Current Issues in Mathematics Education

Materials of the American–Russian Workshop

Moscow State Pedagogical University

Teachers College, Columbia University

November 18–20, 2016

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Edited by Alexander Karp

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Preface

The present publication comprises a collection of articles by the participants in the Russian-American workshop "Current Issues in Mathematics Education," which took place on November 18-20, 2016 in New York. The workshop was organized with support from the Eurasia Foundation in the form of a grant presented to the Moscow State Pedagogical University and Teachers College, Columbia University. Participants in the workshop included faculty from both institutions, as well as invited guests and colleagues, and doctoral and masters students of Teachers College, Columbia University. The collection opens with an introduction by A.P. Karp, followed by the articles in alphabetical order. Articles written in Russian were translated into English for this publication (conversely, for the Russian edition, articles originally in English were translated into Russian.)

The materials presented here mirror to a large extent the events of the workshop. They cannot, however, capture the extensive discussions of the papers and the problems they raised, which followed each of the presentations. A video recording of the workshop is available for those interested in that aspect of the proceedings. At the same time, publication of the papers delivered by the principal participants will likewise permit the reader to follow the discussion, as it were, by tracing the similarities and differences in the specific problems encountered in either country.

To be sure, the issues named and discussed here do not make up an exhaustive list. Mathematics education today faces a host of challenges, and ideas concerning their origins and remedies are just as numerous. We must continue the discussion, facing head-on the difficulties and setbacks. Neither shall we attempt to isolate ourselves from the experiences of other nations, but rather try to use those experiences to our advantage wherever possible. It is our hope that the materials presented here will prove useful in that regard.

We would like to extend our sincere thanks to Julia DeButts, Sergey Levchin and Juliana Fullon for their assistance in organizing the workshop and preparing the materials for publication.

Alexander Karp

Reflecting on the Current Issues in Mathematics Education

Alexander Karp
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This article, like the rest of the collection, considers the challenges facing mathematics education. Education in general and mathematics education in particular have always faced and will continue to face challenges: indeed, it could hardly advance save by overcoming difficulties, some of which go back thousands of years, yet when students complained of the hardships of learning, and teachers complained of students' laziness. To be sure, every age also ushers in its own particular problems, its own ways of dealing with problems old and new, and its own accomplishments. At the same time some problems may be regarded as universal, not associated with a particular region (even if they are manifested differently in different parts of the world), while some problems are endemic to particular regions and countries. In the present volume we intend to reflect the state of affairs in two countries: the U.S. and Russia. The contribution of these two nations (despite their many differences) to the international advancement of mathematics and technology is evident. It seems well worthwhile, therefore, to compare the different perspectives of people engaged in mathematics education in these two countries. In this introduction we will present a general overview of the emerging challenges, which will be discussed in greater detail in subsequent essays.

It should be noted straightaway that many of today's changes and challenges manifest itself in many spheres at once, consequently their discussion will invariably spread to several sections. Two of these changes deserve special mention: rapid technological advancement and fundamental social change.

The first is obvious. Today's student, whether in Russia or in the U.S., will not be taking notes in the classroom, preferring instead to record the lesson on his phone, or she will complain that the lesson does not come with a slide-show

presentation that could at least be photographed, since (horror!) it was not posted to the Internet. One Russian author (Suvorov, 1993) has given a very personal account of learning to sell watermelons as a child, for which purpose he was furnished with a table, telling him how much to charge for per weight of watermelon (in 50 gram increments). These days one could hardly find such a thing: everyone has a calculator. American schools tell their students in all earnestness to look on the Khan Academy website (https://www.khanacademy.org), where they will find all the classroom materials explained (presumably better than it had been done by the teacher). Technological progress has influenced not only the forms and methods of instruction, but also its content and philosophy.

Social changes are less obvious and more ambiguous. There is no question that the number of mathematics students across the world has grown considerably over the past century. But it is also true that even today not everyone gets to study mathematics, to say nothing of basic literacy. Still, that is not a problem one faces in Russia or in the U.S., where for some decades now all children are formally instructed in mathematics. Yet there is disparity to be found at a deeper level: a student in a prestigious private school in New England and her peer at a village school in Siberia, or a girl from a school in South Bronx and a boy from a specialized lycée in Moscow are all taught mathematics, but the knowledge they end up with is very different indeed. Changes in the regard are happening very rapidly, and not always in ways that increase the opportunities of all children. Social processes are, moreover, paralleled, as it were, by discussions of these same processes, which often function as a kind of echo chamber. In any event, there is a clear recognition of the problem and, consequently, of the need for a solution.

When discussing contemporary social and socio-political phenomena, we cannot gloss over organizational issues, including questions of authority (educational, among others), and the initiatives, as well as their interplay, of various groups involved in mathematics education. All of this may seem at first to be somewhat removed from what takes place in the classroom, and yet its influence on the actual classroom experience is profound.

In what follows we will discuss various concrete manifestations of the described changes and problems.

Why do we study mathematics?

In reply to this question a Russian reader will probably recite the words of the eminent 18th century scientist Mikhail Lomonosov, words that have graced (and still do) practically every mathematics classroom in Russia: “Mathematics needs must be learned for that it sets the mind in order.” The reality, however, is that for all the prominence of the so-called “formal approach” (Pchelko, 1940; Young 1906), which emphasized the importance of mathematics for general

development, mathematics was taught in school because without it one simply could not perform certain essential tasks. Neither navigation nor trade is possible without it, nor could a military fire its shells or build its fortifications without math. And that is why Lincoln (Ellerton & Clements, 2014) or Pushkin (Karp, 2007a) were taught arithmetic and other such subtleties.

Now we find out that a technique like long division, to give but one example, the teaching of which has been perfected over the course of centuries, has no practical application. No one is using long division today. Nor will a navigator solve right triangles to calculate his position at sea, and even the engineer will entrust it to a computer to make his calculations, forgetting all his schoolboy learning.

Why then should we study mathematics in general, and its various branches (algebra, geometry, trigonometry) in particular? Any number of arguments have been put forth on this account (see e.g., Gonzalez and Herbst, 2006 on the reasons for studying geometry). But if an adult may be persuaded by the argument that mathematics is, indeed, the foundation and formation of rational thought, that is much too abstruse for a child. And yet one must have not only a ready answer to this question, but also one that is persuasive to the student. To be sure, one could always fall back on the argument that "you cannot get through college without mathematics." Yet, however valid, this argument seems to be lacking teeth.

Educators are most commonly told that they ought to demonstrate as much as possible the practical advantages of mathematical skills: confronted with all the ways in which mathematics is applicable in the real world, the student will naturally want to study it. Here we might point out once again that, paradoxically, as the importance of mathematics in everyday life increases, its use by the average consumer decreases significantly. Consequently, it is far from certain that children, however impressed they may be by the widespread use of mathematics in everyday life, will decide that this is a subject that they must necessarily study.

The present author believes that the "adult" answer to the question "Why study mathematics?" has already been given: To bring up a civilized citizen and to separate out those who will go on do devote their lives to mathematics and its applications—this could only take place if every child is given the opportunity to explore the subject in some detail. (We must not be afraid of this "separating out"—indeed, the process is principally one of self-selection by those who, upon reaching a certain point of maturity, can make this sort of decision based on their accumulated experience of mathematics.) As for children, mathematics must be made interesting to them, thus rendering moot the question "Why do it?"

This simple answer, however, prompts all manner of discussions about what is interesting to children, and how one is to go about engaging their interest. Once

again, there have been many attempts to answer these questions, resulting in a variety of teaching methodologies, all of which deserve our attention.

Who should study mathematics?

Picking up from the preceding discussion, one might suggest that only those who are interested in mathematics ought to study it. And if a seven-year-old from the South Bronx, or from some lesser-known neighborhood of Moscow or St. Petersburg does not wish to spend his time learning math then it is his fault and his loss. The hypocrisy of this argument has long been apparent. A child is formed in the process of learning, and by holding the child fully responsible for his or her choices from the very outset we automatically shut the door to those segments of the populations that were previously denied or restricted in their access to education. The obvious truth—that a child has a limited (albeit continually increasing with age) responsibility for his or her choices, and that we must afford access to education for all children up to a certain age, including the possibility of "starting over" for previously unsuccessful students—extends to mathematics no less than to any other subject.

The crucial factor is that a child must be offered real possibilities, from which he or she will be able to choose. In reality many children never get to make the choice to study mathematics, because they are never properly acquainted with the subject. In many cases, in the U.S. as well as in Russia, nominally universal access to education masks the fact that the quality of this education prevents even the most talented and motivated student from advancing beyond a relatively low level. These "problem schools" have developed along rather different lines in Russia and the U.S. In the latter one often hears about the so-called "urban" schools, where the majority of the student population is African-American or Latino, while in Russia, on the contrary, city schools are generally better than those located in the far provinces (there are, of course, exceptions to this rule; moreover, lately one hears more and more complaints about schools in Russian districts populated by recently arrived migrants). In both cases, however, the problem of the quality of education is taking on political importance. Schubring (2012) has put forward a historical analysis of the idea of "mathematics for all." It would be a mistake to believe that this idea has been fully realized in either of the two countries in question. Consequently, we might wish to examine this problem from both perspectives.

To this end we will have occasion to reiterate some relatively obvious truths, since the proposition that certain demographics have no need for mathematics keeps popping up again and again in various guises. And the more veiled this position, the more perilous it is. These days one seldom hears about initiatives to restrict access to education for some particular group, either in Russia or in the U.S. On the contrary, these are frequently couched in expressions of concern for

these same groups. It is therefore essential to unmask their true import, separating them from valid concerns and initiatives.

The practice of subject-specific education, i.e., of relatively early curriculum specialization is popular in Russia, and is based on the perfectly valid idea that different children are interested in different things. Indeed, by the time children are sixteen, or even sooner, they are perfectly capable of deciding independently that they are interested in history more than in mathematics (for example), nor would it be wise to stand in the way of that decision. It must be kept in mind, however, that a child has a right to change his or her mind, and to this end a system must be flexible enough so that it neither deprives children completely of the study of mathematics nor prevents them from resuming a more general course of study if at some point they should wish to do so. We should also make sure that students are not compelled to make such a choice at an increasingly early age, and—more importantly—that this decision is left to the child, and is not made by someone else, e.g., by the school, to satisfy some administrative criteria. The introduction of distinct curriculum tracks into the program is typically accompanied by various pronouncements on the respect for the rights of children. We must make sure that reality is consistent with these lofty ideas, and that they do not result instead in thousands of students' being deprived of mathematics education.

The perfectly valid claim that today's society is made of people from very different cultures (whatever meaning one gives to that word), and that these cultures must be respected in equal measure, is often used as an argument for teaching some sort of "special math" in a place like the South Bronx (to use our old example), which, it is acknowledged, will be perfectly useless in college, but is somehow said to be better suited to the local population. There is no question that a teacher in St. Petersburg would do well to point out to her students that the great mathematician Euler, who lent his initial to the mathematical constant *e*, lived and is buried not far from their own school. Similarly, a teacher might mention other names, honored by other cultures in other places, and in general draw parallels between the subject matter at hand and the cultural values of various peoples. And the same time, it is important that such digression would not displace fundamental course material.

Attempts to exclude students from the study of mathematics may appear in the guise of concern for their personal lives, "Girls don't need math!"—or for the student body as a whole. Walker (2003) relates an incident in which an African-American student was not allowed to transfer to a more advanced class because his teachers liked his "positive influence" on students in the weaker section.

In other words, the problems we are facing today go beyond the socio-economical, political and organizational, into the methodological and ideological (which includes deeply rooted stereotypes).

Who teaches mathematics?

Teacher training is becoming an increasingly prominent topic of discussion. An entire discipline has emerged in the recent years, concerned with the question of teacher preparation (see e.g., Ball et al, 2008, 2009). A glance at Russian textbooks for prospective teachers (e.g., Stefanova and Podkhodova, 2005) also reveals considerable changes in this regard. What courses should future teachers be taking? What is the proper balance between courses in general education, methodology and mathematics proper? Do we need courses aimed at expanding what Ball, et al. have called the "mathematical horizon," even if it is not likely to find immediate application in teaching mathematics at the secondary level? And if so, does the future mathematics teacher really need to take a course in general topology? What about homotopic or differential topology?

The present author had gone through two student teaching, the second, more extensive of these requiring him to teach twenty lessons. By comparison, a student teaching in the U.S. typically consists of hundreds of lessons. Which is the right way? How should a prospective teacher be acclimated to the realities of the school environment?

Not so long ago a debate flared up in the U.S. about whether a middle school teacher (7th-8th grades) ought to be familiar with differential and integral calculus. The debate grew out of a study that showed that the majority of American middle school teachers are not proficient in these subjects, unlike their Chinese peers. Mathematics teachers in Russia, even those who will go on to teach elementary school, are taught calculus (how well they know it by the end is another matter), but they are far behind the Americans when it comes to using a graphing calculator. In Russia, as in China, there is traditionally greater focus on the "deep understanding" of the material, as Liping Ma had termed it (1999), while in the U.S. a teacher's grasp of the material typically stays at the level of a high-school graduate (Cooney, & Wiegel, 2003). What lessons are we to learn from all this accumulated experience, so different in the two countries? Any advancement in this field would surely be valuable.

At the same time we must bear in mind that teachers being trained today will be teaching not only tomorrow, but also thirty years from now. How should we train teachers for a future school, of which we know so little today?

How should the study of mathematics be organized?

Whatever preparation a teacher ultimately receives, his or her principal training takes place in the classroom. What awaits teachers there? There has been a lot of discussion in American newspapers lately on whether a teacher's performance ought to be evaluated on the basis of students' test scores, and whether these scores should be made public. Teacher performance review is also part of the Russian school system. Moreover, today it is a much more sophisticated affair than it has been in the past. Administrators on either side of the ocean invariably

draw parallels with the world of business, where performance is king: one is either getting results (in this case: level of student knowledge, measured by some objective criteria, such as tests) or not. One could argue (and some do) about the objectivity of these "objective criteria" or one could approach the problem differently turning to some case studies.

Possibly the biggest success story in mathematics education in the U.S., at least of the past several decades, is that of Jaime Escalante (1930-2010), which has been turned into a popular film. An immigrant from Bolivia, Escalante found work as a teacher in a middling American public school. In time, despite some opposition from the school administration, he began to offer his students more advanced mathematics course, asking more of them, but also spending many more hours with them to help them get through the course material. Gradually, in a school where students were known to struggle with regular mathematics course, dozens of students began passing AP Calculus, a college level course in differential and integral calculus. Students who never suspected that they could master these subjects, because they were too difficult or "boring," ended up studying them. The unprecedented success (a Russian reader would have to imagine that a run-of-the-mill provincial school is suddenly sending dozens of students every year to the regional and even national Math Olympiads) made Escalante into a celebrity. In addition to the film, he was the subject of the book "The Best Teacher in America" (Matthews, 1998) and countless articles. At the same time there's trouble on the horizon. It turned out that he was breaking all manner of rules: enrollment in his courses could reach as high as 60 students, while union rules permitted no more than 35. To be sure, Escalante was not getting paid any more for the extra students, but rules are rules! Staying late at school also became a problem. Thankfully, there were no accusations of impropriety, but it meant that the security guard had to stay late as well! That wouldn't do. In the end Escalante left the school (which had meanwhile changed principals), followed by other teachers, his disciples, and the school went back to way it was, when nobody took advanced placement courses.

A good Russian teacher striving for real results, rather than simply doing his routine job, was usually better off than Escalante, if for no other reason than a general lack of order. The legendary St. Petersburg teacher A.R. Maizelis (Karp, 2007b) also stayed late with his students, for which purpose he had his own key to the building. What would have invariably raised questions in the U.S. did not seem to bother anyone in Russia, at least in those days. Perhaps the situation there will also change some day…

These stories certainly do not fit into any business model. What sort of free enterprise is it, when one could hardly take any initiative? Working extra hours (even not for money, but for results) is forbidden. Any organizational change is forbidden. If we extend our parallel with the world of commerce, the school begins to look like a medieval trade guild, which dictated to its members when

to put out the lights and what tools to use. The guild structure collapsed under pressure from external forces, with independent craftsmen beyond the guild's reach setting up their own businesses with no regard for guild statutes. So too in education we see the emergence of out-of-system structures (which must be taken into consideration when discussing mathematics education), which are gradually displacing schools.

This, in turn, raises several questions. First of all, how should we evaluate the quality of these structures? It is easy enough to tell whether a private tutor, hired to help a poor student improve his grade, is doing the job or not: if the grade stays low, the tutor is not effective. But what if the out-of-school structure is not supplementary, but rather entirely supplants regular schooling? Is it even capable of serving this function? Today in the U.S. parents can opt to homeschool their children and design their own curriculum. But a child must be taught a whole range of subjects, not just mathematics. This requires a certain kind of logistics. Are parents prepared to meet this challenge?

We might imagine some future scenario, in which parents, perhaps with some assistance from a special service, are able to hire teachers from anywhere in the world, teaching remotely via the Internet. The result is a course of study that is at once individualized for a particular child, but is at the same time driven by a particular teacher. This scenario is still only a remote possibility today, but it is clear that we must be more mindful of the importance of good teachers, facilitating their initiatives and their reach. To be sure, this alone will not clear the ranks of poor and mediocre teachers so that we would no longer need to worry about them, just as we will still have to worry about students, whose parents are not sufficiently committed to seek out the best possible education for their child. But by emphasizing the importance of a good teacher and thereby changing the entire decision-making process we could improve the entire educational experience. We have to stop obstructing the work of would-be Escalantes.

This feeds into another concern for students who wish to learn more. The present author once had the occasion to write a letter, along with his colleague, in support of students at a certain New York school, who petitioned their school to offer a pre-calculus course (the equivalent of a 10th-grade algebra course in a Russian school). A teacher was willing to offer the course, but the principal found that it would not be a good use of the school's resources, and that less advanced courses, already offered by the school, would be sufficient for the students. Unfortunately, even our letter could not sway the principal's mind. It is a paradoxical situation: on the one hand the school system professes deep concern for students who have limited access to education, and on the other hand those wishing to broaden their access are refused.

It is worth noting that the various specialized schools in New York, geared toward the gifted student, have far fewer openings (the difference is on the orders of magnitude) than there are eligible students, who have passed the requisite exams. (One could take issue with the exam, of course, but that is beside the point.) As a result there are no places even for the socially and economically disadvantaged students, who have no other means of getting a better education.

Russia is rightly proud of its track record in supporting students, who evince an interest in mathematics, including those hailing from far-lying regions. By way of example we might mention the practice of distance education (Marushina, & Pratusevich, 2011). But it would but naive to think that Russia faces no problems in this regard.

We have outlined only two of the areas of the concern when it comes to the organization of mathematics education. There are many others, including the problem of education inequality that we discussed before. New technology, as well as new perspectives on the problems facing education—perspectives emerging in the context of ongoing social developments—can help us meet these challenges. Considering the experiences and discussions of other countries seems to us especially useful in this regard.

What should be taught in school?

Contrary to popular belief, mathematics curricula have changed significantly over the years. There are also notable national differences. It would be impossible, for example, to compare Russian and American courses in geometry, trigonometry or even algebra, even though many (though not all) propositions are commonly found on both sides. Certain topics are not covered at all in one or another country. Dmitry Faddeev, an important Russian research mathematician, used to say that when it comes to secondary-school mathematics one should always operate on the principle of "guilty, until proven innocent," i.e., every topic must be presumed unnecessary until proven otherwise. Taking into consideration teaching practices in other countries should point to a similar conclusion: just because something is done a certain way in your country does not mean that it is the only way of doing things. One has to compare and consider, and even if one's own practice ultimately turn out to be the better way for your country, it doesn't imply that it is flawless. And these flaws must be addressed.

Both countries, moreover, are facing the need for radical curriculum reform. As we discussed earlier, many of the technical skills being taught today will certainly turn out to be impractical. Does that mean that they ought to be purged from the program? Perhaps not, or at least not always so, because the connection between skill and conceptual understanding (which no one is trying to get rid of) is rather complex: understanding without skill is hardly possible. But there is no question that the program is in need of reform, especially when we consider

newly emerging topics and changing needs. It is doubtful that anyone has a precise idea of how this is best accomplished, but there are plenty of initiatives and suggestions in this regard, and these ought to be promulgated widely.

Finally, in both countries we are witnessing the emergence of various programs aimed at improving education for a particular subset of the population: e.g., for the gifted, or those requiring a more in-depth treatment of a particular subject, or those who respond better to alternative learning methods, etc. All of these practices must be examined.

How to make the teaching of mathematics more effective?

We have saved the questions most pressing for the practitioner of mathematics education to the very end. And there are plenty of them. Here also we include the entire complex of issues subsumed under the topic "lesson planning and organization." We are clearly moving away from the long-dominant model, where the teacher addresses the entire class, and the class works as a unified whole. Paradoxically, we are returning to a system where, as in the middle ages, several groups of students, housed in a single classroom, may work relatively independently, or even when students can work individually on their own projects. How does one plan one's lessons and organize material in such a setting? What is the teacher's role? What must the teacher do to maximize the students' learning experience?

Learning through problem-solving has long become a popular slogan, and a great deal has been invented and understood in this regard (see e.g., Karp 2015 on the changes in the organization of problems in the late 19th- early 20th centuries). But today one also hears about new kinds of problems, about a new system of organizing them—appropriately, these are subjects of ongoing discussion in both countries.

Technological discoveries have the greatest impact on mathematics education, inasmuch as they have changed out understanding of the goals and the content of mathematics education. But they are also highly significant for mathematics education in that they have ushered in new methods and resources: there are countless examples of this, from new ways of presenting material to new methods of understanding and evaluating student progress.

All these and many other aspects of mathematics instruction and techniques of improving its effectiveness must be discussed and examined in detail.

