

Chapter 6

The Time Value of Money: Annuities and Other Topics



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Chapter 6 Contents

- Learning Objectives
 1. Distinguish between an ordinary annuity and an annuity due, and calculate present and future values of each.
 2. Calculate the present value of a level perpetuity and a growing perpetuity.
 3. Calculate the present and future value of complex cash flow streams.

Principles Used in Chapter 6

- Principle 1: Money Has a Time Value.
 - This chapter applies the time value of money concepts to annuities, perpetuities and complex cash flows.
- Principle 3: Cash Flows Are the Source of Value.
 - This chapter introduces the idea that principle 1 and principle 3 will be combined to value stocks, bonds, and investment proposals.

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Ordinary Annuities

- An **annuity** is a series of *equal dollar* payments that are made at the end of equidistant points in time such as monthly, quarterly, or annually over a *finite period* of time.
- If payments are made at the end of each period, the annuity is referred to as **ordinary annuity**.
- Example 6.1 How much money will you accumulate by the end of year 10 if you deposit \$3,000 each for the next ten years in a savings account that earns 5% per year?
- **Could** solve by using the equation for computing the future value of an ordinary annuity.

The Future Value of an Ordinary Annuity

$$FV_n = PMT \left[\frac{(1 + i)^n - 1}{i} \right]$$

- FV_n = FV of annuity at the end of n th period.
- PMT = annuity payment deposited or received at the end of each period
- i = interest rate per period
- n = number of periods for which annuity will last

Easy to make errors when using the Equation.
Very Easy to handle using Financial Calculators

Example: Future Value Ordinary Annuity

$$FV_n = PMT \left[\frac{(1 + i)^n - 1}{i} \right]$$

$$\begin{aligned} FV &= \$3000 \{ [(1 + .05)^{10} - 1] \div (.05) \} \\ &= \$3,000 \{ [0.63] \div (.05) \} \\ &= \$3,000 \{ 12.58 \} \\ &= \mathbf{\$37,740} \end{aligned}$$

This is really messy

Future Value Ordinary Annuity (calculator)

- Using a Financial Calculator (Much Easier)
- **Enter**
 - N = 10
 - I/Y = 5.0
 - PV = 0
 - PMT = -3000
 - FV = **\$37,733.67**

Solving for PMT in an Ordinary Annuity

- Instead of figuring out how much money will be accumulated (i.e. FV), determine how much needs to be saved/accumulated each period (i.e. PMT) in order to accumulate a certain amount at the end of n years.
- In this case, know the values of n , i , and FV_n in equation 6-1c and determine the value of PMT .

Solve for PMT in an Ordinary Annuity

- Example 6.2: Suppose you would like to have \$25,000 saved 6 years from now to pay towards your down payment on a new house.
- If you are going to make equal annual end-of-year payments to an investment account that pays 7 per cent, how big do these annual payments need to be?
- Using a Financial Calculator.

$$N=6; \quad I/Y = 7; \quad PV = 0; \quad FV = 25,000$$

$$PMT = -3,494.89$$

Checkpoint 6.1 – Class Problem

Solve for an Ordinary Annuity Payment

How much must you deposit in a savings account earning 8% annual interest in order to accumulate \$5,000 at the end of 10 years?

Checkpoint 6.1: *Class problem*

If you can earn 12 percent on your investments, and you would like to accumulate \$100,000 for your child's education at the end of 18 years, how much must you invest annually to reach your goal?

Solving for Interest Rate in Ordinary Annuity

- Solve for “interest rate” that must be earned on an investment that will allow savings to grow to a certain amount of money by a future date.
- In this case, we know the values of n , PMT , and FV_n when using the financial calculator and we need to determine the value of I/Y .

