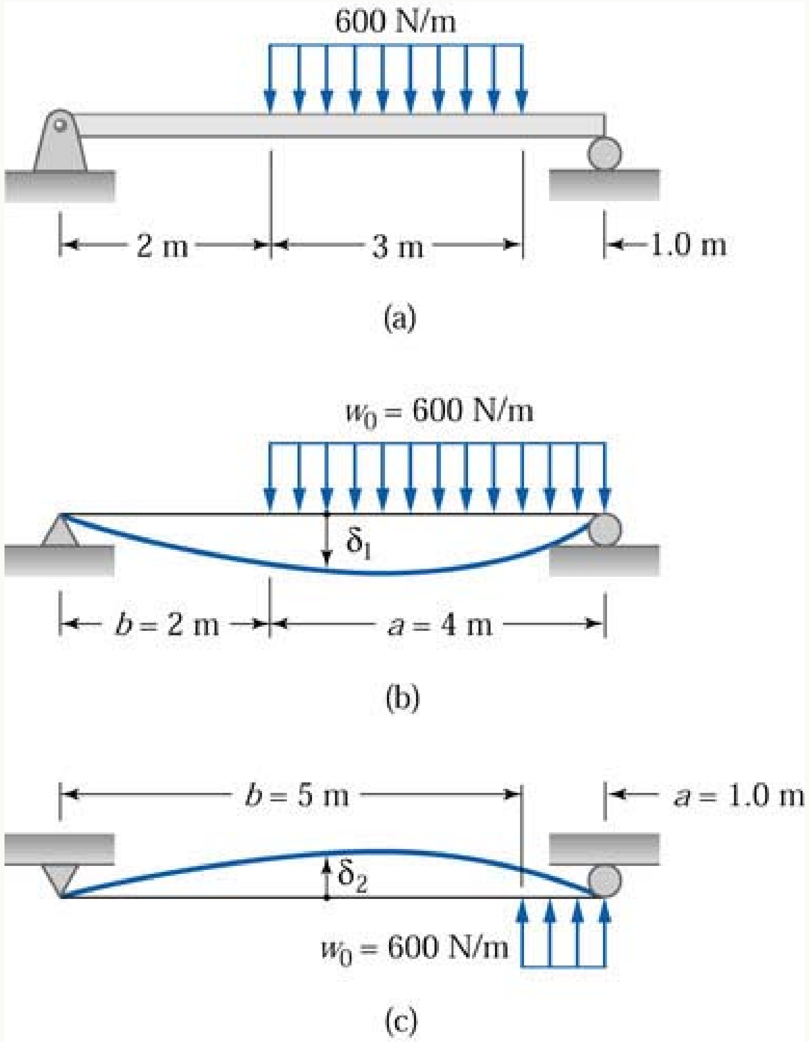


Figure 6.12(a) through (c)

Solution

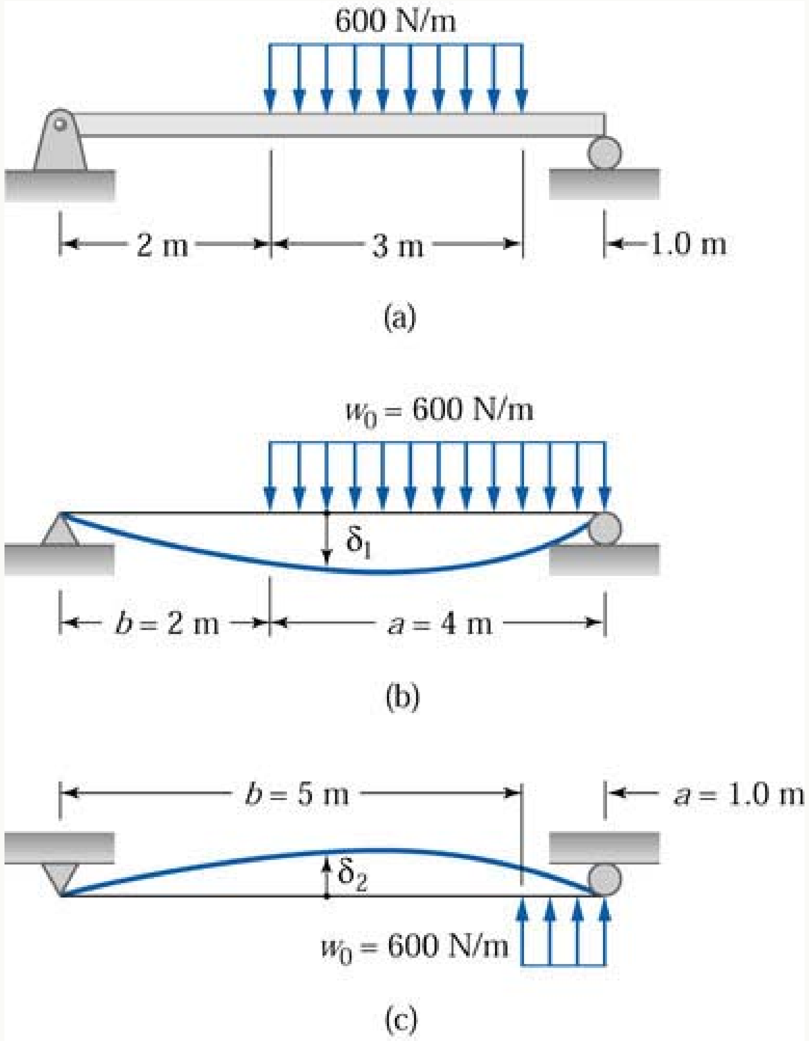
The given loading can be analyzed as the superposition of the two loading shown in Fig.(b) and (c). From Table 6.3, the midspan value of $EI\,\delta$ for the beam in Fig.(b) is