In lieu of a conclusion

This paper is introductory in nature: its aim is to outline the diverse and acute problems facing mathematics education today. In naming them (of necessity briefly, and certainly not exhaustively), the author has made no attempt to offer his own solutions or even outline existing solutions. Indeed, not even the

chapters that make up this volume can make such a claim. As the reader shall see, they are far more concrete than the present article, in the sense that they speak of concrete experience and address certain parts or aspects of the problems at hand. Perhaps one ought not expect some kind of general and uniform answer to all raised questions: as in, we must do it this way! The answer to the challenges we face must be the whole totality of accumulated experience. It is preferable to have a look even at the relatively small advances within the context of the general picture. If the discussions and accomplishments presented in this volume should help our readers in their own thinking and practice, then we shall have achieved our goal.

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MathKit and Math Practicum

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The conventionalism of school mathematics and its remoteness from real life (as well as from mathematics proper) has always been the subject of discussions and one of the main reasons for numerous reforms of mathematical education in different times and in different countries. It's hardly possible to eliminate these shortcomings (if they should be regarded as such at all), because whatever progressive teachers and educators argue, for the overwhelming majority of students and their parents the most important goal is still to pass the graduation exam, which opens the door to university. However, we should aspire to get rid of these faults, if only partly; otherwise interest in the subject will reduce so dramatically that no practical benefits from studying math will have any mitigating effect. This explains the unending search for teaching methods that can stir up interest in mathematics in quite diverse categories of students, even if applied only in small doses. One of these methods, which was suggested long ago and comes in different forms, is to try to simulate for students a professional mathematician's research process. This includes, among other elements: the setting and conducting of mathematical experiments, the observation of their results, their analysis, the building and testing of conjectures, and requires the ability to derive conclusions from observations and support them by logical arguments. This kind of activity often helps to solve both problems: to make the learning process more interesting and bring it closer to real life. The tool that allows school students of various backgrounds to perform their own mathematical experiments and investigations was created about 25 years ago. This tool is dynamic geometry programs, which, having begun with geometry, expanded their applicability practically to the entirety of secondary school mathematics (and more), so that today, we usually call such software *interactive*

mathematical systems (IMS). Two pioneer IMS, *The Geometer's Sketchpad* and *Cabri*, were enthusiastically accepted by the international mathematical community and recognized as the most successful finding yet in the area of educational information and communication technology. Since then, a lot of similar programs have been developed; perhaps the most wide-spread of them today is *Geogebra* (which owes a great portion of its popularity to being freeware). The authors of this paper are directly involved in the creation and development of *MathKit* (MK), a Russian interactive mathematical environment whose first version was released more than ten years ago (see Matematika, 2015). Its distinguishing feature is a well thought-out, convenient interface with original additional construction and editing tools and means for customization, developed on the basis of years of practical usage of various IMS. Naturally, the specific examples that we discuss below are realized in *MathKit*, although in principle they can be transferred into most other interactive systems.

During the last 10-15 years the proportion of educators, teachers, and students familiar with interactive mathematical systems increased enormously, as plentiful collections of ready-to-use dynamic models and extensive literature have appeared. Unfortunately, their practical use in a math class is still very limited. Of course, nobody expects that these resources, or technology in general, will be able to instantly solve all problems of modern mathematical education. But we can't ignore the fact that computers, the internet, and various gadgets have become an integral part of our life, especially since technological advances can really add a new quality to the educational process. So what are the main obstacles to the mass introduction of IMS when teaching mathematics?

In order to use computers systematically in a class, at least two conditions must be met. One is of technical nature: the availability of necessary hard- and software. Of no less importance is the possibility of an effortless and brief transition from the ordinary way of conducting a lesson, to a lesson using technology, and back. The second condition is related to the content of the material: additional time costs, in preparation for lessons and during the lessons themselves, are inevitably required from a teacher wishing to actively use computer methods and resources in his or her work. They must be made up for by a greater efficiency with these materials compared to traditional teaching. This means that computer models and activities must ensure better or, at least, the same results as ordinary forms of teaching, but with a smaller expenditure of time. This is possible thanks to an increase in students' motivation and interest, connection to additional information channels, and new problem settings offered by technology-based lessons.

Currently, technical equipment in schools includes, as a rule, a computer with a projector in each classroom, but the organization of regular individual class work with computers is more difficult: computer labs with a sufficient number of desktops are usually given to lessons of computer science or informatics, and so

teachers would have to count on either mobile computer labs or on the use of personal laptops or tablets. Both these options have their drawbacks; one way or another, in Russia, as well as in many other countries whose schools the authors had a chance to visit, they are quite rare. Therefore, for the time being the prevailing form of using IMS in the class is the lecture format, which is ill-suited for promoting active work by all students. At the same time, the main value of IMS and the learning activities they support is the tools they provide for performing mathematical experiments and explorations, which are, basically, primarily individual kinds of activity.

We will describe an interesting and successful form of using IMS on a regular basis, which was developed in the Advanced Science and Education Center (AESC) of Moscow State University in the framework of a special subject, 'Mathematical Practicum.' This form is free from technical and organizational problems and involves all students in individualized research activities. In addition to some specific notions and competences that students acquire by doing the Practicum tasks, they also receive the firm skill of using the software, and an understanding of its capabilities; it becomes their working tool for solving "normal" mathematical problems. But before we proceed to the assignments of Mathematical Practicum, a few words about the school, AESC.

The Physics and Mathematics Boarding School No. 18, affiliated with Moscow State University, was opened in 1963; later it was included directly into the structure of the university as one of its departments (AESC) and received the second name, Kolmogorov School. The main goal of this school is to provide students who have no access to major university centers with the opportunity to uncover and develop their abilities in math, physics, and other sciences. The experience of this and a number of other similar schools, opened at the same time, in searching for and teaching gifted children proved to be very successful, and in time, many schools modeled after them were organized in many countries all over the world, including the USA. From the very beginning these schools served as a perfect field for experiments in development of new forms and methods of education, including new topics in school curricula, and had a considerable impact on the evolution of mathematical education in regular schools. Mathematical Practicum is a good example of these new forms.

This subject was introduced to the curriculum of Boarding School 18 by one of its founders, the famous mathematician Andrei Nikolaevich Kolmogorov, soon after it was opened. Generally speaking, the Practicum is conducted as follows: a new assignment with the required theory is presented in a lecture; then worksheets with theoretical material and variants of a task are handed out. Normally, the number of variants suffices for all students in a class to get different tasks; some tasks are intended for small groups. The tasks are assigned for 2-3 week periods and are done after regular classes. The results are discussed at a concluding lesson. The tasks were of a practical nature: students had to plot graphs, draw

diagrams, make calculations or a model of some intricate polyhedron, draw a wall paper pattern of a given type, etc. A detailed description of the Mathematical Practicum, which has become a distinctive feature of the course of mathematics in Kolmogorov School for many years, of its ideology, history, and specific tasks can be found in Vavilov (2013). Unfortunately, after two successful decades, the practicum ceased to exist as a separate mathematical subject for a number of organizational and methodological reasons. One of the most important reasons was the growing spread of personal computers and the corresponding software that made many assignments almost meaningless or outdated in their original form. For example, initially, calculations had to be performed "by hand," with tables or slide-rule, and functions were plotted point-by-point on plotting paper. Even though this kind of activity still has a certain value for studying math today, a modern school student, equipped with state-of-the-art computers and gadgets, sees it as an obvious anachronism. The idea of regular practical mathematical assignments had never lost its attractiveness, but old tasks had to be reformulated and new ones devised in such a way that technology should only enhance their efficiency rather than kill them. The redesign began a few years ago; as its first result, the Practicum has returned to the curriculum of Kolmogorov School as a separate subject. New Practicum assignments are based on *MathKit,* but of course in most cases *MathKit* can be replaced by other dynamic software. In particular, a large number of miscellaneous virtual lab works, many of which stem from assignments of the Practicum, formed the core of an extensive collection of digital educational resources (Dubrovskiy et al, 2004), in which they are implemented as sketches of *The Geometer Sketchpad 3*.

The distinctive features of the updated computer-mathematical practicum in the Kolmogorov School, deeply connected to the ideology of IMS, are a greater portion of constructive problems, a broader space for self-study and self-control, and for experimental and research activity. From a teacher's viewpoint, the digital implementation of these tasks makes work easier: students send their completed tasks to the teacher by email, and all that is required to check them is to shift the initial data and make sure that neither the whole construction nor any piece of it is destroyed. As a rule, the correctness and stability of the construction confirm that a student has successfully learned the accompanying theory; sometimes theoretical material is included in the examination on the corresponding topics. We give a number of new practicum assignments below. It goes without saying that such assignments do not require the introduction of a special subject in the curriculum; they can be given in the context of regular math subjects: geometry, algebra, or pre-calculus. The Practicum in AESC includes only 5-6 assignments of this sort during the academic year.

Now let's proceed to the examples. The first three are devoted to constructions in space, or more exactly, on the images of spatial figures, the topic for which

dynamic models are most willingly used by teachers. Notice that some dynamic geometry programs have a special component for the modeling of three-dimensional objects and various constructions on them. However, the advantages of using such models in teaching solid geometry are rather questionable: is it possible to learn, for example, the construction of cross-sections if the computer does this for you? We think that for the purposes of a school course, constructions in "two-dimensional software" can do the job and, moreover, are more instructive (Dubrovsky, 2004; Dubrovskiy, 2003).

Construction of cross-sections. The development of spatial imagination is both one of the main goals of solid geometry as a school subject and fundamental to learning it successfully. From this point of view, the construction of sections plays the most important part in the study of solid geometry. Unfortunately, the time allotted to this topic in the curriculum is definitely insufficient. The problems on the construction of sections, as well as any other construction problems, remain on the outskirts of the school geometry curriculum and, as a consequence, are not included in exams. In particular, they are absent from the unified national examination, which has to be passed by every student to receive a certificate of secondary education and/or enter a university. Perhaps one of the reasons for this lack of inclusion is that such problems are rather difficult to check. In this practicum the sections are constructed on dynamic models of polyhedrons that can be rotated around two axes. This is a case where the problem of checking is practically removed. One doesn't need to trace all the steps of the construction: if the construction is not destroyed under the rotation and variation of the points that specify the plane of the section, which can be established in a few seconds, then we can be sure that it is correct. It is important that while the sections are built exactly the same way as if they were drawn on paper, the changeability of the image aspect enables students to get a more or less realistic picture of what happens in space in the course of the constructions, which helps them to control their actions. This practicum consists of a series of 10-15 tasks of increasing difficulty, starting with the simplest questions from a standard textbook. Each task contains a simple polyhedron (a tetrahedron, prism, or cube) and three points on its edges or faces; the student has to draw the section through these points, i.e., the polygon in which the plane passing through them intersects the polyhedron. In the course of their work on the tasks, students must discover for themselves the main techniques of these kinds of constructions and learn how to use them. This practicum allows teachers to minimize the time in the classroom devoted to the topic without any loss in the quality of learning. The first of such series of assignments was created for the collection of digital resources (Dubrovskiy et al, 2004) mentioned above; since then it has been successfully tested many times in many schools, and even in different countries, in the form of home and class lab work.

Cross-sections 2. A few months after the first section-drawing assignment, a second one is given to refresh and practice this important skill. It contains only a single, but much more difficult, task for each student, which is also performed on a rotating model. The answer to one of these tasks is shown in Figure 1. This particular task is formulated as follows: a rotating model of a regular triangular prism is given; through each edge of the prism a plane is drawn parallel to a diagonal of lateral face (this condition defines the planes uniquely); the student has to draw the image of the convex polyhedron bounded by these planes and its section by the plane passing through three points given on the edges of the prism.

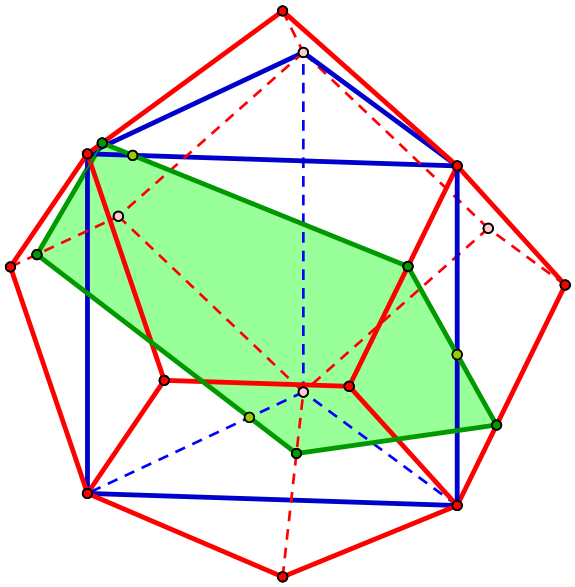

Fig. 1

We can be sure that a student who successfully performs this task has perfectly learned all the tricks of this trade.

Images of polyhedrons. A generic task of this practicum reads like this: given a Cartesian frame and a polyhedron (regular or semiregular) in a certain position with respect to this frame, the student has to draw the parallel and central projections of the polyhedron after it has been rotated around a given axis by a given angle. Performing this task, the student will have to use skills and facts from most chapters of solid geometry: about regular polyhedrons, properties of projections, distances and angles in space, and isometries of space. In the introductory lecture, students briefly get to know how to describe spatial rotations with matrices and how to compute the coordinates of the projections of points from the coordinates of the points themselves.

The results of the work on this practicum were pleasantly surprising. Students learned by themselves to multiply matrices and gained an understanding of the

geometric sense of this operation, studied the matrix-related tools of *MathKit*, and, in many cases, even went above and beyond and constructed dynamic models that allow the user to arbitrarily change the rotation axis and angle, as well as the position of the center of projection. Figure 2 shows one such model with the sliders that control the edge length r of the polyhedron (in this case, a hexagonal antiprism with equal edges), the angle φ specifying the position of the prism's axis, and the angle α of the prism's rotation around this axis; the parameter d relates to central projection, not shown in the figure.

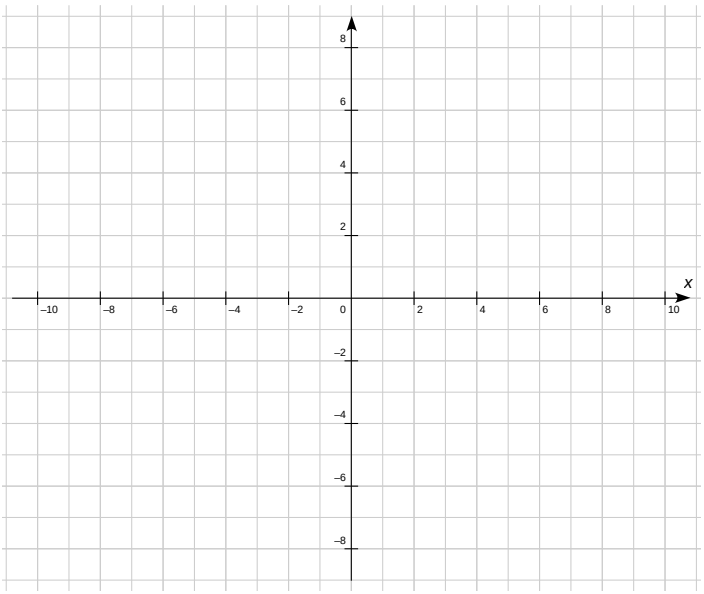

Fig.2

The practicums described above have an important feature: they are easily adaptable both to the level of the class and to each individual student. If the first practicum on sections is based on standard assignments, the simplest of which are found in the curriculum of ordinary schools, the second one requires more spatial imagination and ingenuity, while the third one extends considerably beyond the scope of standard syllabus. At the same time, within one practicum, students receive not only the basic problem, the compulsory minimum, but also more interesting additional advanced tasks, which require some self-study of necessary theoretic material; these tasks are graded separately.

Restoring "eared polygons." Suppose that a polygon is drawn in the plane; mark the midpoint of each of its sides. Now let us erase the polygon, leaving only the marked midpoints. Is it possible to restore the erased polygon, and if so, how? This well-known problem became a prototype of the tasks of this practicum. Another source is the following theorem attributed to Napoleon (this and similar theorems are discussed in Yaglom, 1962, e.g., Problem 21):

the centers A1, B1, and C1 of regular triangles erected externally on the sides of an arbitrary triangle ABC are vertices of a regular triangle (Fig. 3).

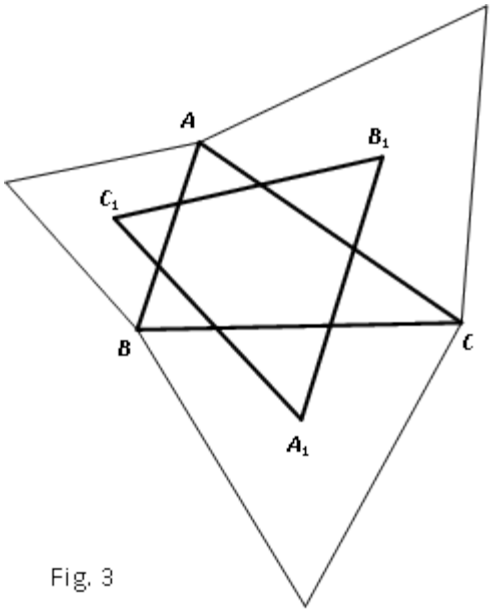

Fig. 3

Napoleon's theorem can be transformed into a construction problem similar to the problem about midpoints described above: given three points A_1, B_1, and C_1, to restore a triangle ABC such that the given points are the centers of regular triangles constructed on its sides. It is not difficult to compose plenty of diverse problems of this type. For instance, in "Napoleon's problem" we can assume that the given points are the vertices of the "ears" (the regular triangles on the sides) or take the center of one "ear," the vertex of another one, and the midpoint of the remaining side of triangle ABC (see Fig. 4), and so on. In the most general case, the triangle ABC has to be restored from the points A_1, B_1, and C_1 such that each of the triangles A_1BC, AB_1C, and ABC_1 has a given set of angles. Also, one can try to restore not only triangles, but polygons with any given number of sides (as in the "midpoint problem").

Let us explain the mathematical background of these problems.

If, in the settings of Napoleon's problem (Fig. 3), we successively perform counterclockwise rotations through 120° around the points B_1, A_1, and C_1, then the point A will first be taken to B, then to C and, finally, returns back to its initial position: that is, it's the fixed point of the composition of these three rotations. It can be shown that for any arrangement of the centers, this composition is a translation, which has a fixed point, i.e., is the identity map if and only if the centers of rotations form an equilateral triangle (Notice that this implies Napoleon's theorem). Since in this case any point of the plane is a fixed point of

the composition, vertex A of the restored triangle can be chosen at random, which means that the problem has infinitely many solutions. In the case where the points A_1, B_1, and C_1 must be the vertices, rather than centers, of the "ear" triangles, we should consider the composition of the 60° rotations around these points. This composition is a 180° rotation, a half-turn, and therefore, it has a unique fixed point. It follows that in this case our problem has a unique solution for any arrangement of the points A_1, B_1, and C_1. In a similar way we can consider other problems of this kind, in which rotations can be replaced by other transformations, such as spiral similarities, or rotational dilations, and reflections. As in the examples above, such problems either have a unique solution for any given points, or, in the general case, have no solutions, but for certain special arrangements of the given points have infinitely many solutions. Studying these special arrangements in the second case, one can discover interesting geometric facts, such as Napoleon's theorem mentioned above.

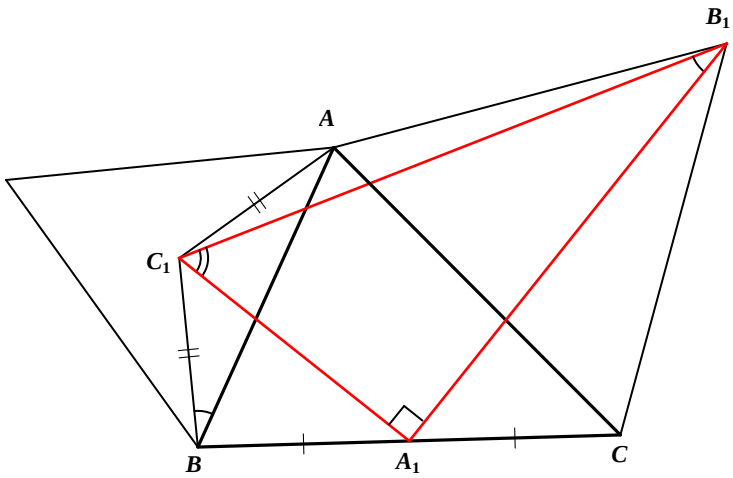

Fig. 4

Each variant of the "Restoring eared polygons" practicum consists of two assignments: in one of them the student has to restore a triangle, in the other one, a quadrilateral; besides, one of them belongs to the first of the two cases described above and the other one, to the second case. In addition to the construction proper, students must explore the solvability and the number of solutions to the problem. The practicum concludes the topic *Similarity transformations of the plane* in the AESC 10th grade curriculum and, in addition to its teaching function, constitutes a weighty part of the exam in this topic. A detailed analysis of two relatively simple problems on "polygon restoring" and teacher guidelines for the use of the corresponding models included in the *MathKit Collection* (Matematika, 2015), can be found in Dubrovskiy (2011, 2012).

Using MK, the assignments of this practicum can be offered at three levels:

- at the *experimental and research* level, applying geometric transformations corresponding to the definition of the "ears", a dynamic model is constructed that allows students to find out experimentally which of the situations described above is related to the task in question and find the solution approximately, by trial and error;
- at the *constructive* level the student has to find a geometric construction that gives all solutions if they exist; it's worth mentioning that in most cases there exists a construction that does not involve compositions of transformations;
- at the "*theoretical*" level a full analysis of the task is given (which boils down to the study of the fixed points of the composition of corresponding similarity transformations) and the construction is performed on the basis of this analysis.