Solve for Interest Rate in an Ordinary Annuity – Example

- Example 6.3: In 20 years, you are hoping to have saved \$100,000 towards your child’s college education. If you are able to save \$2,500 at the end of each year for the next 20 years, what rate of return must you earn on your investments in order to achieve your goal?
- Using a Financial Calculator
 $N = 20$; $PMT = -\$2,500$; $FV = \$100,000$
 $PV = \$0$
 $I/Y = 6.77\%$

Solving for the Number of Periods in an Ordinary Annuity

- You may want to calculate the *number of periods* it will take for an annuity to reach a certain future value, given interest rate.
- Example 6.4: Suppose you are investing \$6,000 at the end of each year in an account that pays 5%. How long will it take before the account is worth \$50,000?
- Using a Financial Calculator
I/Y = 5.0; PV = 0; PMT = -6,000; FV = 50,000
N = **7.14 years**

The Present Value of an Ordinary Annuity

- The present value of an ordinary annuity measures the value today of a stream of cash flows occurring in the future.
 - PV annuity reflects how much you would (should) pay (today) for a constant set of cash flows that would be received each period for a fixed number of periods and given a constant interest rate (required rate of return).
- For example, compute the PV of ordinary annuity to answer the question:
 - How much should be paid today (lump sum equivalent) for receiving \$3,000 every year for the next 30 years if the interest rate is 5%?

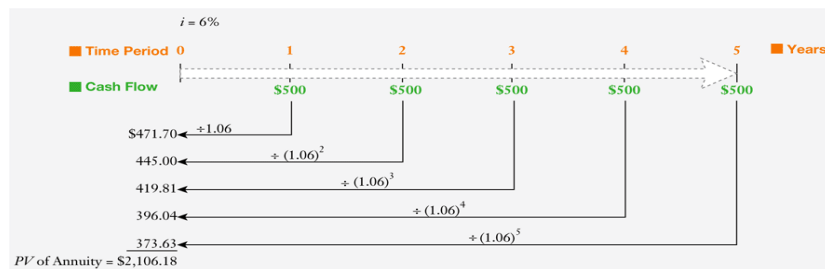
The Present Value of an Ordinary Annuity – Example with Logic

- Figure 6-2 shows the lump sum equivalent (\$2,106.18) of receiving \$500 per year for the next five years at an interest rate of 6%.

Figure 6.2

Timeline of a Five-Year, \$500 Annuity Discounted Back to the Present at 6 Percent

To find the present value of an annuity, discount each cash flow back to the present separately and then add them. In this example, we simply add up the present value of five future cash flows of \$500 each to find a present value of \$2,106.18.



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6-15

The Present Value of an Ordinary Annuity – Example with Financial Calculator

- Compute the present value of an ordinary annuity in which \$500 per year is received for the next five years at an interest rate of 6%.

$$\text{PMT} = 500; \text{N} = 5; \text{I/Y} = 6; \text{FV} = 0$$

$$\text{Solve PV} = -2,106.18$$

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6-16

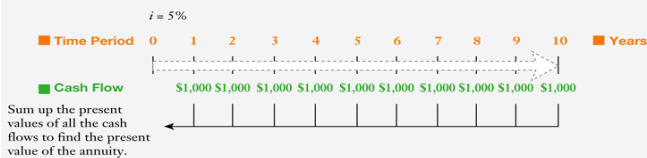
Checkpoint 6.2 – Class Problem

The Present Value of an Ordinary Annuity

Your grandmother has offered to give you \$1,000 per year for the next 10 years. What is the present value of this 10-year, \$1,000 annuity discounted back to the present at 5 percent?

STEP 1: Picture the problem

We can use a timeline to identify the cash flows from the investment as follows:



STEP 2: Decide on a solution strategy

In this case we are trying to determine the present value of an annuity, and we know the number of years, the dollar value that is received at the end of each year, and the number of years the annuity lasts. We also know that the discount rate is 5 percent. We can use Equation (6–2b) to solve this problem.

Amortized Loans

- An **amortized loan** is a loan paid off in equal payments – consequently, the loan payments are an annuity.
 - Examples: Home mortgage loans, Auto loans
- In an amortized loan, the *present value* can be thought of as the amount borrowed, n is the number of periods the loan lasts for, i is the interest rate per period, *future value* takes on zero because the loan will be paid of after n periods, and *payment* is the loan payment that is made.

Amortized Loans - Example

- Example 6.5 Suppose you plan to get a \$9,000 loan from a furniture dealer at 18% annual interest with annual payments that you will pay off in over five years.
 - What will your annual payments be on this loan?