$$EI\delta_1 = \frac{w_0}{384}\left(5L^4 - 12L^2b^2 + 8b^4\right)$$

$$= \frac{600}{384}\left[5(6)^4 - 12(6)^2(2)^2 + 8(2)^4\right] = 7625 N \cdot m^3 \downarrow$$

Similarly, the midspan displacement of the beam in Fig.(c) is

$$EI\delta_2 = \frac{w_0 a^2}{96}\left(3L^2 - 2a^2\right) = \frac{(600)(1)^2}{96}\left[3(6)^2 - 2(1)^2\right] = 662.5 N\cdot m^3 \uparrow$$

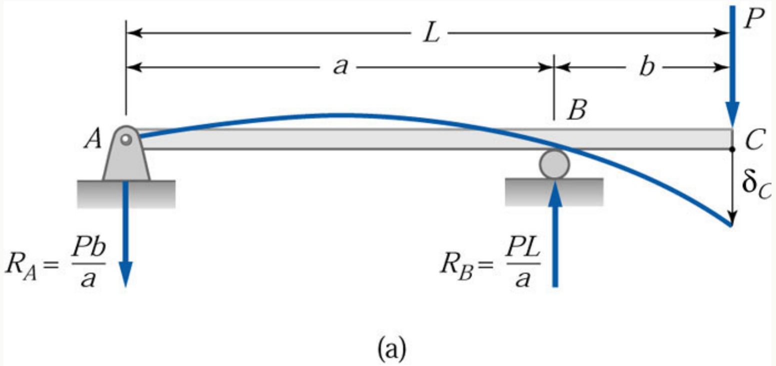


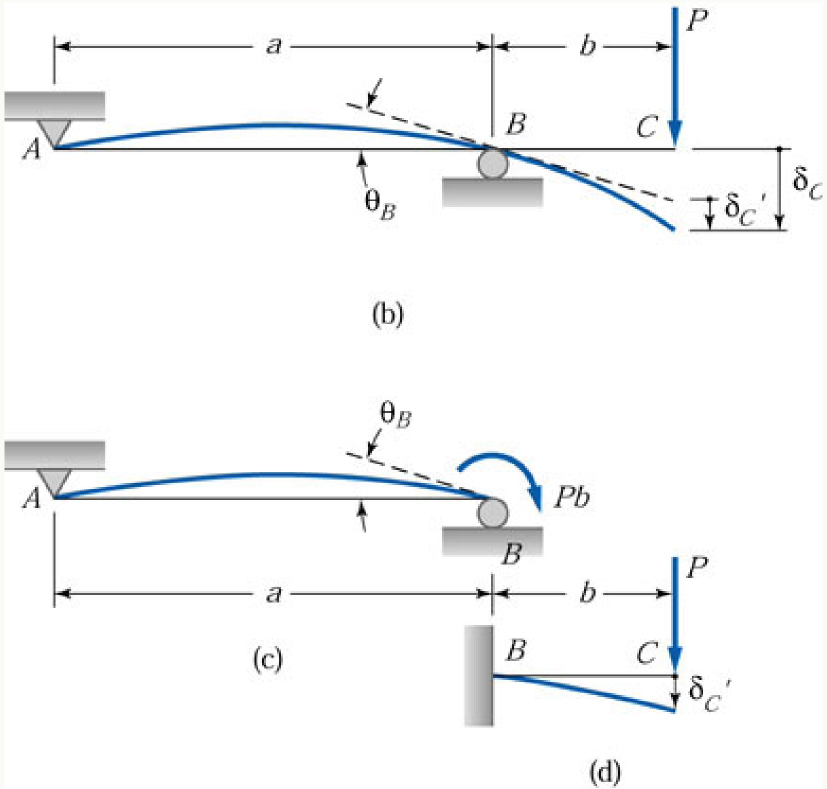
The midspan displacement of the original beam is obtained by supposition:

$$EI\delta = EI\delta_1 - EI\delta_2 = 7625 - 662.$$

$$= 6960 N\cdot m^3 \downarrow$$

Answer





Sample Problem **6.13**

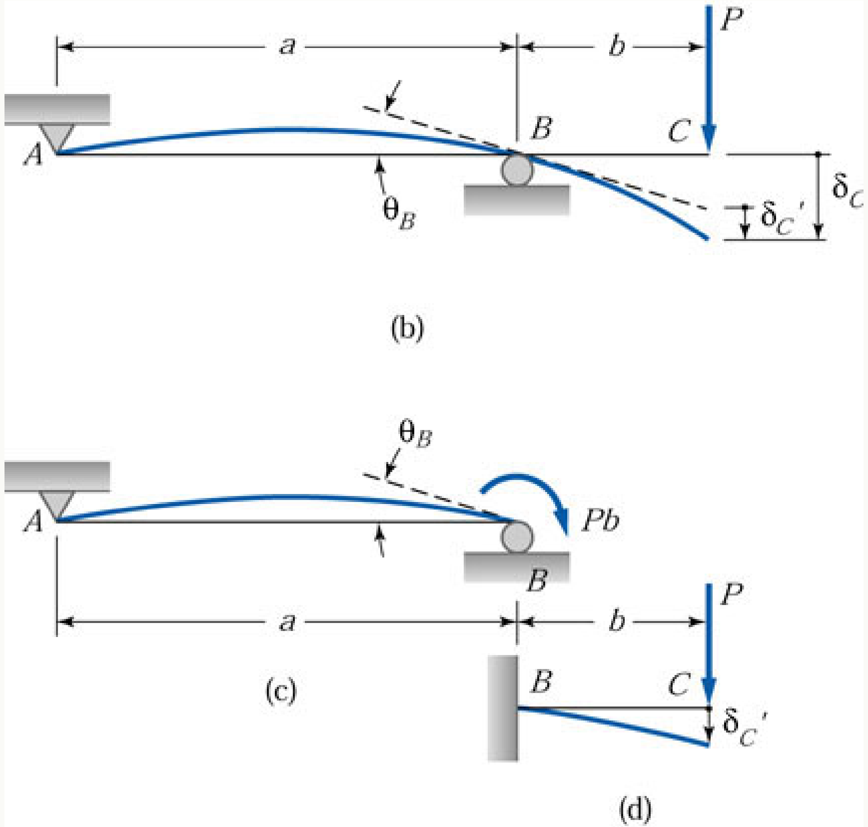
The overhanging beam *ABC* in Fig. (a) carries a concentrated load *P* at end *C*. Determine the displacement of the beam at *C*.

Solution

From the sketch of the elastic curve in Fig.(b), we see that displacement at *C* is

$$\delta_C = \theta_B b + \delta'_C$$

where θ_B is the slope angle of the elastic curve at *B* and δ'_C is the displacement at *C* due to the deformation of BC.



We can obtain θ_B from the deformation of segment *AB*. Shown in Fig.(c). Using Table 6.3.

$$\theta_B = \frac{(Pb)a}{3EI}$$

From Fig.(d) and Table 6.2, the displacement due to the deformation of *BC* is

$$\delta'_C = \frac{pb^3}{3EI}$$

Therefore, the displacement at *C* becomes

$$\delta_C = \frac{pba}{3EI}b + \frac{pb^3}{3EI} = \frac{pb^2}{3EI}(a+b) = \frac{pb^2L}{3EI}\downarrow \qquad \textit{Answer}$$