Students of the Kolmogorov School must perform their tasks at the 2nd or 3rd levels. Special bonus points are awarded to solutions in which the existence condition in the second case (with infinitely many solutions under a special choice of the given points) is formulated as a separate theorem (modeled after Napoleon's theorem), thereby presenting their own small mathematical discoveries. For example, studying the construction in Figure 4 one can come up with the following statement:

> *if points B_1 and C_1 are a vertex and center of equilateral triangles constructed on two sides of an arbitrary triangle ABC, and point A_1 is the midpoint of the third side, then $A_1B_1C_1$ is a right triangle with acute angles 30 and 60°.*

Such discoveries, which require a complex understanding of abstraction and mathematical culture, can be made only by a few exceptional students.

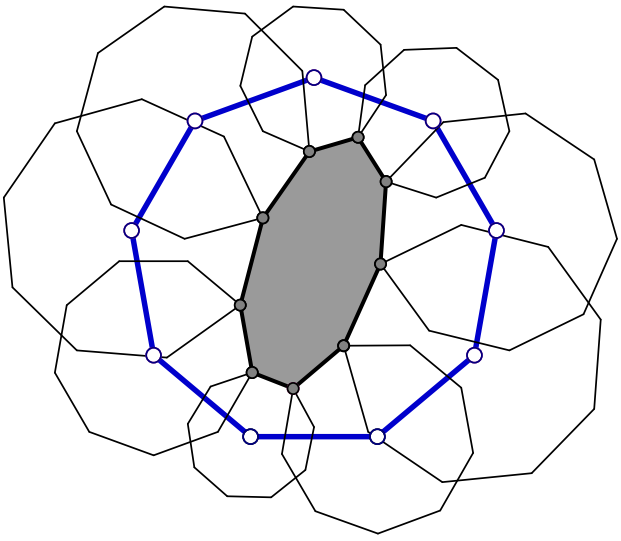

Fig. 5. The centers of the regular heptagons erected on the sides of the shaded heptagon, form a regular heptagon.

A good, albeit rare example is the recent achievement of Nikita Bashaev, a student of AESC, who extended his work on the practicum to find a new proof of the Napoleon-Barlotti theorem, which gives a condition for the centers of regular n-gons constructed on the sides of an arbitrary n-gon to form a regular n-gon (Fig. 5). This proof and an accompanying result are soon to be published in a highly respected in Russia popular science and math magazine *Kvant* (The American reader is familiar with *Quantum* magazine, published by NSTM since 1990 to 2000; the majority of *Quantum* publications are translations of *Kvant* papers.)

We'd like to emphasize again that we want to demonstrate real practicum tasks used in the Kolmogorov School. We are aware of the fact that the mathematical level of most of these tasks is quite high: they are mainly intended for students that have shown great motivation and skill in studying mathematics. But the form of these tasks admits a variation of topics and difficulty level within very broad limits. For this paper we have composed two practicums that are more accessible, perhaps, to students of public schools.

Properties of functions. It is well known how important the ability to give an example (or counterexample) is in calculus. A student can be perplexed by even a simplest question of this sort, when asked, for example, to find a function that has no derivative at exactly two points. This practicum is supposed to help develop the simplest skills of this kind. Whereas in ordinary problems on function plotting the function is given and one must analyze and plot it, we reverse the order of work (and this must make the tasks more interesting): students receive a list of properties of some function and have to find a function with these properties, that is, its formula. Then, using the software, they must plot its graph and use that to check if the function really has the given properties. Each task of the practicum consists of a series of separate questions of growing difficulty, each of which admits a number of correct answers, which also distinguishes them from standard problems.

Here is an example of such a task:

Find functions satisfying the properties listed below and draw their graphs. Each question must be answered on a separate worksheet.

1. *f(x)* is an even function;
2. *f(x)* is a periodic function with period 4;
3. the natural domain of *f(x)* is $[0,5)$;
4. *f(x)* has domain $(-\infty,+\infty)$ and range $[0,5)$;
5. *f(x)* increases on the intervals $(-\infty,0]$ and $[3,+\infty)$ and decreases on $[0,3]$;
6. *f(x)* has infinitely many zeros on the segment [0, 1], but is not identically zero;

7. *f(x)* has local minima at points 0 and 2 and has no local maxima;
8. *f(x)* has a horizontal asymptote $y = 3$ and no other asymptotes;
9. *f(x)* has a vertical asymptote $x = -2$ and no other asymptotes;
10. *f(x)* has an oblique asymptote $y = x + 1$ and no other asymptotes.

Tasks can differ not only in their conditions on functions, but also in permissible ways of description of the unknown function *f(x):* in the simplest cases students can simply guess the solution (its formula) and check it using the computer-generated plot; or one can take a function of a given form with variable coefficients (e.g., a polynomial) and properly pick its coefficients; one can try to form the answer as the sum or product of some standard functions or their more complicated combinations, to use transformations of graphs, and so on. The specific manner in which the function is created is controlled by the set of tools included in the model. Thus the difficulty level is determined both by the questions and by the tools. On a worksheet, students must show the graph and those of its features given in the question: critical points, asymptotes, etc. The desired look of the work is presented to them together with the task. A possible answer to question 7 above is shown in Figure 6.

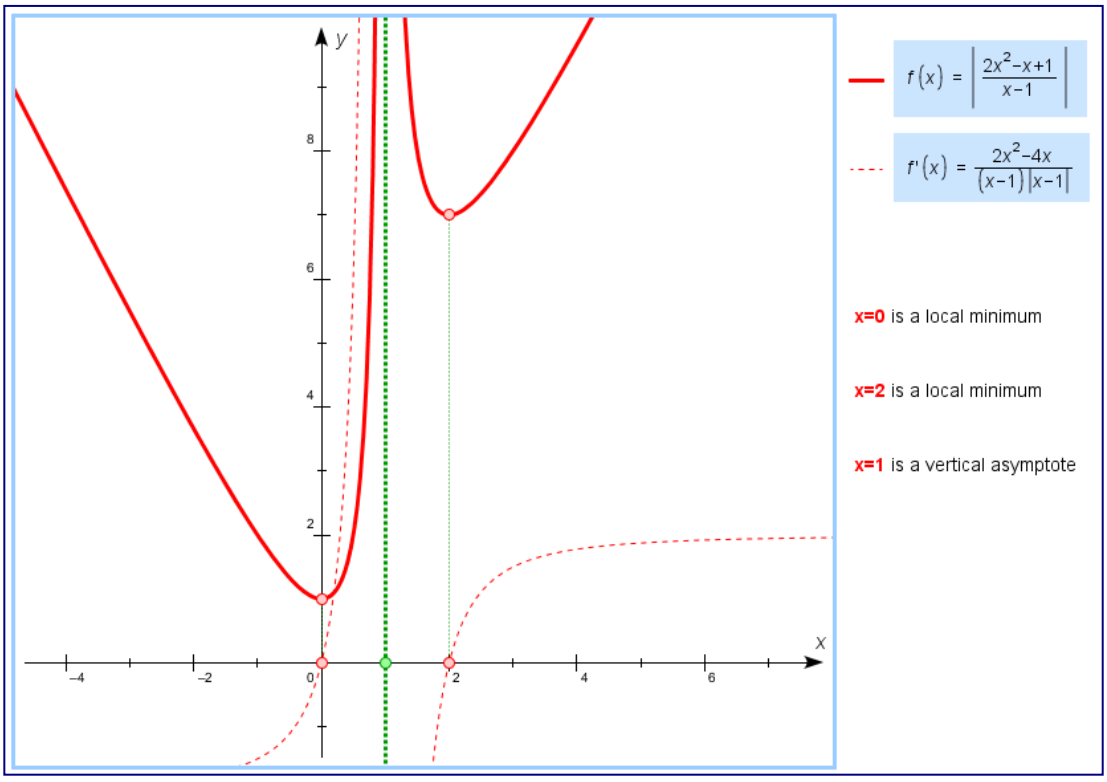

Fig. 6.

Probabilities of random events. The aim of this practicum is to create a mathematical model of a random experiment and its application to problem solving. The practicum is given after students get to know the classical and geometric definitions of probability and learn to use elementary formulas for computing probabilities (addition and multiplication rules, the law of total probability). In the introductory lesson the basic schemes of probability experiments are discussed: random sampling with and without replacement, random points on a line or in a domain, Bernoulli trials. The lesson also shows how to implement each of the schemes in MK and how to collect the results of random trials and process them. Each task is divided into a number of stages:

- the construction of a model in MK;
- solving of a few simple theoretical problems and verification of the answers using the model;
- experimental solution of one or more complex problems that are too difficult to solve analytically (not excluding the possibility that some students will come up with analytical solutions even for these problems, which will make it only more interesting to check these solutions against the model).

Thus, in doing any of the practicum tasks, students go through three stages of a mini-investigation: they construct a computer model, verify that it correctly reflects the given situation, and, finally, use it to obtain new results.

Let us give an example of such a task.

> Ten persons must split randomly into two equal teams to play football. They decided to use a coin: They take turns casting it and either join the *Heads* team or the *Tails* team. As soon as at least one of the teams is full, the process stops, and all the players who had not tossed the coin join the other team (It is easy to notice the connection of this process with well-known Banach's Matchbox Problem).
>
> (1) Create a model of this tossing in MK.
>
> (2) What is the probability that the 1st and 2nd players get in the same team? Same question for the 1st and 5th players? 1st and 6th? 1st and 10th? 9th and 10th?
>
> (3) For what pair of players is the probability of getting on the same team the greatest? For what pair is this probability the smallest? What happens with these probabilities if there are 2n players and $n \to \infty$?

A fragment of a model created by a student to solve this problem is shown in Figure 7.

There are a few more practicums beyond the bounds of this paper. For diversity, let's mention a couple of algebraic ones: in one of them students study the dependence of roots of a reduced quadratic equation on its coefficients (which

are represented by a point in a separate frame), in the other, the methods of approximate calculations of the roots of equations. A large number of "paper" assignments from the records of the Mathematical practicum in the Kolmogorov School are still to be reworked. However, we think that even the examples

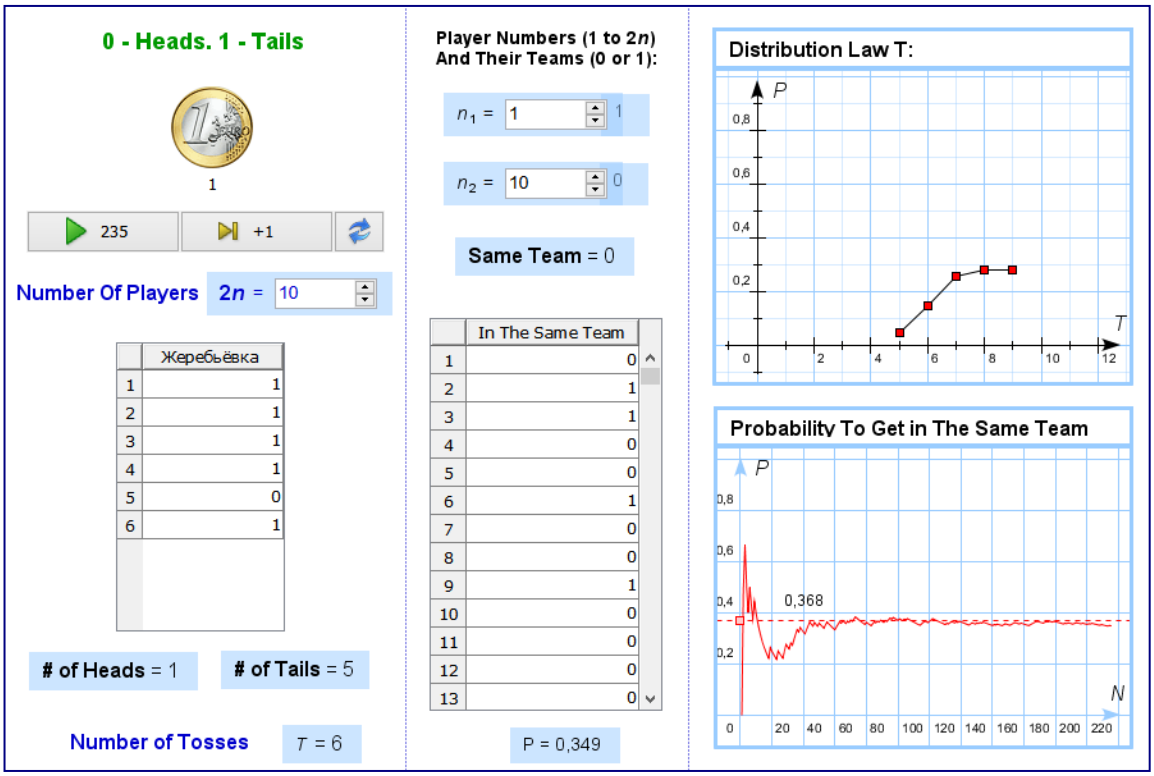

Fig. 7

considered in this paper testify that this form of work, combined with the practical character of tasks, can give a new quality to teaching math in high and, especially, at an advanced level. The practicum can serve both to review old material to ensure that it has been properly learned, and to self-study new material, and its difficulty can be adjusted to the class or to every student. It allows student work to be individualized and opens a field for them to apply their creativity. The digital format of tasks makes it easier to distribute them among students, for students to do them and teachers to check them. Teachers and students have the opportunity to assess the benefits of dynamic systems and develop the habit of using them in teaching and learning whenever they have the opportunity. And, finally, the computer mathematical practicum is favorably assessed by students: in a survey traditionally conducted at the end of the academic year, it received one of the highest grades.

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Making Cultural Change

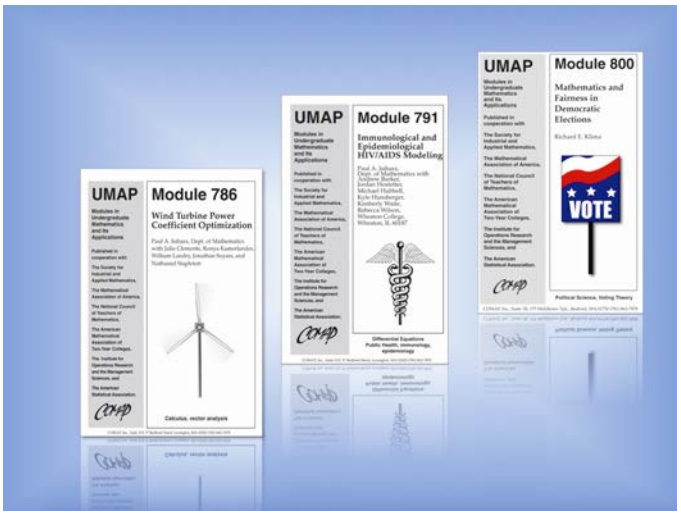
Sol Garfunkel
COMAP

Here is a revisionist personal history. Roughly 50 years ago I decided that U.S. mathematics education needed to undergo a radical change. In my opinion mathematics education needed to morph from being primarily concerned with the education of mathematicians to being primarily concerned with the mathematics education of the general public. Leaving aside the wisdom of that belief, assuming it as a working article of faith – then what?

If you skip the many starts and stops and enthusiasms of youth, I came to another article of faith – that average students would learn more mathematics if they saw a reason to learn that mathematics. – a reason in the real world. And that led me to mathematical modeling. Not having any real experience with modeling and applications (I was a mathematical logician by training) I was fortunate to have Henry Pollak as a mentor.

But the question remained, how does one go about such a paradigm shift in an educational system so resistant to ANY form of change. One more article of fait – you can't beat something with nothing. In other words, it is not enough (unless you are French) to simply argue a philosophical point of view that things must change, you need to show people, in concrete terms, what you mean by change. And so we began making 'stuff'. We built an organization to create teacher and student materials that embodied our philosophy.

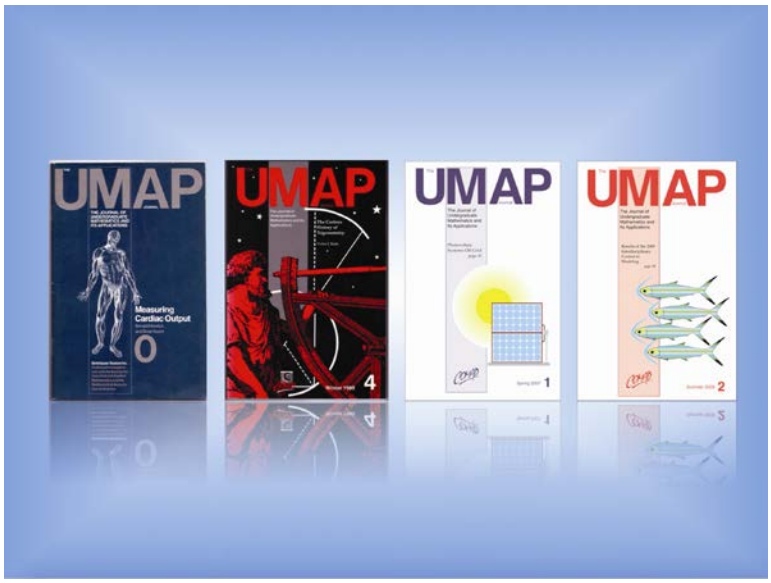
We began small both in the sense of a small working group and in terms of what we produced. We created curriculum modules that were meant to be covered in one hour of undergraduate class time. These modules could range over the entire undergraduate mathematics curriculum and they were meant to be self-contained. Each taught some aspect of mathematics through a real application or model. They were reviewed not only by academic mathematicians, but by teachers and practitioners in the field of application.



Sample of UMAP Modules

Of course, there is no complete set of materials that can present all of the uses of all of undergraduate mathematics. And new applications are being discovered all of the time. So, we needed a process that could continue and be self-sustaining. Moreover, we wanted to locate this work squarely within the center of the academy – having it be part of the academic rewards system.

To do both of these things we founded the UMAP Journal. This is a peer reviewed journal containing articles about modeling, new applications of mathematics, and actual student ready materials. It is now in its 37th year and just like any research journal authors and reviewers consider it part of their academic responsibilities to work for it. And publication figures into tenure and promotion decisions.



Sample of UMAP Journals

Some more history – the UMAP work was begun in the late 70s with initial funding from the National Science Foundation (NSF). In those days essentially all federal funding for mathematics education came from NSF to colleges and universities. There was little or no work K-12. That all changed on 1983/84 with the publication of a series of reports starting with "A Nation at Risk". That report lauded the U.S. math and science education at the tertiary level but pointed out deep deficiencies at the school level. This gave rise to new funds being made available for elementary and secondary STEM education.

COMAP hoped to build on our success at the college level and soon received a number of grants to produce modular material in applications and modeling at the secondary level.

Samples of Secondary School Module Projects

This work helped provide a philosophical foundation for the Standards produced by the National Council for the Teaching of Mathematics (NCTM) in 1989. That document gave rise to many curriculum experiments at the K-12 that have had a profound effect on the introduction of real world problem solving at all academic levels. One of the high school programs was designed by COMAP, called Mathematics: Modeling Our World.

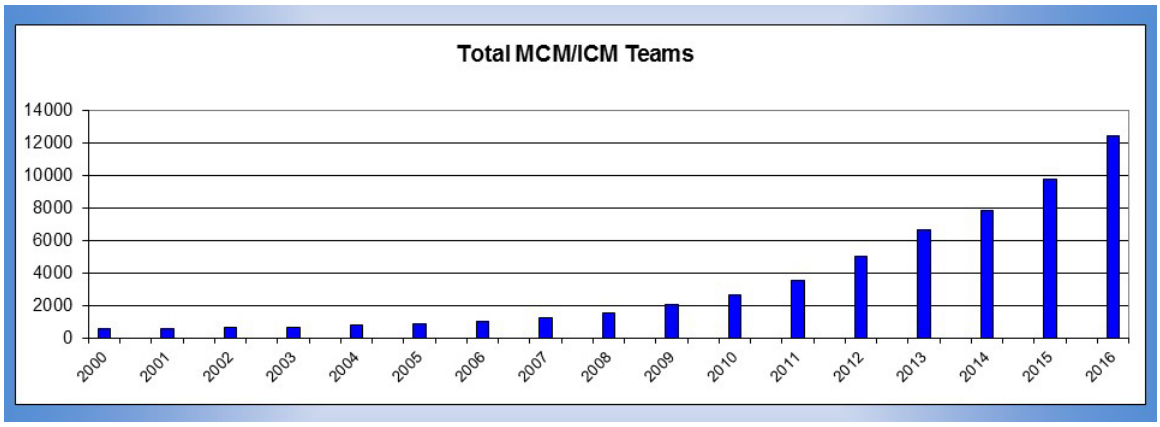
Mathematics: Modeling Our World - Com plate High School Curriculum

I want to be careful here not to oversell. Much of what is taught today and how it is taught is very traditional. But there is no doubt that with technological change and a grudging recognition of the importance of taking a more interdisciplinary approach to math and science curricula things are changing – even for math majors and gifted students.

One last trip back in time: In 1984 COMAP received a small 3 year grant from the Department of Education to found a university modeling contest – the Mathematical Contest in Modeling. In its first full year of operation MCM had 90 teams from 70 U.S. colleges participate.

Flyers for the Undergraduate Modeling Contest

In 1999 we created a sister contest called ICM with problems of a more interdisciplinary nature, requiring working knowledge of fields outside of mathematics. Up to this point participants were mostly from U.S. universities and the contests experienced modest growth. In 2005 that all changed when teams from China began to enter the contest in serious numbers.



Growth Chart for the MCM

In 2016 there were 12,734 teams registered, 12,200 from China. When we wrote the original grant for the contest we said explicitly that the purpose of the contest was not to reward bright students. Rather, the raison d'etre was to promote the teaching and learning of mathematical modeling. And there is no question but that this strategy has been successful. These competitions and various clones have been directly responsible for the addition of modeling courses into university math programs in many countries – notably in China and the U.S.