- Using a Financial Calculator
 $N = 5; I/Y = 18.0; PV = 9,000; FV = 0$
Solve PMT = -\$2,878.00

The Loan Amortization Schedule

Year	Amount Owed on Principal at the Beginning of the Year (1)	Annuity Payment (2)	Interest Portion of the Annuity (3) = (1) × 18%	Repayment of the Principal Portion of the Annuity (4) = (2) – (3)	Outstanding Loan Balance at Year end, After the Annuity Payment (5) = (1) – (4)
1	\$9,000	\$2,878	\$1,620.00	\$1,258.00	\$7,742.00
2	\$7,742	\$2,878	\$1,393.56	\$1,484.44	\$6,257.56
3	\$6,257.56	\$2,878	\$1,126.36	\$1,751.64	\$4,505.92
4	\$4,505.92	\$2,878	\$811.07	\$2,066.93	\$2,438.98
5	\$2,438.98	\$2,878	\$439.02	\$2,438.98	\$0.00

Loan Amortization Schedule Logic

- We can observe the following from the table:
 - Size of each payment remains the same.
 - However, Interest payment declines each year as the amount owed declines and more of the principal is repaid.
- Amortized **Loans with Monthly Payments**
 - Many loans such as auto and home loans require monthly payments.
 - Requires converting n to number of months and computing the monthly interest rate.

Amortized Loans with Monthly Payments - Example

- Example 6.6 You have just found the perfect home. However, in order to buy it, you will need to take out a \$300,000, 30-year mortgage at an annual rate of 6 percent. What will your monthly mortgage payments be?
- Using a Financial Calculator
 $N=360$; $I/Y = 0.5$; $PV = 300,000$; $FV = 0$

Solve PMT = -1,798.65 per month

Checkpoint 6.3 – Additional Complexity and Concept Integration

– Determining the Outstanding Balance of a Loan that will be Refinanced

Let's say that exactly ten years ago you took out a \$200,000, 30-year mortgage with an annual interest rate of 9 percent and monthly payments of \$1,609.25.

But since you took out that loan, interest rates have dropped. You now have the opportunity to refinance your loan at an annual rate of 7 percent over 20 years.

You need to know what the outstanding balance on your current loan is so you can take out a lower-interest-rate loan and pay it off. If you just made the 120th payment and have 240 payments remaining, what's your current loan balance?

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6-23

Checkpoint 6.3 – Logical Solution Flow

STEP 1: **Picture** the problem

Since we are trying to determine how much you still owe on your loan, we need to determine the present value of your remaining payments. In this case, because we are dealing with a 30-year loan, with 240 remaining monthly payments, it's a bit difficult to draw a timeline that shows all the monthly cash flows. Still, we can mentally visualize the problem which involves calculating the present value of 240 payments of \$1,609.25 using a discount rate of 9%/12.

STEP 2: **Decide** on a solution strategy

Initially you took out a \$200,000, 30-year mortgage with an interest rate of 9 percent, and monthly payments of \$1,609.25. Since you have made 10 years worth of payments—that's 120 monthly payments—there are only 240 payments left before your mortgage will be totally paid off. We know that the outstanding balance is the present value of all the future monthly payments. To find the present value of these future monthly payments, we'll use Equation (6-2c).

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6-24

Checkpoint 6.3 – Final Solution

STEP 3: Solve

Using the Mathematical Formulas.

Using Equation (6-2c), we'll solve for the present value of the remaining monthly payment. To find n , we multiply the number of years left until the mortgage is paid off (20) times the number of months in a year (12). Thus n becomes 240. The future value will be equal to zero because the loan will be fully paid off in 20 years. The payment will be \$1,609.25 as given above. In effect, the present value of the payments you still need to make is how much you still owe.

$$PV = PMT \left[\frac{1 - \frac{1}{(1 + \text{annual rate of interest}/m)^{\text{years} \times m}}}{\text{annual rate of interest}/m} \right] \quad (6-2c)$$

where m = number of times compounding occurs per year and # years is the number of years. Substituting annual interest rate = .09, number of years = 20, $m = 12$, and $PMT = \$1,609.25$ into Equation (6-2c), we get,

$$PV = \$1,609.25 \left[\frac{1 - \frac{1}{(1 + .09/12)^{20 \times 12}}}{.09/12} \right]$$

$$PV = \$1,609.25(111.145)$$

$$PV = \$178,860.02$$

Using a Financial Calculator.