But interestingly, the school curricula have been more resistant to change. In part this is due to a phenomena that Is common across many countries. Namely, at he university level institutions and faculty have a great deal of flexibility in what we teach and when. Often professional mathematics societies can take a leadership role. And while these organizations may be conservative and slow to change they are in no way as political as the school leadership. Because in almost all cases the school program is set by a Ministry of Education which is part of the federal bureaucracy. Moreover, in many countries there is an end of school test which is very-very high-stakes. And the curriculum is tied very very tightly to the content of that test.

So, how do we influence that system? One way is to continue to strengthen what we have done. As the MCM/ICM receives greater prestige at the university level, secondary schools become more interested (especially the elite schools). The next obvious move is to institute high school contests. In the late 90s we began the HiMCM contest which is a secondary school version of MCM. We

have made slightly different rules to accommodate high school scheduling, but the problems remain essentially of the same open ended nature and the contest like MCM is a true team experience with significant time allotted to work on the problem. To note the contrast, HiMCM has about 950 teams, over half of which are now from China.

But this is not enough. To influence the political establishment at the school level we need more. As a consequence, in 2015 we established the International Mathematical Modeling Challenge (IMMC).

Logo of the International Mathematical Modeling Challenge

This challenge is more of in the Olympiad mode. In particular, each participating country is allowed to enter up to two teams. Each team consists of four students. The Organizing Committee of IMMC which I chaired of which Vladimir Dubrovsky is a member does not tell the countries how to choose their representative teams. The problem selection and grading is done by a separate expert panel. Because again scheduling is an issue, teams are permitted 5 consecutive days to work on the problem – but those five days can start anywhere in the contest period which runs from mid-March to mid-May. Grading takes place in early June and the Outstanding teams are invited to an awards ceremony. Last year this ceremony was held in conjunction with ICME in Hamburg where the Outstanding teams presented their papers to an international audience.

Organizing Committee and
Expert Panel

In the first year of operation we had 17 teams from 10 countries. In last year's contest we had 40 teams from 23 countries.

Indications are that in 2017 these numbers will rise significantly.

2016 Participating Countries

To give you an idea about the nature of the IMMC problems, the 2015 problem as on planning a movie production and the 2016 problem on insuring a track meet for record setting performance bonuses.

Posters for the 2015 and 2016 Contests

So, IMMC represents an attempt through the creation of an international prestigious challenge to influence school programs. But how? Imagine that teachers and even some administrators get the message that modeling is important. What do they do, given the fact that they likely do not have a clear idea of what modeling is and is not. To address precisely that issue we created GAIMME – Guidelines for Assessment and Instruction in Mathematical Modeling Education.

GAIMME Cover

Quoting from the GAIMME Preface:

"A major reason for the creation of GAIMME was the fact that, despite the usefulness and value in demonstrating how mathematics can help analyze and guide decision making for real world messy problems, many people have limited experience with math modeling. We wanted to paint a clearer picture of mathematical modeling (what it is and what it isn't) as a process and how the teaching of that process can mature as students move through the grade bands, independent of the mathematical knowledge they may bring to bear."

Also through the good offices of Teachers College Columbia we have produced three Modeling Handbooks. The purpose of these documents is to give teachers at the elementary and secondary levels a set of modeling experiences and problems that they can use with their students. And these materials should be effective in teacher training programs, both pre- and in-service. Of course, this is only a beginning. As I mentioned at the start of this talk – cultural change is difficult and cannot occur overnight. We have begun by increasing the understanding of the importance of mathematical modeling. We have shown by example how mathematical modeling can be introduced at ALL educational levels. And now we need to convince teachers that this paradigm shift is not only desirable but doable and give them our complete support.

Contact Information

Using Technology in Mathematics Education

Irina Ovsyannikova
Moscow State Pedagogical University

At the present time, using technology in the classroom has become such an essential part of daily learning activities that it is no longer a choice – it's required. Technology helps teachers prepare their students for the real world, which has become increasingly more technology-dependent; it is now essential for students to be tech-savvy.

Over the course of this year, I've taken part in at least 15 educational events in different regions of Russia which were dedicated to using Technology in Education. The purpose of my work is summarizing experiences, creating and conducting courses of professional development for educators, and making the use of technology efficient and exciting for teachers and students. The first thing that I found out during my trips across Russia was that our country is really very big. My journey during this year covered almost 30,000 miles. So, if we talk about using technology in mathematics education, we need to realize that the situation in Moscow differs from other regions in many aspects: infrastructure, teachers' awareness, and more.

The second thing to mention is that all teachers, throughout our country, struggle with the same issue: it's really difficult for them to integrate new technology into curricula. Although many schools are equipped with computers and other technology, a surprising number of teachers are unable to use that technology effectively. The outcomes that have been achieved through investment in technology are comparatively small (Lim et al., 2013). In the majority of Russian schools, for example, teachers are still using interactive whiteboards as the screen for a projector. Meantime, SMART Boards were invented in 1991 and hardly can be called the latest innovation. From my point of view, the major problem is that teachers work in isolation; they aren't accustomed to using technology for communication, the exchange of experiences and professional development.

Despite all these facts, during my trips I found some impressive examples of using technology in education and I'd like to share them below. And, of course, I will also pay some attention to my own experience.

Benefits of using technology in a classroom

Technology is very beneficial for visualization. Today, the Internet provides a large variety of different resources that can make some fundamental concepts clear and demonstrate the application of mathematics to real-life situations.

One of the good examples of such resources is dynamic mathematics software like *GeoGebra*. Firstly, *GeoGebra* can be used to visualize mathematical concepts as well as to create instructional materials. Secondly, *GeoGebra* has a potential to foster active, student-centered learning by allowing for mathematical experiments and interactive explorations, as well as inquiry-based learning. Additionally, students can gather positive experience with mathematical experiments, which increases their motivation to deal with mathematical content (Durmus and Karakirik, 2006).

GeoGebra isn't especially popular in Russia yet, but teachers who have used it once highly appreciate its academic value. *GeoGebra* gives a lot of opportunities for discovering ideas; since it provides free and cross-platform software, we can be sure that our students have access to it at any time.

Using technological advances, teachers can present lessons in ways best suited to the cognitive styles of their students. For example, the use of video, audio, and text can mutually reinforce concepts and enable students to engage with the same ideas in multiple ways.

The second reason (in addition to its contributions to visualization) for using technology in mathematical classroom is its motivational power. Technology can keep students engaged and motivated in learning, even when solving routine tasks, or memorizing something (like the multiplication tables).

Gamification, defined as the use of game mechanics, dynamics, and frameworks to promote desired behaviors, has found its way into many domains, such as marketing, politics, and health and fitness, with analysts predicting that it will become a multi-billion dollar industry (MacMillan, 2011).

Games have remarkable motivational power; they utilize a number of mechanisms to encourage people to engage with them, often without any reward, just for the joy of playing and the possibility of winning. With the continuing advances in the technological world, the implementation of game-based tasks in curricula became easy, fast, and flexible. Even assessment procedures turn to fun activities with cloud-based solutions like *SMART Lab Monster Quiz* or tools like *Kahoot!* or *Quizizz*.

The first time I tried to implement gamification within my educational process occurred in 2012. I had a class of 8 students, all with failing grades in Math after graduating from elementary school. The headmaster decided to organize a remedial summer course to improve their academic achievements. Suspecting that students would not really love the idea of attending, I designed the tournament named "33 cows." In the beginning of the tournament, every student received their own cow (made of paper), and the goal was to produce more liters of milk during my classes by solving tasks; later they could exchange it for benefit tokens which they could exchange during class. For example, we had a token called "extra time," and it meant that if they need an extra minute during a test, they could give me this token and continue to work on the test for that amount of time. Only a limited number of tokens could be earned per game, and it was not an easy task to earn them. The most expensive token was "immunity," which was an exemption from being asked one in-class question when called upon. One interesting fact was that all students who had enough milk to buy immunity, bought it, but never used it. There was no single case during the course when they used it! Certainly, they may have been lucky and I didn't call on them when they were not prepared, but I hope that the cause was that something had changed in our classroom. Academic results improved, students became more interested in math, they understood that there was a place for fun and success during math classes, they weren't afraid to answer, and they felt smarter. Even this low-leveled gamification (points and rewards) gave great results.

It was my first experience of using game design elements in educational contexts, and my students and I were really encouraged by it. I continued to investigate this approach: last year the course of game-design was developed and we've been successfully working with teachers on it; we have already created few fantastic 'game-based-learning teachers' kits' containing a lesson plan, presentation, and materials for students. During our classes, we used SMART Notebook software, because it changes rapidly according to educational trends and always provides teachers with essential tools for creating lessons. It also has user-friendly interface and opportunity to create engaging activities in less than 5 minutes.

This year we've started a new project with secondary school students: they are developing their own educational games. We hope that students, as they have more experience with educational games, can give us some new ideas about how games can help them learn.

Technology also give us some extra benefits when working with gifted students. The first of them is content differentiation. Gifted students often need to be taught entirely differently, receiving more advanced materials than their peers. Through the use of technology, we can provide them with a wealth of advanced content which is available to tap into at any time, from anywhere. You can use

free online college courses, the free video library of KhanAcademy.org, or create your own video materials, thus making individual work with every person in your classroom possible. In this way, the organization of a lesson where different students receive different instruction becomes simpler, although, of course, we still have to guide student work because even gifted students need our help and support.

The second benefit for gifted students is the opportunity to collaborate widely, communicate, and innovate. Now, using cloud-based solutions, we can organize inquiry-based project work with natural collaboration between students. For example, we often use Google apps as a tool for working together from separate locations simultaneously. We also use SMART amp collaboration software which operates on Google classroom; SMART amp brings the whole class, groups of students, or individuals together in a shared space to work on projects, add multimedia content, and instant message. This is all done using student devices and while tracking contributions from each student, so educators have insight on who is contributing what. And that is important, because it may happen that only one student was really working on a project, while others just helped to present the results.

Working on STEAM projects (Science, Math, Engineering, Art, and Math) became more flexible and effective by using technology. The main difference from STEM projects (as may be guessed from the title) is adding Art at the center of STEM, to make the combination more powerful (Robelen, 2011). In STEAM projects students who are not successful in science but talented in music, art, sculpture, architecture, or design can work together with students who are successful in analytical, logical, mathematical tasks. These kinds of projects help us to organize activities for students so that students who are talented, or just interested, in subjects other than math can productively remain in classes with gifted math students. We also help gifted students communicate socially with their peers, because school is also a kind of social institute and we need to help students to find comfortable positions in that society by improving their communication.

There are some examples of STEAM projects which were realized during mathematical classes. We used SMART amp and Google drive to provide a field for exploration, so that students could work together and share their results with class easily, and even upload it to YouTube and receive feedback from the international community. Already, many students have participated in the publishing process by writing in these channels, viewing themselves as content creators who share their opinions, stories, music, and videos—all with the aim of expressing their creativity and exchanging information. It doesn't actually matter what grade you teach, you always can find an idea for research problems and divide students into teams so that there will be one person strong in math, another one strong in music, and so on. For instance, we discussed the golden

ratio and how it is connected with the beauty: we analyzed some photos of people known to be beautiful, found patterns and then, in a computer lab, transformed the photo to fit exactly in the golden ratio. The results demonstrated that beauty does not necessarily perfectly correspond to the mathematical formula! In 7-8th grades we worked on making kinetic sculpture; it was really challenging task, but nevertheless collaboration between students with different styles of studying gave us great results. In high school, we were engaged in a project connected to abstract art: students first studied the artistic ideas of Kandinsky, Picasso, and Malevich, and then created some kind of painting in the style of their selected artist in *GeoGebra*.

Conclusions

I am an advocate of using technology in education; yet I believe that even if good teachers find themselves and their students on an uninhabited island, they will be able to teach fundamental concepts efficiently (say, by making drawings on the sand). Using technology, however, is the inevitable future of education; we cannot teach in a same way as we taught before. As the famous American philosopher John Dewey said, "If we teach today as we taught yesterday, then we rob our children of tomorrow" (Dewey, 1944). I'm happy and proud that we help teachers to explore new technology, integrate it into the educational process, try out new approaches, and help them be dedicated to their work. Teachers ought to change their teaching styles from time to time to keep themselves excited about their work. Having an excited and enthusiastic teacher is probably not a sufficient condition for having excited and enthusiastic students, but I am absolutely sure that it is a necessary condition.

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Mathematical Education in Russia: Modern Approaches to Math Teacher Preparation

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The current situation in the field of education in Russia is quite dramatic. Focusing primarily on results, rather than on the content of education, has become the norm in educational standards both at high schools and universities. The difficulties with the implementation of new ideas are connected both with a new and unusual formulation of the problem and the rejection of reforms on the part of the educational community. This article focuses on the approach to the training of teachers of mathematics proposed by the team of the Faculty of Mathematics of Moscow State Pedagogical University.

General education

From 2009-2012 Russia adopted new Federal State Educational Standards. There are three standards: one for primary education, which covers grades 1-4; one for basic education, which covers grades 5-9, and one for secondary education, which covers grades 10-11. The Ministry of Education and Science of the Russian Federation (hereinafter the Ministry of Education of the Russian Federation) has provided a gradual transition procedure for education based on the FSES (Federal State Educational Standard). The compulsory transition of all schools to new educational programs based on the requirements given by FSES for basic general education has been implemented since September 1, 2015. In primary education this transition occurred even earlier, in 2011.The transition to education based on the FSES in grades 10-11 is planned for 2020. The latest generation of the FSES includes requirements, perhaps for the first time in Russia, not only for the content, but for the results of education. All educational outcomes are reflected in the FSES in very general terms. They are more fully explained in suggested educational programs for specific levels of general education. It is important to note that the term “program” is used here not in relation to any particular school discipline, but to mean a document that

describes what's happening at a school during particular stages of general education. In the suggested programs there are requirements (also only suggested, not mandatory) for the educational content of disciplines and expected results.

It should also be noted that the standards allow for the simultaneous existence of several programs based on them, but in reality, today there is only one sample program for every level of general education, developed by request of the Ministry of Education of the Russian Federation. We need to recognize that an important step was full public discussion of these sample programs on the Internet before their approval. The standards allow each school to make its own educational program, but schools that have state accreditation are required to take into account the requirements of the officially recommended (suggested) program. In reality, it means that the vast majority of schools follow (or try to follow) the logic that guides the sample program. We must say that the existing division of educational standards and educational programs in the regulatory framework often gives rise to speculation about the meaninglessness of this generation of standards because of their lack of educational content, while from the words above it should already be clear that this is not so. Besides, the FSES contain a description of the requirements for realizing the plans for education, including staffing, material and technical equipment of schools, and the informational and communicational environment.

The results of schooling include not only reaching a specific level of knowledge of the subject matter but also so-called "individual development," that is, the development of moral and civic qualities of the students, and in so-called "meta-disciplinary development," as a general characteristic of the development of learning skills. Of course, the novelty of the approach, combined with the challenges of evaluating the results, creates difficulties. It was much easier to measure the results in a subject matter in the past, now the State Final Examination procedure in grades 9 and 11 provides some formal description of what is required. Moreover, the reality today is that the tasks set by the State Final Examination largely determine the mathematical content that is taught in schools. In particular, there is an official systematic list of requirements for the level of graduates' knowledge and testable elements of the content and a specification — a document describing how this list's requirements must be reflected in the exam variants (FIEM, 2016). We must say, however, that this list was made on the basis of the previous (2004) standard's requirements. But when pupils whose course of study was set by the FSES from the first grade of the Elementary school in 2011 will have to pass the Basic State Exam (BSE, examination given in 9th grade) or the Unified State Exam (USE is offered in 11th grade), it is expected that there will be some changes in the requirements for the content of examination procedures.

The Final State Examination in grade 9, called the Basic State Exam (BSE) must be entirely based on an open bank of tasks, posted on the Internet by the Ministry and well-known in advance. Of course, the real exam's conditions and tasks may differ from the open bank ones in minor details and numbers, but in general the exam's content is known. After grade 9 some number of pupils continue to pursue education in the sphere of work specialties; they go to vocational schools. However, largely due to the fact that the Russian army is filled mostly by draft (and is not a very popular career path), many young men in grade 9 plan to continue their education at universities. By doing this, they hope for a deferment from the draft at least until the end of their studies.

The Unified State Examination in grade 11 (USE) in mathematics has, since 2015, been conducted separately on two levels: basic and advanced. Passing mathematics (as well as an examination in Russian) is mandatory. The tasks for the basic level of the Unified State Exam, like the Basic State Exam, are selected from an open bank of tasks. The advanced level of the Unified State Exam is only partially based on the open bank of tasks. The examination's variant of the USE also has tasks of increased complexity; their exact conditions are unknown in advance. The last task (there were nineteen tasks on the advanced exam in 2016) is actually appropriate for the Mathematical Olympiads, and it can't be solved by a pupil who is not deeply passionate about mathematics.

At the same time each year, several months before the exam some sample examination variants (also known as demo versions) for preparation are published. They are based on the aforementioned list of requirements and the specification's requirements. There are also solutions to these open tasks and rules of their evaluation. This demo version is available with a clear indication that tasks included in it don't cover all the content that will be tested by the exam; the full list of items that can be monitored is listed separately. However, there was an unexpected case that caused a number of complaints — in 2016 it turned out that the real conditions of the advanced-level tasks were different from the demo version in more major ways. The required solutions made use not only of memorizing formulas and algorithms for concrete tasks, but also a nuanced understanding of the subject. What is seen as strange by many in the mathematical community is that the statistical data with the country-wide results, as well as variants, for the exam don't become available to the public even after the end of the annual cycle (in September of the year preceding the next exam) of the Final State Examination. The advanced level of the mathematical exam is important during the university application process. The results of the basic level of the USE are not accepted in colleges where it's necessary to know and use mathematics for further studies. At the same time, virtually all graduates of grade 11 want to enter university, not least because of the army draft. University students aren't required to join the army, unlike almost all other young men who have no health restrictions.

A typical Russian pupil (not a winner of Olympiads) must show the result of three Unified State Exams before matriculating at a regular Russian university (not at the Moscow State University or the St. Petersburg State University, which have their own additional entrance examinations), one of which is always the Russian language USE; the two others depend on the student's chosen course of study and their educational profile. For example, when matriculating at the Moscow State Pedagogical University (MSPU), a future teachers of mathematics, in addition to the result of their Russian language Unified State Exam, must give the results of their Social Studies' USE (as must any future teacher) and the results of their USE in Mathematics, at the advanced level. But applicants who have decided to become professional mathematicians, if matriculating at the Moscow State Pedagogical University, must present the results of their Russian USE, the results of their USE in Mathematics, at the advanced level, and the results of their profile USE in Physics.

Entering the university on so-called "free ride track" (an education paid for with money allocated by the State) happens in the following ways: in each university the number of places in each discipline of education that will be paid for by the State is known in advance. Within the educational areas there is a possibility to distribute places at the university level (for example, to admit more students who wish to become teachers of physics than teachers of history, or vice versa). The decision on the distribution of places is announced in advance. Applicants are allocated to the desired majors (e.g., a teacher of history) according to their USE results. It is important that the number of students who pursue the same major, and whose score on the Unified State Exams were the best, doesn't exceed the predetermined limit.

Each exam (except the USE in Mathematics, basic level) is scored out of 100 points. In 2016 it was possible to be admitted to the Faculty of Mathematics of the Moscow State University of Education in order to receive a degree as a teacher of Mathematics after five years, with 215 points in total for three exams. It's curious that this same year the Faculty of Mathematics of the Moscow State University of Education admitted an applicant — a would-be mathematician (not a teacher), who received 100 points (the maximum) on the USE in mathematics, advanced level.

It is worth saying here that "individual development" and "meta-disciplinary development," both mentioned earlier, are not evaluated at the examinations and actually, it looks like nobody knows how to assess them so far. Unfortunately, the universities have quite a lot of complaints about the applicants and admitted students. Newcomers display a lack of ability to study, to organize their time, and to plan the trajectory of their lives. This has led to an understanding of the necessity of special disciplines to provide a kind of psychological support in the curriculum of many universities. Further, some courses were added to the curriculum to fill the gaps in the standard high school mathematics curricula,

e.g., starting in 2016 the Faculty of Mathematics of the MPSU has added an additional course of study (one day that is 8 hours every week) during the very first semester.

Higher pedagogical education

There are two FSES for higher pedagogical education at the bachelor level (the most recent version adopted at the beginning of 2016) for either a 4 or 5 year course of study: the FSES for master's level (December 2014) and, separately, the FSES for postgraduate studies (August 2014). The difference in length of study is due to the fact that teachers may train to teach one (4 years) or two (5 years) subjects. According to recent surveys (September 2016, see below) in pedagogical universities, there has been an increase in the popularity of the combination of Mathematics and Computer Science. It can be explained by the fact that in school educational standards these subjects are combined into a "single unit," as well as by the fact that in a school in a rural area both mathematics and computer science, most likely, will be taught by the same teacher. Much less common, but also fairly common is the combination of Mathematics and Physics. The Moscow State University of Education prepares students to teach a combination of Mathematics and Economics.

In the FSES of higher education (not only pedagogical studies) the concept of a competence is introduced as the expected result of achieving all of an educational program — a set of disciplines, practices and a final certification procedure. Competences can be of different types (at the moment their typology is changing). The FSES of the bachelor's level of pedagogical education (to be implemented in 2017) requires reaching both universal competences and general professional competences. There are eight universal competences in total; one of them says, for example, that "graduates with a bachelor's degree must be able to manage their time, to build and realize their path of self-development on the base of principles of lifelong learning." Seven general professional competencies are described, an example of them is the following: "graduates with a bachelor's degree are able to monitor and evaluate the formation of pupils' educational results, to identify and correct learning difficulties." In addition, specific professional competences are defined. The new standard lacks a clear definition of them and each university has the right to establish its own wordings, taking into account several aspects, one of which is that the main role must be played by another regulatory document — the Professional Teacher's Standard (The Ministry of Labor, 2013).

This document appeared in 2013 and was approved by the Ministry of Labor and Social Protection of the Russian Federation; it contains the description of the so-called 'labor actions' of a schoolteacher, e.g. the actions that teachers perform in the course of their work. The structure of the Professional Teacher's Standard is such that it describes not only the labor actions of any teacher, but, separately,

the labor actions of a teacher of mathematics. The purpose of this document (as opposed to the FSES, which is approved by the Ministry of Education of the Russian Federation) is to reflect the employer's (and the state's) point of view on the quality of preparation done by any working teacher, not only today's graduates. It is currently being tested in a number of Russian regions, and in 2017 the new teacher's certification based on the Professional Standard is supposed to be launched all over the country. In their description of 'professional competence' the Ministry of Education of the Russian Federation proposes the following structure: to describe the student's knowledge and skills before achieving the competence, then to describe the knowledge, skills and experience that student will be able to demonstrate after studying, as a result of achievement of the competence. Knowledge, skills and experience can be achieved on several levels: the so-called minimum acceptable, basic, and advanced. To elaborate why students must actually achieve this level of experience we must link it up with some of the labor actions provided in the Professional Teacher's Standard. Finally, we the one should be able to specify in the description of the competence where specifically the knowledge, skills, and actions (experience) mentioned appear in the curriculum and where specifically they appear in the framework of studying some disciplines and practices. But the most important aspect is how their achievement must be evaluated.

The suggested educational programs of higher education are also not finished: many of them are still in development, including pedagogical educational programs.

At the moment the Moscow State University of Education, in collaboration with the Ministry of Education of the Russian Federation, is developing an educational program for teachers of mathematics on the bachelor's level according to the competency-based approach. In summer of 2016 the Federal Methodical Union in Education and Pedagogical Sciences (an association of pedagogical universities of the country) by order of the Ministry of Education of the Russian Federation has developed a list of professional competences (and relevant knowledge, skills and actions), that define the requirements for the preparation of teachers in basic and secondary education. The list doesn't consider the school subject matter (mathematics, science, or social studies) of teacher preparation; the requirements are for the preparation of all teachers of primary and secondary school.

So what do we need to include in preparation programs for teachers of mathematics so that the education they receive is relevant? For what labor actions typical only for teachers of Mathematics do we need to prepare them? And how can we do it? Another serious issue is the debate about the ratio of higher mathematics and school mathematics (so-called elementary mathematics) in programs of teacher preparation.

A separate, important issue is the content of the Teacher License Examination — the procedure that confers the right to start working as a teacher. The traditional format of graduation includes two phases: the state examination (which usually checks only the theoretical knowledge of students) and the defense of a thesis.

Public opinion

On behalf of the Russian Federation's Ministry of Education, a team from the Moscow State Pedagogical University has held a series of interviews. The results are quite revealing. Here are only some of the many observations and conclusions those interviews provide.

We have interviewed more than 40 universities, offering programs on engineering and natural sciences as key recipient of mathematically well-prepared applicants. Questions were mainly concerned with the quality of preparation of high school graduates coming to study.

According to the respondents, in most cases (over 90%), university professors have to repeat parts of the standard high school program with their first year students. Many students can't construct graphs of basic elementary functions, don't know their properties, don't know the basics of probability theory, don't know geometry, trigonometry, logarithms, and more. And these deficiencies haven't become smaller over the past three years. To fill these gaps, a number of universities have organized remedial courses in mathematics for their first year students (2-4 hours per week in the first semester).

In terms of content, the most important factor for the continuation of study in engineering (and natural science) at university is, according to the respondents, learning the elements of mathematical analysis and functions, while theory of Probability, Mathematical Statistics, and History of Mathematics were considered less important.

The majority of respondents (80%) recognized that learning mathematics is important for the formation of analytical and systemic thinking. As a subject in school promoting the development of skills in experimental researches, communication, and teamwork, math was perceived as less important.

The following problems with the organization of mathematical education at school were noted: "coaching" for passing the Unified State Exam and the lack of proofs-based geometry (the first-year students don't show the ability to prove).

Finally, according to the respondents, graduates of engineering and natural science faculties are not being prepared for a possible future work at a school.

Another survey of over 40 teachers of mathematics at schools across the country was conducted. We must say that here the sample wasn't quite so random; the survey was conducted mostly among teachers at "good" schools according to educational rankings, and among participants in professional contests. This was

a conscious choice of the survey's organizers, based on a hope for more constructive feedback. Respondent teachers supported the idea of changes in the educational standard of pupils' mathematical preparation, but about half of them (45%) believe that the content of mathematical education must remain unchanged, and changes must relate to technology, tools, and methods of preparation; about a third (34%) of respondents admit that the possibility of changes in the content of mathematical education may be necessary according to the current context. According to a quarter (27%) of respondents, the discipline's requirements provided in the suggested program of general education (grades 5-9) are fully compliant with the standard; another quarter (25%), in contrast, believes that the requirements of this program need to be clarified, and a fifth of respondents (20%) note minor inconsistencies. According to a quarter (26%) of respondents, the subject demands of the suggested program of general secondary education (grades 10-11) are in compliance with requirements of the standard, but a quarter (24%), believe that the requirements in this program need to be clarified, and a quarter (26%) note minor inconsistencies.

The majority of respondents (84%) use some educational techniques that are different from the traditional approach: such as differentiated education, organization of problem and research situations in the educational process, preparation of students for independent task solving and fact-finding, flipped lessons, and others. However, about a half of respondents (46%) rarely use ICT in practice, preferring to teach by traditional means; one in seven (15%) use ICT for every lesson, and one in four (27%) use ICT every week. Supporting materials both for the BSE and the USE (lists and specifications) are used by about a third of respondents, while the demo version and open bank tasks are used by almost everybody (92%). About a half of respondents (49%) assign the solving of demo tasks and tasks from open bank of the BSE/the USE in the final year, about one hour per week, another third (31%) offer a one- or two-hour training session for each topic.

Also, a survey was conducted which involved more than 160 professors of mathematical departments, of both pedagogical and some classical universities participating in preparing future teachers of mathematics. The results are below.

As was noted above, most universities, as presented by respondents, are now offering a double major bachelor's degree for the preparation of teachers of mathematics with such second directions as Informatics, i.e. Computer Science, Physics, and Technology.

The intermediate exam session is usually held twice a year at the end of the semester, but some universities use a modular course structure which means that evaluation may be conducted more frequently — for example, twice a semester, every two months.

Both teaching methods and general cultural preparation of future teachers of mathematics take about 25% of time; while subject specifics take about 50 percent. Courses relevant to teaching methods begin in different universities at various time — mostly in the 2nd or 3rd year of studies — and end in the 4th or 5th year (during the 5 year period of study). The majority of responses refer to students involved in a continuous (teaching) practice in schools at the same time as they study methods of teaching, pedagogy, and psychology. Teaching practice lasting about 4-6 weeks, sometimes 8-10 weeks, and is held usually during the 3rd or on the 4th year, but may start by the 2nd year and also can take place in the 5th year (if during the five year course of study).

Many universities use rating systems (93%), business games (32%), or case-studies (36%) as means of controlling the formation of competences.

The procedure of the Final State Examination at universities in many cases (73%) consists of a state examination and defense of a thesis, and sometimes only of the defense of a thesis (27%).

The study of the BSE/the USE in a university is considered a part of teaching methods course (52%) or as an elective course (56%). In most cases (60%) future teachers are advised to teach an elective course at school for 1 hour per week throughout the year on topics of the BSE/the USE. Respondents believe that the current system of pedagogical education prepares teachers well to implement the following labor actions at the beginning of their teaching activities (the wordings are in accordance with the Professional Teachers' Standard): the formation of capacity for logical reasoning and communication, for using this capacity (73%); the formation of specific knowledge and skills in the field of mathematics and computer science (93%).

At the same time, it is noted that universities are only marginally preparing teachers of mathematics to perform such labor actions as: the formation of the ability to comprehend the basics of mathematical models of a real object or processes, readiness for application of modeling the construction of objects and processes, determining or predicting their properties (so say near 44% of respondents); the formation of the mental model of mathematical situations (including a spatial image) (a bit more than 50%); the formation of a material and informational educational environment that is conducive to the development of the mathematical abilities of each child and implementing the principles of modern pedagogy (58%); the formation of the intellectual ability to overcome intellectual difficulties, to solve fundamentally new tasks, and to show respect for intellectual work and its results (49%); the identification of doubtful and improbable data together with pupils (53%); cooperation with other teachers of mathematics and computer science, physics, economics, languages, and other disciplines (56%); the development in pupils of the initiative to use mathematics (52%); the professional use of elements of the informational educational

environment, taking into account the possibility of the application of the new elements of such an environment, which are absent from a particular educational organization (55%); the use of informational resources in work with children, including distance learning resources to help children to learn and to use these resources independently (49%); and assistance in preparing pupils to participate in Mathematical Olympiads, competitions, research projects, intellectual marathons, chess tournaments, and pupils' conferences (47%).

Professors of pedagogical universities rarely use special math software or Internet resources in their work; about half of the respondents do not use them at all, and only 10% use them in each lesson. More than half of the respondents note a positive effect associated with the use of visualization and interactive capabilities (with the help of ICT) on the comprehension of complex issues during the study of mathematics. In many pedagogical universities some additional disciplines were introduced to the preparative program in order to eliminate some gaps in school mathematical education, which often lasted 36-72 hours (2-4 hours per week during the first semester). Due to poor high school mathematical preparation, the majority of respondents have to change (reduce) the amount of university course material. The inability of school graduates to reason logically, to prove, to understand abstractions, and to see the possibility of their use behind abstract objects in a particular situation are all noted, as are a poor understanding of the concept of function, methods of transformation of algebraic expressions, and solving equations and inequalities.

There is a significant difference in the amount of elementary mathematics teaching during the preparation of future teachers of mathematics for primary and secondary schools in different universities — the spread is from 2 to 30 credit units (the total for a year of study at a university is 60 credit units).

According to the respondents, the drawbacks in the organization of teaching method preparation include: an insufficient volume of teaching practice, a lack of grounding in elementary mathematics in the preparing program, and an unwillingness of teachers to properly monitor work with students during teaching practice.

We note that in all surveys both teachers and university professors traditionally complain about the lack of time for teaching students. Besides, according to many respondents in all conducted surveys, the content of mathematical preparation at school in recent years was greatly simplified, disconnected from real life, and overloaded with routine tasks.

The interpretation of the survey results is, of course, very important. Should the claim, for example, that learning probability at school is unimportant to the engineering and natural sciences' university programs bring us to the conclusion that we need to exclude this section from every school's program? Or how should we treat the fact that a university's community doesn't consider school

math a venue for experimental research, communication, and teamwork? Should we agree with these opinions or should we try to oppose them with meaningful counterexamples? We do not go into the further discussion of these issues here, limiting ourselves to reporting the results only.

Competences of a teacher of mathematics

The international and Russian experience, the Russian regulatory framework in the field of education, and the results of surveys conducted in Moscow State University of Education all highlighted some specific features and areas of knowledge of a teacher of mathematics, and in the future we plan to build a program in math and teaching methods aiming to develop them. A theory of teacher preparation also assumes that school teachers will do what they have been taught at university. Thus, there is an obvious reason to acquaint students with the best practices of work with pupils and to encourage students to participating in such work, even at the university level.

Problem solving should be a common element of all the activities, supporting the tradition of the Russian mathematical school and, at the same time, a step toward an approach to education, declared in the FSES of general education at all levels, that sees problem solving as the primary way of understanding mathematical content.

The team of the Faculty of Mathematics of the Moscow State Pedagogical University attempts to form and to evaluate the following competences of a teacher of mathematics:

- **An ability to teach mathematical modeling and computer experiments.**

The results shown by Russian pupils in PISA in mathematics are lower than the results they showed in TIMSS at the same age. A well-known cause of this gap is the practical orientation of PISA tasks, and the need to build mathematical models of real-life situations. It turns out that Russian pupils are not able to do it well. Besides, in many cases, school mathematics can't be taught dogmatically, but must be taught as an experimental discipline, contributing to the development of scientific intuition, curiosity, and initiative among learners.

During the teaching of mathematical modeling in school (and the teaching of future teachers) it is advisable to adopt new approaches, including those based on so-called 'research tasks,' modeling by using a computer.

- **An ability to teach mathematical reasoning (a proof).**

The Russian (Soviet) education from the 18th century onward has a strong tradition of teaching geometry. Geometry enables the connection of:

- visualizing the representation of mathematical objects;

- deductive constructions (evidences), the usage of heuristic considerations, speculation, hypothesis testing supported by visual images during the construction of evidences;
- written (and in certain extents, oral) presentation of mathematical constructions and proofs in a wide, mathematically meaningful field;
- algebraic modeling of visual configurations;
- the most important aesthetic, historical, and cultural aspects.

This tradition of teaching geometry helps Russia to maintain a relatively high place, for example, in TIMSS. Geometry is a kind of a tool for forming the capacity for logical thinking in pupils. Thus its value extends beyond just the study of mathematics.

Modern ICT tools offer significant potential for mathematical experimentation in geometric material as a basis for hypotheses, which then are convincingly substantiated.

Another section of the content in a suggested school program which helps students to develop the skills of mathematical reasoning is, of course, mathematical logic. Logic is also the basis of the content of the school discipline of Informatics (Computer Science). In this section, deep mathematical statements may be presented in simple terms, which often allows for visual interpretation (Semenov, 2002), .

- **An ability to teach the manipulation of formal mathematical structures (formulas, algorithms and etc.).**

This is probably the most difficult responsibility of a teacher of mathematics. The intrinsic value of mathematics as a discipline—that it is intended to nourish the development of logical thinking in general—is recognized by everyone. This recognition can play (and in fact often plays) a cruel joke upon us: the mathematical content of the discipline becomes only a secondary goal. But is there really a deep meaning in the solution of equations, a need to transform logarithms and trigonometric expressions? How deep is mathematical purpose behind these exercises? Do we need to repeat them endlessly, if many of the details of real future tasks will always be entrusted to the computer?

We must say here that notions of mathematical lessons as something that could be particularly beneficial (in comparison with other disciplines) for work on the development of human thinking in general are without sufficient scientific basis and are, rather, simply beliefs, (Star, 2013). Moreover, some researchers who are highly respected in the world of mathematical education say that, in general, what subject material (mathematical or not) we teach does not matter, but it is important how we do it (Schoenfeld, 2014).

Thus, even taking into account the need to prepare school pupils to pass the BSE/the USE and their need to learn all the subject content provided by a suggested program, it's still important to focus the attention of both school and university students, who will be future teachers, on mathematical aspects that have important applications in today's and tomorrow's world.

- **An ability to teach probability and data analysis.**

It's one of the biggest challenges for Russian school education today. We must say that the first decision to teach the foundations of the Probability Theory was made in pre-revolutionary Russia at the Second Congress of Teachers of Mathematics. Subsequent events in our country have led to the appearance of a so-called *command* economy; they also have made massive probabilistic thinking unnecessary for the state ideology. Despite the outstanding works in the field of the Probability Theory by Soviet mathematicians, including Kolmogorov and many of his pupils, a tradition of study in mass education did not emerge. Probability courses came into modern schools about fifteen years ago, thanks largely to the efforts of E. A. Bunimovich and V. A. Bulychev.

Today, tasks on the Theory of Probability are included in the USE, but, nevertheless, we must understand that no section on probability has been included in the average university's course of teaching methods. There was simply no one to develop it. Teachers who are working at schools didn't study probability as a part of school mathematics. As a consequence, they are afraid of teaching it, because it's the most inconvenient section for them. Mathematics educators at universities are the same people today as they were twenty years ago. At the same time, the value of probabilistic thinking for a successful life in the modern world can't be overestimated. The modeling of probabilistic events and analysis of the data using a computer are perfect ways to make the development of this section interesting for all participants.

- **An ability to teach the use of computer tools and ICT.**

Do we see people using blackboards somewhere in the fields of industry or of scientific research? Is it true that they now prefer the tools of computer visualization and modeling? What about students? Do we see a contradiction here?

One of the more prominent Soviet and Russian mathematicians, A.G. Kushnirenko, has recalled in a recent interview his experience in 1993 of teaching school teachers geometry by using a computer at the University of Pennsylvania, USA. In particular, he said that by 1998 the necessity for such a course had disappeared because all teachers began to look for a way to learn how to work with the appropriate software independently. Using programs of computer geometry became standard, so without them it was impossible to get a job.

Unfortunately, the state of affairs in Russian school and pedagogical universities is still far from this. Meanwhile, all the necessary means, including domestic world-class software, exist today.

- **An ability to teach mathematical communication.**

Communication occupies a special place in educational mathematical activities, in particular, the exchange of mathematical information. Today, it's clear that the standard school course of study must include a variety of communication opportunities: work in small groups, discussions in large groups, and presentations of individual and group projects. In all of these activities, a central theme is the pupils' development of common communication and organizational abilities, along with the ability to use mathematical language in communication. Teachers must be able to organize student activity in such a way as to allow them to see the difference between the mathematical language and the language that people use in everyday life, and to appreciate the precise nature of mathematical language.

A special role in students' education must be played by mathematical circles (clubs) — a wonderful tradition of domestic schools. Organization of circles by faculties of mathematics in pedagogical universities seems to be the most natural way of developing the communication skills of future teachers, in addition to direct teaching practice at school. Among the Faculty of Mathematics of the Moscow State University of Education in 2016, the circle operates following the system of N. N. Konstantinov: pupils receive a sheet of paper with tasks, and after solving the task invite a professor (or another student, in our case) to look over their solution. The teacher's task in the emerging dialogue is to make sure that there are no errors in the solution, or to point them out and at the same time leave an opportunity for students to change a solution to the right one. In addition, team mathematical school competitions are held by the faculty.

Assessment of a teacher of mathematics

Of course, after the priority change in teacher education with more attention paid to the result of education, obvious changes should occur in teachers' assessment procedures. In the traditional form, the Final State Examination looks like an inevitable ritual. Students learn answers to previously known questions from the basic sections of higher mathematics and repeat them on the State Examination. Then the defense of their thesis follows. Provided that a student showed the necessary diligence in the process of studying and preparing for their diploma, they merely need to paraphrase a prepared text.

This procedure is very formal. But the main problem is that it's unclear exactly how these ritual actions are related to the future work of a teacher. The first step in changing this model in the Moscow State University of Education will be made this very year — there will be tasks of increased complexity on the State

Exam. Further, it is obvious that many components of the procedure of certification should be reevaluated: including, for example, a student's lessons (recorded on video) during teaching practice.

Perhaps these changes will include the appearance of teaching method problems on the certification exam, such as those, for example, which are offered for current teachers in the annual Creative competition (MCCME, 2016) within the walls of the MPSU.

Conclusion

The content of this article is, to a large extent, a mission statement for the future; much remains to be understood and to be done. The transformation of teacher education in Russia, including the preparation of teachers of mathematics, is mostly in full swing, often difficult, and generally unavoidable. The team of the Faculty of Mathematics of the Moscow State University of Education is developing itself, on the one hand as the inheritor of Vygotsky's tradition, but, on the other hand, as the heir of many outstanding mathematicians, scientists and teachers who cannot possibly all be individually listed. The team makes every effort for the realization of these ideas.

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Implementation of the Conceptual Framework for Russian Mathematical Education

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The conceptual framework for the development of mathematical education in the Russian Federation (in what follows, 'the Framework,' see Ministry (n.d.)) was adopted by decree no. 2506-p of the Government of the Russian Federation on 24 December, 2013. In subsequent years, this conceptual framework established a pattern which was used in quite a few conceptual official documents concerning other areas of education.

The Framework was developed in accordance with an executive order of the President of the Russian Federation (May 2012), in the period of time when Federal State Educational Standards (FSES) were being implemented in Russia at various levels of education. While on the whole the vector of educational development prescribed by the FSES was pointing in the direction of progress, the FSES were formulated in quite a general way. It was particularly important at that time to develop a conceptual framework which would be consistent with the FSES but would provide clearer guidelines.

It is also important that hundreds of school and university teachers and administrative staff took part in the discussion of the Framework. The draft framework was also repeatedly discussed at meetings of the Scientific and Methodological Council on Mathematics at the Russian Ministry of Education and Science.

After the FSES and the Framework were adopted, the next stage of clearing up the concept of education began: a nationwide discussion of model syllabuses for various courses, including mathematics, was organized by the Moscow City Pedagogical University on the order of the Russian Ministry of Education and Science. It involved thousands of participants who took account of the main points of the Framework. One result of this discussion was the adoption of a two-level model mathematics syllabus for secondary schools. This was extremely

important; before that, it was officially assumed that the content of education and the requirements regarding the level of primary school (1-4) graduates' training must be the same in all schools. The standards for all schools were theoretically unified. On the other hand, *de facto* there were schools with in-depth courses on some subjects (for instance, mathematics) for grades 7-9. Adopting two model syllabuses in mathematics (basic and in-depth ones) was an important precedent.

Another important precedent was the adoption of two versions of the Unified State Examination in mathematics, which had been proposed by both school and university mathematics teachers for several years. (It could be even more natural to have two versions of examinations in the Russian language). The existence of a specialized (in-depth) version of the examination is also important for another reason, which we are now going to explain.