Enter 240 9.0÷12 -1,609.25 0

N **I/Y** **PV** **PMT** **FV**

Solve for 178,860.02

Using an Excel Spreadsheet.

=PV(rate,nper,pmt,N) or with values entered =PV(.09/12%,240,-1609.25,0)

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6-25

Checkpoint 6.3 – Assess new Payments

– Compute New Payments with 7% interest rate on the refinanced Loan

• Using a Financial Calculator (New Loan)

N=240; I/Y = 7/12; PV = 178,860.02;
FV = 0

Solve PMT = -1,368.70 per month

Original Loan monthly payments = \$1,609.25.

Save more than \$200 per month by refinancing!!!

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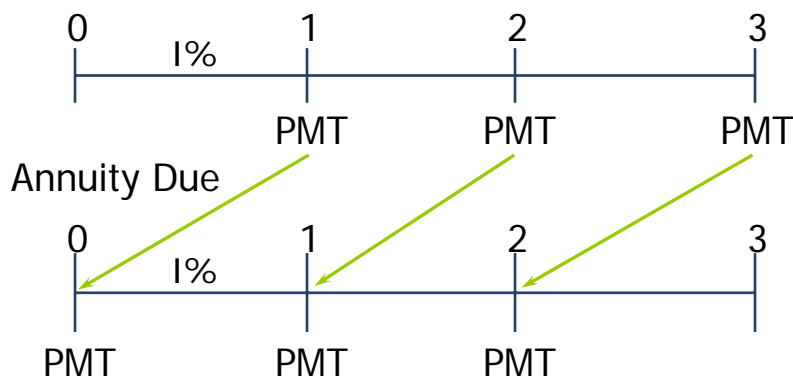
6-26

Annuities Due

- **Annuity due** is an annuity in which all the cash flows occur at the beginning of the period.
 - For example, rent payments on apartments are typically annuity due as rent is paid at the beginning of the month. (pay in advance of using resource or asset)
- The examples and logic will illustrate that both the future value and present value of an annuity due are larger than that of an ordinary annuity.
 - In each case, all payments are received or paid earlier.

Difference between an ordinary annuity and an annuity due

Ordinary Annuity



Solving for FV: 3-year Ordinary Annuity of \$100 at 10%

- \$100 payments occur at the end of each period but there is no PV.

INPUTS	3	10	0	-100	
	N	I/YR	PV	PMT	FV
OUTPUT					331.00

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5629

Solving for FV: 3-Year Annuity Due of \$100 at 10%

- Now, \$100 payments occur at the beginning of each period.

$$FVA_{\text{due}} = FVA_{\text{ord}}(1 + I) = \$331(1.10) = \$364.10$$

- Alternatively, set calculator to "BEGIN" mode and solve for the FV of the annuity:

BEGIN					
INPUTS	3	10	0	-100	
	N	I/YR	PV	PMT	FV
OUTPUT					364.10

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5630

Annuities Due: Present Value

- Since with annuity due, each cash flow is received one year earlier, its present value will be discounted back for one less period.
- Solution approach on Calculator is to compute present value of $n-1$ period ordinary annuity and then add 1 payment at face value.

Perpetuities

- A **perpetuity** is an annuity that continues forever or has no maturity. For example, a dividend stream on a share of preferred stock.
- There are two basic types of perpetuities:
 - **Growing perpetuity** in which cash flows grow at a constant rate, g , from period to period.
 - **Level perpetuity** in which the payments are constant rate from period to period.
- Used to value infinite life “stuff” like financial assets (common and preferred stock) and corporations.

Present Value of a Level Perpetuity

$$PV = \frac{PMT}{i}$$

PV = the present value of a level perpetuity

PMT = the constant dollar amount provided by the perpetuity

i = the interest (or discount) rate per period

PV of Level (constant) Perpetuity (example)

- Example 6.6 What is the present value of \$600 perpetuity at 7% discount rate?

$$PV = \frac{PMT}{i}$$

- **PV = \$600 ÷ .07 = \$8,571.43**

Checkpoint 6.4 – Class Problem

Compute Present Value of a Level Perpetuity

How much would you pay, today, for a perpetuity of \$500 paid annually that can be transferred to all future generations of your family? The appropriate discount rate is 8 percent?