It is absolutely clear that the USE is not just an unbiased evaluation of the results of teaching a course, nor (which could be better) of the competencies in this subject that the graduate has acquired as a result of all his education – at school or out of school. Just as powerful as the social importance of the USE (whether positive or negative) is the impact that the USE has on the content of education. Of course, by content we do not mean just the list of topics learned, but also what components of that knowledge are tested in assessments, which can be, for instance, the solution of a geometric problem new to the student with a detailed proof, or their `close-to-textbook' knowledge of the proof of a theorem, or their ability to choose the correct numerical answer from a list suggested in the test.

The fact is that, whether we like it or not, students will be tutored to `pass the exam.' (taught to the test) One of the main drawbacks of the USE, as it was initially designed, consisted of forgetting this obvious detail. Consequences were not slow to arrive. The USE was focused on "algebra with elements of calculus" and contained many tasks where the correct answer was to be selected from a list. Immediately, the mathematics that was actually taught in schools began changing. In particular, the share of geometry taught dropped drastically. Thanks to the efforts, in particular, of this author, this development was halted and geometry was reinstalled. However, the problem was not entirely solved.

It was an element of the USE from the start that at the beginning of each school year a so-called "demonstration version" of the exam was published on the Web. This is very useful to give students in their last year and their teachers an idea of what the actual USE looks like, how complicated the tests are, and so on. However, teachers discovered almost immediately that the actual USE tests were going to be very similar to the "demo-version." And this meant that they should concentrate on solving the problems from the demonstration version and similar problems, which could be found in numerous "USE training books," and should put aside the standard school textbooks and problem books. They turned to what

can be called "cramming." Of course, this did not occur because the designers of problems were "too lazy" to invent different problems or even because they wanted to ensure that tests of the same level were equally hard. The reason was that an actual variety of test problems would result in a serious decline in the USE scores. But the examination is already criticized for overly simple problems and simultaneously, low threshold numbers of solved problems. (Surprisingly, USE is chosen as the target of these criticisms and not secondary school mathematics training as it has developed independently of the USE)

In the specialized (in-depth) USE we can clearly see a departure from these practices. Its tests are reasonably diverse in many respects. Quite naturally, some teachers are critical of this diversity. Their criticisms should be taken into account, not with the result of diluting test diversity, but rather in the direction of reducing the technical complexity of examination tests.

The mathematics USE will be improved further and all the essential modifications will be discussed 4-5 years before their implementation. One of the desired trends, which was tested in the 9th grade, could be an appraisal system which stimulates the study of all the components (in 9th grade these are arithmetic, algebra, geometry, and real mathematics). We shall gradually move away from the demonstration version to the diversity of actual variants of the examination. As concerns other problems and prospects, see (Semenov, 2014).

The USE is just one of the components ensuring the high quality of education. The teacher's qualifications are the key element. Looking at the three main lines of development in the Framework, the line of "Human Resources" is crucial for improving the situation with both "Motivation" and "Content." Teachers' training is being modernized nowadays. One of the main principles of this modernization is encouraging pre-service teacher education in schools. In particular, concrete work in schools, including supervising school study groups, checking homework, offering extra lessons to students who lag behind and, of course, giving lessons on one's own are indispensable components of teacher education. Such practical work is particularly efficient now, when student teachers' work, as well as some elements of regular teachers' work and schoolchildren's activities, can be recorded using portable video equipment or just a high-quality mobile phone. These videos are then used for further analysis and underlie investigations of psychology, pedagogy and methodology. Moreover, not only work in secondary or high schools but also internships at day-care centers and primary schools can be quite useful for understanding the origins of problems that arise in the education of schoolchildren. It is no less important for future teachers to keep on solving mathematics problems—first of all, problems from the standard school mathematics course. Solving problems, analyzing them, and reflecting on them are all necessary for pre-service teachers; in combination with internships these must comprise the main part of their education in a subject field.

One can ask a natural question: and what about university mathematics? Where is higher mathematics, algebra, calculus, and so on? Here is my answer. I am against requiring student teachers to be able to present memorized, but poorly understood proofs of theorems in classical university courses while neglecting the fact that they cannot solve fairly easy problems from the high-school course (even if these are in fact USE problems). We think it important that a student actually adopt all the knowledge she/he is assumed to have learned. This requirement of academic integrity may look trivial, but it is too often violated.

On the other hand, we must ensure that capable students get help with learning areas of mathematics that attract their interest. This can be accomplished with the use of open educational Internet resources in combination with individual tutoring by professors at a pedagogical university or, when necessary, teachers from other universities, in the framework of an educational network

Mathematical education of primary school teachers is of particular importance. We often hear from mathematics teachers that this is the root of many problems. This is where we can see massive gaps in elementary mathematical literacy, with such consequences as ideas, dismissed by the Framework, that some children are "incapable of learning mathematics" and can be "scientifically" labeled as having dyscalculia. Attention to this line of mathematical education does not mean that future primary school teachers should be taught analytical geometry or tutored to solve trigonometric inequalities. A major component of their education must be developing in them a capability to recognize the concrete hurdles that a particular child may have in solving a particular problem, as well as the general difficulties the child has in mathematics. One tool for developing this capability is solving a broad range of mathematics problems for primary schools, including problems at an Olympiad level (for instance, borrowed from the Kangaroo contest or the Kvantik journal), and identifying the actual or potential intricacies of their solution.

In a project which we are carrying out on behalf of the Russian Ministry of Education and Science, the prospective content of mathematical education is designed as being relevant outside mathematics. This means that the problem-solving strategies developed on the basis of mathematics will be used in a broad spectrum of circumstances. Here we are based on the Russian and international heritage and practice, which is expressed in a saying attributed to Lomonosov: "One should learn mathematics, if for no other reason than because it sets one's mind in order," and where we could also mention Poincaré, Freudenthal, Lakatos, Polya, and others.

It is underlined in the Framework that a considerable share of human's mathematical activity is related to work in the field of information technologies. The integration of mathematics and informatics in primary school, which is stipulated in the FSES and is often implemented in actual education, underpins

the corresponding orientation of students. In particular, a possible approach is to engage students, on a permanent level, with the solution of educational programming problems, which are gradually becoming more practice-oriented, and at the same time with learning new elements of programming and in teaching informatics to younger students. This concept is internationally promoted by the World Information Technology and Services Alliance (WITSA n.d.). On the other hand, the New Technology Initiative was adopted in the Russian Federation (see http://asi.ru/nti/) . Presently, a Conceptual Framework for School Technology Education is being developed, where links between modern technologies (primarily, information technologies) and the main subjects in the science and mathematics curriculum are consistently actualized.

Referring to the need for leaders, the Conceptual Framework proposes establishing a number of centers of excellence in research and education, which will keep up in all respects with the best world centers. This includes appropriate wages and living conditions for the leading international experts in mathematics, their colleagues and Russian professionals of various categories, the corresponding infrastructure, and so on. In certain respects, such centers will work like the Princeton Institute for Advanced Studies or similar centers in places such as Europe, China, and India. Such centers will be established with federal aid in St Petersburg, Moscow, Novosibirsk and Kazan'. They will also be created in Ufa and Ekaterinburg, at the expense of the regional governments.

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Goals and Challenges of the Mathematical Olympiads of Today: Science, Sport, University Admission, or Status?

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In this paper we will speak about present-day mathematical Olympiads in Russia. Olympiads are considered an efficient tool to search for and select talented young people and involve them in mathematics. Nowadays, people speak about a crisis in the Olympiad movement; the features of that crisis will be analyzed below.

Historical sketch of mathematical Olympiads

Mathematical Olympiads developed gradually. In the Middle Ages some mathematicians shared with other mathematicians problems which they had been able to solve, but whose solution could not be easily guessed: the famous contest between the Italians Antonio Maria Fior and Niccolo Tartaglia (see Guter and Polumov, 1980) and the solution by the Frenchman François Viète of a problem put by the Dutchman Adriaan van Roomen (see Struik, 1987), could be called proto-Olympiads.

The first-ever Olympiad for *gymnasium* graduates was carried out in 1894 in Hungary, on the initiative of the Hungarian Mathematical Society and famous physicist Loránd Eötvös (see Kurschak, 1963).

In Russia the first Olympiads were convened in 1934, in Leningrad (now St Petersburg); one of their organizers was the remarkable geometer and corresponding member of the USSR Academy of Sciences, B.N. Delaunay. The first mathematical Olympiad in Moscow was organized in 1935, and its organizing committee was headed by the corresponding member of the USSR Academy of Sciences P.S. Alexandrov and included professors of mathematics

from Moscow State University. Problems offered at Moscow Olympiads were published later in many books (see for example, Galperin and Tolpygo, 1986).

Other countries also organized Olympiads. The first International Olympiad for high school students was held in 1959 in Romania. In 1961, the Ministry of Education of the Russian Soviet Federative Socialist Republic (and later on, the Ministry of Education of the USSR) started to convene annual mathematical Olympiads for high school students (All-Russia, and between 1967 and the collapse of the Soviet Union, All-Union Olympiads). This National Olympiad consisted of five tours:

1. school level,
2. district (town) level,
3. regional (republican, territorial) level,
4. zonal level,
5. final tour.

In 1992, All-Union Olympiads were abolished; the All-Russia Olympiad for high school students (ARHSO) inherited the same 5-tour structure. Starting in 2009, when the regional tour (called the "federal regional tour" at that time) was eliminated, ARHSO was reduced to 4 tours.

ARHSO is organized by the Ministry of Education and Science of the Russian Federation. Its first tour identifies the best participants within a high school; the winners are allowed to participate in the second tour. The second tour determines the best high school participants within a municipal district; the winners are allowed to take part in the third tour. The third tour ascertains the best participants within a region; the winners may participate in the fourth tour. Finally, the fourth, concluding tour determines the roughly 100 best high school students of Russia. (A team of 6 students to represent Russia in the International Olympiad is selected from these 100 people through a further, intensive, selection process.)

Also in 2009, the Moscow Mathematical Olympiads (MMO) ceased to be regarded as tours of the All-Russia Olympiad; however, they are held annually, maintaining the tradition.

In addition to the ARHSO, the Moscow Mathematical Olympiad (MMO), and the St. Petersburg Mathematical Olympiad (SPbMO), there are plenty of Olympiads in Russia of varying difficulty and coverage. One of the most notable is the International Tournament of Towns (TT), which has been organized since 1980. Many universities organize their own Olympiads, which (under certain conditions) are taken into account when considering admission.

On the whole, the Olympiads have greatly influenced secondary school mathematical and scientific education. However, nowadays the Olympiad movement is undergoing several major crises:

- a crisis of substance,
- a crisis of format,
- a crisis of ethics.

Below we will speak about the character of these crises and possible ways to meet them.

Mathematical Olympiads as a tool to search for talented youth and attract them to research

In this section, we will consider the **crisis of substance:** we believe that problems offered at Olympiads are not so beautiful and novel as they once where, since the main emphasis is now placed on the sporting component. Here is what we mean: one of the most important tasks of mathematical Olympiads is to search for talented youth and encourage their early involvement in mathematics. Competitions stimulated the development of the mathematical circles (clubs) and increased requirements for the mathematics background of high school students. The principal role here was played by the tours of the All-Russia Olympiad for high school students.

Each year the methodological board which devises problems for the All-Russia Olympiads faces the serious challenge of inventing completely new, interesting problems of reasonable difficulty, which can be solved using techniques taught in high school and with the condition that children attending mathematical circles would not have distinct advantages over those who cannot attend such circles. While in the past it was possible to put forward ideas which were novel for high school students (the pigeonhole principle, the concept of invariants and semi-invariants, colorings, etc.), currently all these concepts are fundamental to circle's programs (Fomin, Genkin, Itenberg, 1996). Finding a new idea is quite difficult and is considered a great success. Hence, the solutions of problems offered at present-day Olympiads are combinations of known ideas, while the difficulty of a problem depends on the number and variety of the ideas required for its solution (see Kanel-Belov, 2011).

Being competitive, students and supervisors of circles focus on standard ideas for solving Olympiad problems. This places an artificial constraint on the range of problems considered in circles. Many important and interesting topics (from the point of view of the mathematical sciences) fall outside that scope, because they are "not needed for Olympiads." Fortunately, some supervisors have enough foresight to address such artificial limitations, with success. However, they constitute only a minor part of all circles.

A search for talented high school students is not the ultimate goal of an Olympiad. The participation in the Olympiad movement should be a first step in pursuing higher-level mathematics, because the school curriculum is not aimed at that level of study and has no suitable means for it. At the same time, new ideas appear inevitably in the study of mathematics, but, as a rule, they are too difficult for high school Olympiads—this calls for a "reshaping" of such ideas (if and when it is possible).

This is the crux of the **crisis of substance:** the departure from the mathematics and a metamorphosis of Olympiads from a festivity of mathematical beauty into a sports competition in the high-speed combination of standard ideas.

Mathematical Olympiads as a tool to select engineering students

Here we will speak about the **crisis of format:** the concept of an Olympiad in mathematics became very broad, which has resulted in a considerable loss of quality. We must take a wider view to explain this phenomenon. About 10 years ago, the Uniform State Examination (USE) was introduced in Russia. All high school graduates must take this examination at the same time and solve the same problems. It is considered both a school-exit exam and a university entrance exam. Additional entrance exams were prohibited (after a while, some exceptional universities were permitted to organize additional university-specific tests, but were obliged to take the results of the USE into account). School graduates were enrolled in universities on the basis of their USE scores.

Under the current circumstances in Russia, the USE does not provide a completely unbiased picture. With the aim of admitting the applicants in order of merit, universities won the right to take into account high school students' Olympiad results as well. The most active universities organize their own Olympiads. These activities were streamlined when the *List of the Russian Council for Olympiads for high school students* was created. The List defines the Olympiads that ensure admittance benefits for their winners. Each Olympiad on the List is ascribed a level: first, second, or third. Admittance benefits depend on the level. This means that if two Olympiads have the same level, then a university must provide the same benefits to the winners of each of those Olympiads (the benefits at different universities may be different).

For example, the aforementioned MMO, SPbMO, and TT are first-level Olympiads on the List. As of 2016/2017, the List contains, in total, 29 Olympiads in mathematics: 9 first-level ones, 11 second-level ones, and 9 third-level ones. Universities specify their benefits at the end of the academic year, before the start of the entrance period. Two kinds of benefits are possible: university entrance is granted without entrance tests or the maximal USE score is given. The List and the levels are renewed annually. ARHSO remains off of the List; the universities must admit ARHSO-winners without exams.

The introduction of the USE created stress among applicants and their parents, since the fate of an applicant depends on a single examination. This is why many regard an Olympiad as a "backup solution," insurance in the case a student fails with the USE. To a great extent, this is indeed so, but unfortunately many applicants cherish groundless hopes of winning an Olympiad on the List.

On the other hand, some organizers of Olympiads, while fulfilling formally all the requirements to be included on the List, have considerably reduced the complexity of problems offered at their competitions; sometimes such Olympiads cannot be distinguished from ordinary university entrance exams of the past or from an examination in a school with an advanced course of study in mathematics.

Masses of high school students now prepare for Olympiads on the List, often not because of a sincere interest in mathematics, but at request of their parents, who take this problem to heart long before their child leaves high school. Many teachers and education managers were proactive and organized appropriate courses or offered individual private tuition to prepare students for Olympiads. On the one hand, the increase in demand for high quality mathematical education may be looked upon as a positive fact. On the other hand, the situation is now so tense that, for a high school student hoping to enter a good university, it is considered odd not to prepare for or not to participate in an Olympiad from the List.

This is the essence of the **crisis of format:** nowadays the Olympiads are not viewed as a natural way of working with talented children, but rather as a kind of backup entrance exam.

Mathematical Olympiads as a success criterion for an educational institution

In this section, we will speak about the **crisis of ethics:** winning an Olympiad is now formally treated not only as a measure of success for the student, but also for their school. In turn, this warps the motivation of a high school student who participates in an Olympiad and the motivation of a school teacher who teaches a talented student.

In order to understand this phenomenon, we must recall certain details of Russian education. The rating of Russian high schools is compiled each year and the list of the schools taking the first 500 positions (TOP-500) is published. It is an honor to be on this list; moreover, in some regions schools are granted extra funding for high places. This means, in particular, that in each school the management desires to have a good rating.

The rating of a particular high school is calculated on the basis of the total score, which is obtained by summing the points that the school received for specific achievements. Good results on All-Russia Olympiads for high school students

are a significant achievement. The good result of a student at a regional tour can bring 1-3 points to the high school (depending on the value of the award); a good result on the concluding tour can bring 5-10 points (again, depending on the value of the award). For comparison, the good result of a student on the USE can bring 1 point maximum to the school, which is much less than they get from a victory at an Olympiad. According to unofficial estimates, 50-70 points are considered sufficient to be in TOP-500 (such information is never published officially).

The substantial contribution that successes at Olympiads can make to the total score of an educational institution leads to an excessive degree of attention paid to students that can attain Olympiad victory. Such high school students often get considerable concessions in academic activities (for example, they are allowed to miss some lessons or school in general in order to get prepared for an Olympiad). Some schools attempt to entice high school students from other (no worse) high schools for the purpose of increasing their total score; such an invitation can be more attractive if they arrange special conditions not compatible with the requirements of general education. There is also the converse phenomenon, when schools which are poorly suited for the needs of talented high school students often try to obstruct their transfer to better schools. The interests of the students are not taken into consideration.

This situation is aggravated by the fact that the rating also has an effect on the assessment of the work of the municipal departments of education, which puts an additional pressure on high school officials. Moreover, students are also under pressure, since their participation at an Olympiad involves the unwelcome emotional stress of being responsible for their alma mater school. So upholding the honor and prestige of their school is now not just a noble right, but also an uneasy responsibility.

This is the essence of the **crisis of ethics:** participation in an Olympiad ceased to be a voluntary matter for each high school student—instead, it attracts unnecessary attention from the administrative staff, coming into collision with the doctrine of optimal development of children.

Thus, there are currently three forms of crisis in the Olympiad movement in Russia:

1. **Crisis of substance.** The lack of challenging new mathematics in problems offered at Olympiads, with the consequence that mathematical circles are now aimed at preparing for Olympiads instead of the in-depth study of interesting topics

2. **Crisis of format.** The erosion of the concept of a mathematical Olympiad—they are now looked upon as obligatory for senior students

3. **Crisis of ethics.** The interests of students are pushed into the background; school officials race for points in the school rating.

Measures to address the crises in the mathematical Olympiad movement

The crises mentioned above have been discussed for many years by the Olympiad community. There has been some positive development in each aspect, even though no ultimate solution is in view. Now we briefly discuss the apparent changes.

Addressing the crisis of substance

The mathematical content of problems offered at Olympiads is improved by inviting actively working researchers (Doctors of the Sciences and professors who were themselves winners of Olympiads in the past) to participate by sitting on methodological boards. However, scientists and mathematicians cannot spend a great deal of effort and time on such projects, so only a few research mathematicians have joined the Olympiad movement.

On the other hand, winners of Olympiads who are upperclassmen (at high schools) or undergraduates (at universities) are in fact ready to view mathematics as a field of research. For such students there are summer (and not only summer) workshops with research mathematicians who are ready to familiarize young people with their research fields. These workshops have become increasingly popular: we can mention the Summer Conference of the Tournament of Towns (SCTT) and the Summer Conference "Modern Mathematics" (Russia), the Summer Workshop (Lyon, France), and the Workshop in Bremen (Germany) as just some of these events.

Most of these workshops offer lectures or cycles of lectures delivered by researchers; however, SCTT has an entirely different format. The Summer Conferences of the Tournament of Towns invite the best participants of the Tournament of Towns Olympiad, who are asked to solve at their discretion a few relatively long series of problems on the same topic. This approach allows one to focus on an area and approach some unsolved problems in mathematics. Problems are chosen by professional mathematicians specifically for SCTT; the same mathematicians participate at SCTT and carry on conversations with high school students. Occasionally at an SCTT conference or after it a high school student who got interested in a particular topic makes a breakthrough and solves previously unsolved problems. The materials of Summer Conferences of Tournaments of Towns were published (Konstantinov, 2009).

Addressing the crisis of format

Levels were introduced to differentiate Olympiads in accordance with how difficult it is to win the top places. The first level threshold is relatively high. Thanks to this, the first level is given to Olympiads with really interesting, nonstandard, hard problems; it can be considered as a sort of quality test.

The traditional popular Olympiads whose quality is no lower than that of the All-Russia Olympiad (the Moscow Mathematical Olympiad, the St Petersburg Mathematical Olympiad, and the Tournament of Towns) have reworked their formats and now meet all the List requirements to ensure the maximum preferences to their winners.

There is no doubt that the organization of the second- and third-level Olympiads fits the quality standards: they generate considerable interest among their participants, the problems they offer are absolutely correct and are marked by a certain novelty. Therefore, any competition granted an inclusion on the List deserves consideration.

Addressing the crisis of ethics

Of certain interest is the new ruling that the points "gained" by a high school student who has changed school should be divided between the old and new schools. This rule, which applies to the student's first year in the new school, is aimed at distributing points as fairly as possible and at reducing the "profit" from recruiting a new high school student in comparison with "nurturing" the school's own students. However, this is only a half-measure--in both the literal and figurative sense.

The majority of the teaching community puts the interests of a student first. This allows high school students to move along an optimal educational path, even when this requires a change of school. The law ensures the right of a student (or, more precisely, of a parent or a legitimate representative of a person under legal age) to choose a school (provided that the school can admit the student and the student fulfills all the entrance requirements). Hence nothing prevents pupils' mobility. At the same time, we see a trend towards a decrease in this mobility due to the technological developments that allow a students to get high-quality education in any Russian town.

Multidisciplinary mathematics competition at AECS MSU

Being aware of the imperfections of the system of existing Olympiads, and in cooperation with like-minded souls, we worked on the implementation of our vision for an "ideal" mathematical competition. In this section we will elaborate on the format of such a competition.

Alongside individual Olympiads, there are many mathematical team tournaments in Russia. Such tournaments grant no benefits and feature greater freedom in choosing the subjects of problems. In particular, some tournaments offer problems requiring the knowledge of higher mathematics or specific topics beyond the high school level (this is prohibited for ARHSO and for Olympiads on the List). Thereby, this prompts the idea that one can (and should) study mathematics beyond the high school level. This is an attempt to fill the gap between the high school and the university programs.

Such team competitions are mostly “mathematical battles”—a very popular game played with two teams. However, there are other forms of tournaments. One such form was implemented by the author and his colleagues at AECS (the Advanced Educational Scientific Center at Moscow State University) from 2008-2014, under the name "Multidisciplinary mathematics competition." AECS MSU continues to convene multidisciplinary competitions in a slightly different format. As we are going to describe only the original format, we will speak about multidisciplinary competitions using the past tense.

A mathematical multidisciplinary competition was a team/individual tournament for teams of four 8th, 9th, 10th or 11th graders. Competitions were convened in two age groups, "junior" (8-9th- graders) and "senior" (10-11th-graders). The tournament involved 5 competitions:

- individual competitions
    - in algebra and number theory (written),
    - in combinatorics and logics (oral),
    - in geometry (written);
- team competitions
    - "Mathematical race" (written),
    - "Team Olympiad" (oral).

Individual competitions followed the format of classical Olympiads: 4-5 problems with a 240 minute time limit, written competitions were scored as at the All-Russia Olympiad, after a careful solution check; the scores for oral competitions (0 or 1, depending on whether the problem was solved or not) were assigned by the judges of the contest.

Team competitions featured substantially different formats: a fast written competition "Mathematical race" (4 sets of 3 problems with 20-30 minutes for each set) and a long oral competition, named the "Team Olympiad" (8-10 problems, 240 minutes for the entire Olympiad).

The reasoning behind introducing various formats for competitions was, first, to increase the group responsibility for each student’s personal achievements (the individual scores of every participant were taken into account when computing the team rating) and, second, to help participants develop a taste for teamwork.

An important role was also played by accompanying events:

- an experimental tour which utilized new organizational forms of intellectual and research competitions

- popular scientific lectures in mathematics presented by members of methodological boards
- meetings with representatives of educational institutions

The general principle of the distribution of awards was as follows: "awards should be given for everything which is worth awarding." Hence in each league awards were given in each team competition and in the total team scoring (6 categories total); the best participants in each form were given separate awards for each individual competition and for the highest individual total score (8 categories total). Thus, anyone who succeeded in at least one area was given an award. One can say that this is exactly how science works: results are obtained in particular directions, rather than in all branches simultaneously. The winners were determined according to precise (though quite sophisticated) principles; nevertheless, all the teams were given awards to emphasize that, in research, everyone does his or her bit.

The methodological boards for the multidisciplinary competition included representatives of the Olympiad community and actively working researchers (some mathematicians manage to combine these activities successfully). New ideas in problem solving were received with particular interest, the competitive element being deliberately eclipsed. In addition, each member of the methodological board had the opportunity to deliver a lecture on a mathematical subject of their choosing. Usually, such lectures were attended by a great number of interested students.

Multidisciplinary competitions attracted some foreign teams: in a short period of time teams from Serbia, Korea, Greece, Mongolia, Ukraine, and Kazakhstan took part in tournaments. In pursuing this, our idea was to demonstrate to the participants that, in mathematics, there are no borders between countries and that mathematics is the province of mankind.

The "experimental tour" deserves special mention This competition, which was not graded, was held on the last day of the tournament and available to all interested students. Each year the format of the experimental tour was different. Our idea was to look for a new format, leaving aside the classical concept of an Olympiad. Over the years, experimental tour included the following competitions:

- fast puzzle solution
- mathematical quests
- data retrieval from the internet
- mathematical modeling

and other forms of intellectual activity related to mathematics in some way or another.

The materials of the first three "Multidisciplinary mathematics competition" tournaments were published (Tikhonova, Sharich, 2012), while collected materials of other tournaments are in preparation for publishing.

Conclusion

Despite the great number of controversies in the Olympiad movement, some solutions to the crises are quite possible. In this concluding section we will consider the ones that seem to be the most feasible.

To address the crisis of substance, one should start with a closer and deeper collaboration between Olympiad organizers and actively working scholars. Presumably, one reason why researchers do not participate in the Olympiad movement is that such activity cannot formally be reckoned as a scientific activity, even though these activities are comparable in their complexity. From the scientific point of view, problems devised for Olympiads lie between pedagogy and mathematics, and so fall into neither category. In view of the general trend towards extending recognition to interdisciplinary fields, it may occur in the foreseeable future that a mathematical sub-discipline under the official name of "Olympiad mathematics" will appear (this name is presently used in the Olympiad community, but it frequently sounds odd to the uninitiated).

As technology advances, knowledge becomes more easily accessible. Perhaps the ban on using out-of-school methods to solve Olympiad problems will soon be lifted. This would have resulted in a greater use of higher mathematics in the programs of competitions—the material of the first two university semesters provides prolific sources of new ideas (the increasing popularity of student Olympiads is proof here).

To address the crisis of format, the educational community should move more actively in explaining the true sense of Olympiads to high school students and their parents. Perhaps a separate category of trials will be developed—something between Olympiads and university entrance exams—in order to enable mathematically gifted (but impartially not the best) high school students to demonstrate their abilities and gain some benefit from this.

Changes in state legislation regarding university admission regulations cannot be forecasted. One may only hope that the admittance formalities will be more flexible and the List will not be necessary.

To address the crisis of ethics, one should relax the formalities of citing Olympiad results in the assessment of an educational institution. Of course, such results reflect the performance quality of a school, but the format of scoring is unstable due to the fluctuations that result from sheer luck, health conditions, and other factors that are not directly related to the actual knowledge level of a high school student.

The role of an educational institution in high school education is decreasing. Many parents turn to family education and high graders turn to external studies. Modern technologies and legislation contribute to this development. As a result, the idea of "studying at a particular school" may prove senseless. In this way, the problem of grading schools for the achievements of their students will be automatically eliminated.

On the whole, the Olympiad movement, which is a living, fairly young organism, is undergoing some natural changes. It is difficult to exaggerate its integral contribution to the development of high school mathematical education; at the same time, it must be admitted that nowadays the Olympiad movement only supports the existing system of further education, by helping it to develop quantitatively, rather than qualitatively. New precursor forms of out-of-school activities are actively developing these days, such as conferences of high school students or competitions in mathematical modeling. Olympiads are attractive because of their precise and transparent criteria and impartiality in determining the winners, which are combined with impressive mathematical content. An organic combination of these features and new forms could be a new step in the evolution of the Olympiad movement.

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Mathematical Research Problems in Russian Schools

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Introduction

This article discusses the use of a particular type of problem in the teaching of mathematics. It is generally recognized that solving problems is the principal method not only for learning mathematics but also for developing pupils' critical and creative thinking. Recently, some Russian school teachers have started applying problems of a particular type as well as traditional mathematics problems, in Russia they are known as "research problems." Some advantages of these problems and difficulties of using them follow.

Two examples

We should start with two examples of problems. At first, a traditional problem is discussed, then a 'research problem' based on the same subject matter is presented.

A traditional problem

In the trapezoid ABCD, the diagonals intersect at point O (see Figure 1). The trapezoid bases are 4 cm and 8 cm, and the altitude of the trapezoid is 6 cm. Find the area of the triangle BCO.

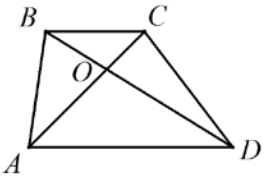

Figure 1.

To solve this problem, students need to take the following steps:

1) identify the similarity of the triangles *BCO и AOD* and to prove their similarity;
2) figure out the similarity factor using the data given (it is equal to bases ratio, i.e. 2);
3) figure out the length of the altitude of triangle *BCO*, drawn to the base BC, taking into consideration that the ratio of similar triangles altitudes is equal to the similarity factor. (This means that the point O divides the altitude that goes through it into segments equal to 2 and 4 cm.);

4) using known formula, calculate the area of the triangle *BCO* (4 cm^2).

This is an example of a good training problem. To solve it students must use well-known facts (the criteria for the similarity of triangles, the properties of similar triangles, the formula for the area of a triangle) and find a way of solving the problem on their own, which is not obvious from the data given.

A detailed solution of the problem (including writing it all down) will take 10-15 minutes depending on how quickly students find the core idea of the solution. If students successfully solve the problem on their own, we may state that they have a secure grasp of basic geometrical facts and can use them to solve rather non-standard problems.

Now, we should give an example of ***a research problem*** on the same subject.

A non-rigidly given trapezoid

The trapezoid bases and the altitude are given. From the list below, what can be figured out with the data given?

1) The lengths of the legs of the trapezoid;

2) The length between the midpoints of the legs of the trapezoid;

3) The lengths of the diagonals of the trapezoid;

4) The distance between the midpoints of the diagonals of the trapezoid;

5) The areas of the triangles which the diagonals divide the trapezoid into;

6) The legs of the trapezoid are extended to the intersection point. What else can you figure out in this construction?

7) What will change in the answers to the questions from 1 to 6 if one of the trapezoid angles is also given?

8) What will change in the answers to the questions from 1 to 6 if one of the legs of the trapezoid is also given (but the angles of the trapezoid are not given)?

Practice proves that even if students are successful at solving a concrete problem they do not always realize what can and cannot be figured out in this construction (in other words, what is constant in relation to possible changes), i.e. a formal solution of a traditional problem often does not give a complete understanding of the situation.

We may correctly solve the traditional problem stated above, figure out the area of the triangle *BOC* and then ask students whether it is possible using the data given to figure out the length of the diagonals. This question will be most likely accepted like a totally different task, because in solving the first one they did not think about the whole situation.

The research problem stated above aims to fully clarify the geometric construction given. Questions are offered to enable a student to determine all of

its basic invariants. This kind of problem results in the students' transition from a conventional approach to a geometric problem to a dynamic one. If the trapezoid altitude is given, the bases of the trapezoid are to lie on the two parallels with the given distance between them. Imagine that one of the bases is fixed on a parallel line, then the second base can move along the second line.

The task is to determine which parameters are constant under these changes. Under such a reformulation of the conditions the first three tasks are easily solved: the lengths of the legs and the diagonals of the trapezoid seem to change when one of the bases is shifted, and the length of the midline is constant.

The 4th and the 5th tasks are more complicated, moreover the solution of the 5th task is counter-intuitive. Although the lengths of the diagonals and the legs of the trapezoid change, the areas of the triangles remain the same. Usually such an unexpected result causes a great emotional reaction. Interestingly, students are shocked by it even if they have already solved the problem in its "traditional" variant, i.e. have found the area of the triangle *BCO*.

The 6th task allows students to choose their own possible invariants of the given construction (for example, the distances between the intersection point of the extensions of the legs and the bases of the trapezoid). This task is even more complicated than the previous one, as it gives us a free choice and therefore puts us into uncertainty.

The 7th task clarifies the quantity of the parameters determining this family of trapezoids. It is enough to define one of the angles in addition, and then the trapezoid is determined so that it is possible to figure out all of its elements. This means that the entire family of trapezoids depends on one parameter.

Finally, the 8th task, which is similar to the 7th but demonstrates that some sets of geometric data can determine finite sets of different objects (in this case there can be two different trapezoids if the leg is longer than the altitude of the trapezoid).

Solving this problem in a classroom may take up to 2 lessons, and all the same the problem most likely will not be completely solved. This time expenditure is quite natural, as such a formulation of the task is much closer to the real work of a scientific researcher who may work at a problem for weeks or even months.

As a rule, problems of this kind excite a special curiosity in students and encourage their creativity, as diverse questions considered naturally become themes for discussion and individual or collective investigation. Besides, to solve problems of this kind it is appropriate to apply different computer programs (for the problem stated above there are programs of dynamic geometry: "GeoGebra", "Live geometry" and others). The advantages of the research problems mentioned have repeatedly been discussed within the Russian mathematical society (Skopenkov, 2008; Sgibnev and Shnol, 2007), although it has not resulted in the mass application of such problems. What is the reason for this?

Research problems in Russian school education

Since the late '80s, when school teachers started practicing research problems in their lessons, two principal types of investigative work have been conducted in schools and partly applied in practice.

1) extracurricular research work chosen and carried out by a student on his own;

2) solving research problems during lessons (individually, in groups, or as an entire class).

Currently, the first type of research has been clearly developed and has taken on some new organizational forms in Russian schools. Many schools in different regions hold school research conferences for students where research papers on mathematics, amongst other topics, are discussed. Some books describing the common methods of such work as well as containing concrete research problems have been published (Sgibnev, 2015, Ivanov and others, 2013, Kulanin and others, 2013). Leaders of this work organize workshops and conferences about this style of research (http://www.mccme.ru/circles/oim/mmks/opyt.htm and http://www.mccme.ru/nir/uir/).

During the past few years new Federal State Educational Standards have been implemented, and as a result project-making and educational research have become a compulsory part of school education so that the majority of Russian schoolchildren are nominally involved in this type of work. Nevertheless, experts in this field say that in practice only a small amount of interesting research is carried out at approximately 20 Russian schools because each of these investigations requires a highly skilled teacher, new ideas, and new tasks every year.

In the overwhelming majority of schools an individual research paper is considered to be a simple semi-compilation, in which a student has showed some originality in conveying the extracurricular material. Of course, such work when performed independently can be useful in many ways. On the other hand, once this kind of work becomes wide-spread and compulsory in Russia, a flood of penny-a-line compilations or stolen papers are uploaded to the Internet. In any case, over the last 20 years, extracurricular student research has become a component of Russian mathematical education with all its advantages and disadvantages.

The same cannot be said about the second form of applying research methods in teaching mathematics during lessons in public schools. There are many publications in methodological magazines (Dalinger 2000), reports made at conferences, and theses devoted to the theme, but no noticeable changes to the way student research is carried out have appeared in public schools. In our opinion, although it may seem paradoxical, an ordinary student having no special gift for mathematics needs such research problems more than a keen

student of mathematics. Students who are fond of mathematics and want to solve more complicated and non-standard traditional mathematical problems derive pleasure in using mathematics when solving such problems. Any non-standard problem either has an element of exploration or is connected with the need to conclude, make assumptions, and try hypotheses. For an "average" student, learning "theoretical material" and "solving" a set of tasks to prepare for an examination, complicated, non-standard, olympiad problems are too difficult. As a result, during mathematics lessons the majority of students do not get to experience the pleasure of discovering something on their own, and, as a whole, their mental skills develop insufficiently. Finally, we should say that conducting step-by-step research in mathematics lessons is closer to the everyday activities of students in math lessons than making a compilation on a given theme. The latter type of work is often undertaken by students as a form of independent activity that is vaguely connected to common tasks in mathematical education, whereas research problems maintain a connection to the basic course and develop different mathematical skills that students need to pass their exams successfully.

Reasons for such limited use of research problems

Firstly, we should try to define main reasons that the elements of exploration are rarely used in public schools, and then offer some ways of changing the situation.

1) Teachers do not have enough time for such work during lessons

Conducting individual or collective student research of the type outlined in a core program and especially extracurricular programs requires additional classes. When a teacher systematically teaches new material to students and gives them a competently considered and systematically organized set of exercises and problems on new topics, it is 2-3 times faster than research requiring a new problem be solved by students either individually or in a group. The current Russian mathematics curriculum includes a vast array of material to be studied, and the authors of these programs calculate the time required for learning the material using traditional tasks and teaching methods. A teacher essentially has no time left over to carry out investigations during lessons. Although these observations cannot be universally true, this is the way the majority of Russian teachers look at the situation.

2) Teachers are inexperienced in teaching research work

Teachers did not carry out research like that when they were children or students. Many school level research problems are too difficult even for strong teachers as the way they are expressed is rather unusual. Special courses are required where teachers can learn to solve such problems as well as how to

deliver research based lessons. In our experience, interested teachers master these skills and acquire a taste for solving research problems quite quickly.

3) There are no research problems chosen according to students' strength

A set of well-selected training tasks is the main instrument of high-quality mathematics teaching. Unlike making exercises for practicing skills or techniques, creating a new interesting, beautiful, thorough training problem is creative and that is why it is an unpredictable process. It is impossible just to sit down and in 2 hours contrive a good set of research problems on a particular theme. The tradition of mathematical education preserves many fine, classically formulated tasks; it has accumulated successes achieved by a few generations of mathematics teachers. However, it is obvious that at the moment there are not enough not very difficult research problems (for some sets of such problems see Sgibnev, 2015). We may hope that in 20-30 years the joint efforts of teachers in different countries will result in a richer collection of such problems, but as of today no Russian textbooks or schoolbooks contain a satisfactory set of research problems that can be regularly applied by a teacher in lessons in a public school.

4) It is difficult to plan and manage a lesson centered around research

An experienced teacher can easily plan lessons according to traditional models. Using traditional methods teachers can estimate the quantity of time they need to deliver the content, differentiate strong and weak students by giving additional, more complicated tasks to those who are more able, and succeed in teaching students to solve problems without any difficulties. When a traditional problem is being solved, an experienced teacher can easily encourage a student to solve the problem by pointing out the next step in the solution. However, it is often harder to do for a research problem, that is why students work on it longer. In other words, a lesson when students carry out independent research requires more improvisation and organizational skills from the teacher, and the latter needs to be more flexible in defining the educational tasks and evaluating whether those tasks are achieved. As teachers are currently overloaded (as I know, an average teaching load is 24 lessons a week), the majority of teachers do not have the resources to regularly prepare more complicated lessons than usual.

5) It is difficult to assess mathematical research

Often research conducted in lessons does not result in a complete solution of the given problem but gives some partial advancement to understanding the situation. For example, one of the results can be the refutation of a false hypothesis suggested by the student. When dealing with research problems, the process of solving is more important than the result achieved. A fixation on this process, and more so on its assessment according to some criteria that have been set in advance often give rise to great difficulties. Giving an appropriate grade to a student on a standard revision test that evaluates concrete abilities and skills connected with a topic learned is far easier than to appreciate at its true value

research that reflects a student's individual train of thought. The system of assessment (the system of giving feedback to students) is as a whole a problem area in Russian schools, because teachers fear that students will not take the tasks seriously enough if they are not evaluated. Thus, research problems are used as recreational techniques of education, which are sometimes necessary for relief but are far from the basic tasks and methods of teaching.

6) The final state exams tasks influence the forms of educational work in lessons

Whatever wonderful proposals for mathematical education are written on the concept of the development of mathematical education in Russia or in the preamble of an educational program, the practical forms of work, types of tasks, and emphasis in teaching this or that theme depend to a large extent on what is on the final examination. Only two final tasks in different variants of the Uniform State Exam in Mathematics (a task with a parameter and a task about some characteristics of a class of numbers) may be called, even at a stretch, research problems. To deal with such problems under the absolute time limit of an exam a student must be able to use many non-standard methods, albeit ones that do not formally exceed the limits of the program. But these tasks are intended for very strong students and do not in any way influence the situation in the public school education system.

Given that, on the whole, mathematical education in Russia is in a rather sad state (according to FIPI reports, approximately half of school-leavers do not study 10-11th grade mathematic), we cannot expect that a lot of tasks with elements of research will be included into the final examinations. Thus, currently the exam does not motivate teachers and students in public schools to solve research problems.

Some ways of resolving the problems

Let us dwell on some reasons for the restricted use of research problems in Russian public schools listed above and discuss the possibilities of changing the current situation.

1) How can a teacher find the time for research problems in lessons?

As stated above, contemporary school mathematics programs in Russia contain a large volume of material, which is why a teacher has virtually no time to apply new styles of work in lessons. Suffice to say that, in the 1980s when "The theory of probability and statistics" was added to the 7-11th grade programs, other areas of the curriculum were not reduced, and in the standard time-table the quantity of lessons was reduced from 6 to 5 lessons a week. Nevertheless, ordinary teachers in a public school can find some time to work on research problems if they really want to and can organize their time accordingly. How to do this? First of all, one should slightly shorten those parts of the course where students overcome merely technical difficulties, such as cumbersome examples on

operations with fractions, transformations of fractional-rational expressions or expressions with radicals, etc. For example, to make all the students of the 6th grade calculate the value of the expression $\left(1.2-\frac{1}{3}\right)\div\frac{1}{15}$ is enough, and teachers can stop at this level. More technically complicated tasks take a lot of time and do nothing for the mathematical development of students.

Then, one can single out from the topics of the main course the best way of studying using independent research. Such topics do not require much more time while the effect of discovering new knowledge on their own is most appreciable. The study of a linear function graph's dependence on the corresponding coefficients falls into this category. The topic is studied in the 7th grade, but since a student of the 7th grade is not required to be able to prove strictly such dependences one needs little time to discover them on one's own. It is very useful for students to conduct an experiment on their own and to try to formulate some natural rules.

Moreover, lessons that traditionally fail may be devoted to research problems rather than the basic curriculum. One example are lessons at the end of a term, when all the grades have already been given, another -- lessons at the end of an academic year, when students are tired of traditional activities, and review is not very effective. Similarly, can be used relief lessons in the middle of technically complicated, hard topics (for example, the topic "Operations with improper fractions"). In other words, if a teacher includes some research problems in a lesson, when it is appropriate from methodological and psychological points of view, he or she may find some time for this even under the contemporary "suppressed" Russian education curriculum.

The situation changes slightly in high school. The most recent changes in the model of the Uniform State Exam in mathematics give teachers more freedom when they work with 10–11th grade students who will not take the profile exam, such as arts students. As soon as the majority of students do not have to spend all their time preparing for the basic part of the Uniform State Examination (which is easy enough), and the contemporary program lets teachers flexibly change the order of the scope and sequence of the 10–11th grades, a teacher can widely apply research problems when suitable for a class – for example, revising the course of the 5–9th grades or teaching new themes.