Present Value of a Growing Perpetuity

- In growing perpetuities, the periodic cash flows grow at a constant rate each period.
- The present value of a growing perpetuity can be calculated using a simple mathematical equation.

PV of a Growing Perpetuity (formula)

$$PV = \frac{PMT_{period\ 1}}{i - g}$$

PV = Present value of a growing perpetuity

$PMT_{period\ 1}$ = Payment made at the end of 1st period

i = rate of interest used to discount the growing perpetuity's cash flows

g = the rate of growth in the payment of cash flows from period to period

Checkpoint 6.5 - Example

Compute PV of a Growing Perpetuity

What is the present value of a perpetuity stream of cash flows that **pays \$500** at the end of year one but **grows at a rate of 4% per year** indefinitely? The **discount rate** is **8%**.

STEP 3: Solve

Substituting $PMT_{period\ 1} = \$500$, $g = .04$, and $i = .08$ into Equation (6-6), we find

$$PV = \frac{PMT_{period\ 1}}{i - g} = \frac{\$500}{.08 - .04} = \$12,500$$

Thus, the present value of the growing perpetuity is \$12,500.

STEP 4: Analyze

Comparing the value of the \$500 level perpetuity in Checkpoint 6-5 to the \$500 perpetuity that grows at 4% per year we see that adding growth to the cash flows has a dramatic effect on its value. To see why this occurs, consider the year 50 payment under both the level perpetuity and growing perpetuity. For the level perpetuity, this payment is still \$500; however, for the growing perpetuity the payment for year 50 is the following:

$$PMT_{year\ 50} = \$500(1 + .04)^{50} = \$3,553.34$$

Checkpoint 6.5: *Class problem*

What is the present value of a stream of payments where the year 1 payment is \$90,000 and the future payments grow at a rate of 5% per year? The interest rate used to discount the payments is 9%.

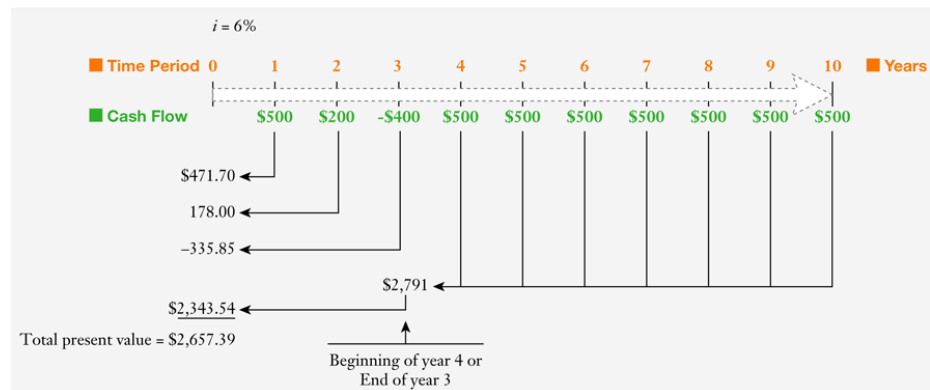
Complex Cash Flow Streams

- The cash flows streams in the business world may not always involve one type of cash flows. The cash flows may have a mixed pattern. For example, different cash flow amounts mixed in with annuities.
- Business analysis solutions involve organizing and valuing these mixed cash flows.
- For example, figure 6-4 summarizes the cash flows for Marriott.

Complex Cash Flow Streams – Important to understand this figure

Figure 6.4

Present Value of Single Cash Flows and an Annuity (\$ value in millions)



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6-41

Checkpoint 6.6 – Class Problem

Compute Present Value of a Complex Cash Flow Stream

What is the present value of cash flows of \$500 at the end of years through 3, a cash flow of a negative \$800 at the end of year 4, and cash flows of \$800 at the end of years 5 through 10 if the appropriate discount rate is 5%?

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6-42