2) How can a teacher find good research problems that an ordinary student can solve?

To understand where to find good research problems we should define what a good research problem is. The wonderful book of A.I. Sgibnev "Research Problems for Beginners" (Sgibnev, 2015) defines a fine research problem in the following way: "A good research problem for beginners is the one with *a natural parameter*, that can be followed during the research, i.e. an easily singled out sequence of particular cases, so that a student realizes in at every step forward

what he can do further" (p. 6). We should add that when working with an ordinary class it is particularly useful to apply program materials well-known to students, but in a non-standard way, while they solve the problem. Furthermore, important criteria for the efficiency of the work with a normal class will be the possibility of moving with short strides as well as with long ones, and the fact that the problem has natural generalizations. One example of a good research problem was given at the beginning of the article, so let us discuss two more examples of such problems that make it possible to better understand the characteristics of "a good research problem" outlined above.

Some examples of research problems

The diagonals of rectangles

1) On a piece of graph paper one draws a rectangle measuring 2x5 squares. How many squares does the diagonal of this rectangle cross? *(The diagonal crosses a square if it enters in the square and does not just go through the apex.)*

2) The same question is for the rectangles 2x6, 2x7, 2x8, 2x9 squares.

3) Define a formula for any rectangle 2x*n* and prove it.

4) Conduct detailed research for the rectangles 3x*n*.

5) Conduct detailed research for the rectangles 5x*n*, 4x*n*, 6x*n*.

6) Conduct detailed research for the rectangles mxn.

7) Summarize the problem.

Of course, you could start with a "zero" question: how many squares cross the diagonal in the rectangle 1x*n*, but in my opinion it is better to start with a non-trivial and interesting question, and the students will have to explore the rectangle 1x*n* independently during the research.

Students achieve the answers for the questions 1 and 2 empirically. It is not difficult to see the pattern as soon as you put the data into the table.

A rectangle	2x5	2x6	2x7	2x8	2x9	2x10
The number of squares that are crossed by the diagonal	6	6	8	8	10	10

Even a weak student can formulate this pattern: if the rectangle is an even number of squares long, then the number of crossed squares is equal to the number of the squares along the side of the rectangle, and when the number of squares is odd, the number of crossed squares is greater than the number of the squares along the side by one. To substantiate this pattern is much more difficult. It is a positive characteristic of the problem: a student who has seen the pattern, but cannot definitely prove it, starts to realize the difference between a hypothesis based on the experiment, and a proved fact.

How can we suggest to students the idea that they need to prove the pattern discovered by them? It is enough to ask them to substantiate the answer for a reasonably large value of n. For example, when trying to answer the question "How many squares will the diagonal in the rectangle 2x1000 cross?" a student will realize that he cannot draw such a rectangle and needs to make some general reasoning for proving the answer. How can students do it? To clarify the situation, let us draw the rectangle diagonal from the lower left corner to the upper right one. Attentive students notice, that if the long side of the rectangle contains an even number of squares, then the diagonal goes through the apex of the squares in the center of the rectangle (See *Figure 2*).

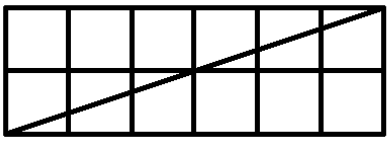

Figure 2

Then they think, firstly the diagonal crosses all the squares of the lower left rectangle of size 1x(n/2), and there are n/2 of them, and then all the squares of the same upper right rectangle, so that it will cross exactly n squares.

But if n is odd, then the center of the rectangle is in the common side of two squares ((See *Figure 3*). That means that (n+1)/2 squares will be crossed by the lower half of the diagonal and the same number by the upper one, in total n+1. So, in order to make a substantiation of the answer a student needs to see from the illustration that the decisive factor in this case is the location of the center of the rectangle. Then they must use both this information and a new object that has not been mentioned in the conditions to prove the statement. This is also an advantage of the problem: the proof requires the invention of a new method, but not a step that is too big; it should be within their powers.

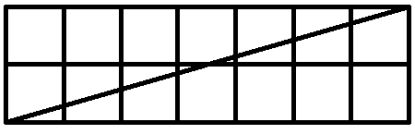

Figure 3

Let us move on to the 4th question: the exploration of the rectangles 3xn. First of all, the task is shorter, and students must guess on their own that they should start from an experiment with small values for n. Secondly, quite quickly (already in the case 3x4) it becomes obvious that the pattern of the previous task does not work directly: the number of intersected squares depends not on whether n is even or odd, but on something else (divisibility by 3). Thirdly, the pattern at this point is more complicated, it is not so easy to notice and formulate it.

A rectangle	3x3	3x4	3x5	3x6	3x7	3x8	3x9	3x10
The number of squares that are crossed by the diagonal	3	6	7	6	9	10	9	12

Fourthly, the proof of a new pattern also cannot be carried out without any changes from the previous task. It is also a positive characteristic of the problem. Strong students, who have quickly solved points 1 to 3, will have to think about the 4th one longer. Weaker students, if they tenaciously carry out the experiments, have the same chance of solving this point of the task as well.

Let us go to the consideration of the common case of mxn. Generally, at first students see that if m and n are coprime numbers, then the number of crossed squares is equal to $k = m + n - 1$. To prove this, we need another new idea, and not all of the students will guess it. We need to calculate how many vertical and horizontal lines should be crossed in order to reach the upper right apex from the lower left one. It is obvious that we need to cross all the internal horizontal and vertical lines, and we get into the first square without any crossings, so that we achieve the formula $k = (m - 1) + (n - 1) + 1 = m + n - 1$. If, though, m and n are not coprime numbers, the pattern is far from being seen easily, especially for difficult cases like 6x9. Finally, a student can guess that the point is in the greatest common divisor of these two numbers, and define a basic formula $k = m + n - d$, where d is the greatest common divisor (m, n).

In a basic school course, the idea of the greatest common divisor is introduced into practice mainly for the addition of common fractions, but there are a very small number of challenging problems using the idea mentioned. In the given geometrical problem the greatest common divisor suddenly gives the key to solving the problem in general, so that it strengthens the importance of basic concepts, as well as connections between different sections of mathematics. The problem also has a natural continuation and generalization. One may consider the diagonals of the rectangular parallelepipeds, divided into isolated cubes. On the one hand, the task turns out not to be a trivial consequence of the problem that has been already solved (one needs to understand what happens when two edges of the parallelepiped are expressed by not co-prime numbers but all three edges are co-prime). On the other hand, the problem is not too complicated after going through the plane.

We should mention that the problem discussed above, with the exceptions of the concept "the greatest common divisor," which appears only on the last steps of the solution, has only a very slight connection with the core school program, and this is its obvious disadvantage under the time constraint mentioned above.

Let us discuss another problem, which has a basic school task for the 7th grade as its source.

Sum of squares of two binomials

How many terms can be in a standard polynomial that is equal to the sum of the squares of two binomials?

Our experience says that if expressed as listed above it confuses even quite strong students. Many of them do not understand what the task is and what the first step towards solving it should be.

Because of that, the problem should be divided into parts with additional questions.

1) An example. Let us open the parenthesis in the expression $(a+b)^2+(c+d)^2$, the sum of squares of two binomials: $(a+b)^2+(c+d)^2=a^2+2ab+b^2+c^2+2cd+d^2$. There are no like terms here, and we get the polynomial consisting of 6 terms.

2) Give an example of the polynomial consisting of 5 terms and equal to the sum of the squares of two binomials.

3) Whether such a polynomial can or cannot have:

a) 3 terms, b) 2 terms, c) 4 terms?

4)* Whether a monomial can or cannot be equal to the sum of the squares of two binomials?

Let us consider why the problem stated above is difficult for the majority of Russian students, although it is based on a subject matter that is usually learned quite well (polynomials and formulae of shortcut multiplication). The main reason is that the absolute majority of tasks in the Russian algebra program are exercises where students should follow a well-known algorithm or work by analogy with problems investigated by a teacher.

Tasks to construct an algebraic expression with a special characteristic are rarely set, although they are useful and "simple" (try to think of the equation with the roots given or the fraction that has no sense if x=2, etc.). This problem does not call for using the algorithm that students are familiar with; it requires some invention on behalf of the students. The only way to begin to solve is to be not afraid to experiment. At the beginning, the majority of students do not understand which monomials they should use in order to achieve the required result. The correct strategy is to put something into the parenthesis and to see what happens. As a result, many students get stuck on some of the simplest cases: 5 terms and 3 terms. After that, the search for the solution for 2 terms is carried out more consciously. Almost any student can deal with these problems within a reasonable time frame. We should mention that in solving the problem students define their comprehension of the concept of a "binomial" more precisely: for example, for a task with two terms many of them offer the solution $(a+a)^2+(c+c)^2$ and only later realize that it is false.

The problem about the polynomial with 4 terms requires a new idea. We need to realize in advance which terms will be similar after the parenthesis are open. If no ideas appear after a while, a teacher can suggest the first square of the

binomial and ask the students to think of the second one. In this case, the problem still remains quite difficult (a possible hint that does not spoil the task is a great advantage of the problem). For example, one can suggest to the class the first square of the following binomial: $(a^3+1)^2$. The advantage of it is that further one can analyze two possible solutions with slightly different ideas.

The first solution. To cancel out doubled products:

$$(a^3+1)^2+(a^2-a)^2=a^6+2a^3+1+a^4-2a^3+a^2=a^6+a^4+a^2+1.$$

The second solution. To create two such pairs:

$$(a^3+1)^2+(a^6+1)^2=a^6+2a^3+1+a^{12}+2a^6+1=a^{12}+3a^6+2a^3+2.$$

Other good prompts for a teacher to use are also possible. For example, the first square of the binomial may look like: $(ab+cd)^2$.

Finally, the last question, whether a monomial can be equal to the sum of the squares of two binomials, is quite difficult for 7th grade students. It is possible to slightly simplify the problem by only taking binomials of the same variable, but even in this case, the proof appears to be almost beyond the capability of the 7th grade. Because of that, in the majority of classes it is reasonable to finish with nothing more than a discussion of the question, suggesting the hypothesis that it is impossible and ascertaining the fact that for the present we cannot prove it.

In my opinion, the analyzed problem is a good example of a research problem: it is based on the core program material and solving it can strengthen the specific subject skills obtained while studying the theme, at the same time it helps to develop exploratory skills.

The discussed problem, as well as the task about a trapezoid outlined at the beginning of the article, show that in order to conduct valuable research during lessons, there is absolutely no need to exceed the limits of the school program and find some additional time for this. By analyzing the basic themes of the school program we can find the material to create research problems, although, as it has already been said, this is much more difficult and unexpected than compiling traditional tasks.

3) How to maintain a well-balanced level of difficulty when formulating a research problem for the whole class?

When a teacher works with a class where the students are of mixed ability, it is crucial to organize the sequence of the tasks correctly. On the one hand, the first questions must be within all the students' grasp, on the other, there should be tasks in the problem that make even the strongest students think, such that solving them is a real joy for the students. The problems listed above meet the requirements.

Sometimes, I have come across materials for carrying out research within a lesson structured so that they looked like detailed reports about every small step. Every step until the end was thought through by the teacher in advance, and even the criteria of evaluating all these steps were compiled (Ho Foo Him, Spario Soon, 2002). There is a hidden threat in that: "a research problem" may become just another regular form of educational work, where you do not need to think – just answer the questions given by a teacher. But strong students often tear away such form of work, because it deprives them of the legitimate pleasure of overcoming obstacles and getting results on their own.

In my opinion, we can obtain some new possibilities with the help of modern computer technologies. Some students need small stages to advance, for others it is interesting to jump over two or three stages on their own. The computer environment let us do both. Teachers need to divide the problem into some quite big steps, and to offer guidance for conducting the next stages (for example, in the form of questions); hints about the necessary new ideas for those who cannot make such steps on their own can be also offered. This way, a teacher can follow not only the progress of every student but also the time this student was thinking without prompts, and the frequency with which each student seeks help. As I know, in Russia there is no yet such a system for carrying out mathematical research in class. I think that the creation of such a system is a goal of the near future.

4) How to teach teachers to deal with such problems?

In my experience, from time to time, many teachers like to change teaching styles, and to start using textbooks with a new approach, or to pose tasks other than in the usual exercise-books. The possibility of trying unusual methods in their lessons is appealing to many teachers. As for learning to use research problems, a teacher needs to have experience solving such problems, which is why in order to train teachers one should start by conducting workshops on how to solve such questions. The workshops I have conducted over the last few years demonstrate that teachers solve such problems with great enthusiasm, although not without some difficulties. The problems that are solved by the teacher on their own while playing the role of a student are more likely to be used in lessons later.

Conclusions

Let us sum up. During the last few decades in Russia, theoretical understanding of the fact that the application of research type problems has many positive effects has matured. Nevertheless, only a small number of such problems are practically used in contemporary Russian schools. To change the situation, we need:

1) to decrease the volume of *the obligatory core* of the school curriculum at the expense of its primarily "technical" parts; to give teachers enough time to use different teaching styles, including research problems;

2) to create a pool of research problems with open access, within the ability of public school students. In particular, one should actively use foreign experience to do this; one could differentiate students of different abilities through this pool of resources.

3) to conduct a series of workshops (webinars) training teachers to use research problems in mathematics lessons.

We should conclude this article with an important remark regarding the way in which new initiatives can develop in Russia. To prevent research problems becoming the next bug-bear of education, they cannot be thrust upon teachers as a compulsory part of lessons. One must give all teachers the opportunity to offer during their lessons any problems (traditional or research), that the teachers find to be the most suitable for their aims and objectives. It is necessary to understand that in a large system (such as any national educational system in any country) significant changes in a short period of time cannot be expected.

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Some Political, Sociological, and Cultural Issues Related to Mathematics Teaching and Learning in the United States[1]

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There are important political, sociological, and cultural issues as well as issues related to curriculum, instruction, and assessment that affect education systems in the United States. Most students attend public schools, which are supported by funds from local and state tax revenue and the federal government. These schools serve millions of students, and, as is commonly known, serve students from the many different racial and ethnic backgrounds reflecting the diverse citizenry of the United States. These schools are located in rural, suburban, and urban settings across the country. Throughout the history of public education, different groups of students have had differential access to schooling depending on their actual and perceived status – for example, poor and immigrant students rarely received any education at all, and then after the Industrial Revolution received only rudimentary schooling focusing on the basics of reading, writing, and arithmetic. Until the 1960s, descendants of slaves in the South, largely kept from learning to read and write throughout slavery, received fewer months of schooling than their white counterparts, because they were expected to labor in the fields most of the year. Further, local, county, and state boards of education, headed by whites, sought to ensure that the education of Black Americans during this time period largely focused on the type of schooling that would be beneficial to agrarian and domestic interests. Girls and women were restricted

[1] Much of the analysis in this paper appeared in Walker, E. N. (2006). Challenging limiting assumptions: High-quality mathematics for underserved students. Working paper for the *Understanding Educational Equity and Excellence at Scale* Invitational Forum (pp. 71-78). Providence, RI: Annenberg Institute for School Reform.

from education throughout much of the era when public schooling was available to men and boys. When coeducational education was instituted, girls and women often were expected to take a curriculum that focused on the domestic sphere, courses and training that would prepare them to be good mothers, wives, and/or domestic servants.

All of this historical context and related issues mean that the present education system is still affected by traditions arising from two centuries ago. Indeed, even the school calendar and school day are largely governed by considerations related to the agricultural calendar and rhythms of public life. Because public education systems arose in cities, towns, and rural areas independent of national oversight, the United States has had a long tradition of public education being managed by local principalities. In short, the curriculum, organization, and priorities of public education are dictated by local school boards, school district superintendents, and state commissioners of education.

This patchwork of multiple school systems has significant implications for the mathematics education of young people in the country. Unlike many other Western countries, the United States does not have a national curriculum. Until very recently, individual states and individual school districts set their own standards and curricula. For mathematics, the National Council of Teachers of Mathematics put forth a standards document in 1989, the *Curriculum and Evaluation Standards*. These standards had a significant impact on states' and localities' standards related to math education, and also curriculum, textbooks, assessments, and professional development for teachers. Beginning in 2009, the National Governors Association and the Council of Chief State School Officers launched the Common Core State Standards Initiative, describing this as an effort to "develop common, college- and career-ready standards in mathematics and English language arts". At the time 48 states (of the 50) agreed to participate. In 2016, 42 states have adopted the Common Core, with some states opting out from the beginning, some states leaving the initiative. The development of the Standards also led to two major organizations designing assessments aligned with the Common Core, the Partnership for Assessment of Readiness for College and Careers (PARCC) and Smarter Balanced Assessment Consortium (SBAC).

Within this context of a standards movement ostensibly designed to improve mathematics teaching and learning for American students, there is substantial evidence that there is differential access to high quality mathematics in schools. The issue of equity is paramount in American mathematics education. In theory, the presence of inequity seems to go against American ideals of fairness and equality. In practice, the presence of inequity has significant and consequential implications for student performance and learning. For example, in the United States for reasons related to historical patterns of discrimination and inequity, Black, Latino/a and Native American students on average score lower on standardized tests than students who are from Asian and White backgrounds.

On the 2015 National Assessment of Educational Progress[2] measuring 8th grade mathematics achievement, for example, Asian Pacific Islander students had an average score of 306, White students 292, Hispanic students 270, American Indian/Alaska Native students 267, and Black students 260 on a 500 point scale. Similarly, there is a pronounced pattern of differential achievement by poor and affluent students: students whose parents graduated from college scored on average 294 points, while students whose parents graduated from high school scored, on average, 268 points[3]. The experiences and education that students receive in school have a great deal of explanatory power for these differences in scores – for example, nearly 1/3 of the variation in high school math performance differences can be explained by differences in course taking. In some schools serving poor and traditionally underserved students there may be those who exhibit mathematics talent and remain unidentified, or they may be known to be mathematically talented, but are constrained by the limits of the school and school system in which they are educated.

There are many factors, in addition to coursetaking inequities, related to these disparities, and a large body of research has developed to explore both causes of the disparities and how to ameliorate them. While many educators and researchers describe the differential scores as an "achievement gap", others have described it as an "opportunity gap" (e.g. Flores, 2007) and an "education debt" (Ladson Billings, 2006). The question many mathematics educators ask, "How can we support the development of mathematical talent for all students, not just a privileged few?" has its roots in the history of differential access to education for many groups in society. In addition a substantial body of research and policy in mathematics education in the United States revolves around the question "How can we improve equity and access to quality mathematics for underserved children (poor students and students from what are commonly termed 'minority' groups)?"

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[2] The National Assessment of Educational Progress (NAEP) is the largest nationally representative and continuing assessment of what America's students know and can do in various subject areas. Assessments are conducted periodically in mathematics, reading, science, writing, the arts, civics, economics, geography, U.S. history, and Technology and Engineering Literacy (TEL).
NAEP provides results on subject-matter achievement, instructional experiences, and school environment for populations of students (e.g., all fourth-graders) and groups within those populations (e.g., female students, Hispanic students). NAEP does not provide scores for individual students or schools, although state NAEP can report results by selected large urban districts. NAEP results are based on representative samples of students at grades 4, 8, and 12 for the main assessments, or samples of students at ages 9, 13, or 17 years for the long-term trend assessments. These grades and ages were chosen because they represent critical junctures in academic achievement. (https://nces.ed.gov/nationsreportcard/about/)

[3] These scores are obtained from
http://www.nationsreportcard.gov/reading_math_2015/#mathematics/groups?grade=8

Asking these questions requires mathematics educators, researchers, and others to reconceptualize the discourse about mathematics underachievers, which largely describes their lack of opportunity, participation, and performance in mathematics. Certainly current reform efforts focused on improving equity in mathematics education are critical to improved performance of African American, Latino/a, and Native American students; however, too often the structural and institutional deficits in US education spur the creation of simplistic mandates that focus on curriculum without regard for pedagogy, and teaching without regard for learning.

Too often, the equity discussion about mathematics education and the reform efforts it inspires focuses on the number of mathematics courses that students take, whether (and more recently, when) students take Algebra, or the availability of Advanced Placement courses in mathematics. While these mechanisms are important, they are all indicators of, essentially, curricular and organizational decisions on which many individuals (especially school district policymakers, administrators, and teachers) have an impact. These 'easily identifiable' equity indicators are signals that equity of access and opportunity *might* exist in a school system or school. But it is not enough to identify these indicators. It is important to examine these indicators in depth to determine if equity is truly present in schools.

Clearly opportunities for underserved students to learn and do mathematics have dramatically improved since the founding of the country. As previously stated, a timeline throughout US history reveals that at various junctures poor students, girls and women, and formerly enslaved and immigrant students had limited access to mathematics. However, major educational inequities continue to have a significant impact on mathematics outcomes. For Black and Latino/a students, in particular, the promise of the Supreme Court's *Brown vs The Board of Education, Topeka, Kansas* decision in 1954 to alleviate school segregation (which codified gross inequity in terms of funding and quality of resources for education) has largely been an elusive one. In deeply segregated schools, these students and/or low income students continue to be less likely to be exposed to highly qualified teachers, extensive resources, or a network of challenging mathematics courses than their White and/or more affluent counterparts. For example, a 2015 report about New York City schools found that 39% of high schools do not offer algebra II and both physics and chemistry[4]. In integrated schools (schools serving a diverse population of students including White students) African American and Latino/a students may find themselves, despite their achievement, "tracked" into lower level mathematics courses than their Asian or White counterparts. These experiences reflect lowered expectations and can translate into lowered outcomes.

[4] http://www.chalkbeat.org/posts/ny/2015/07/23/report-many-nyc-high-schools-dont-offer-advanced-math-and-science-courses/

Although there is evidence that students experience inequities in mathematics beginning in elementary school, the middle school years, in particular, mark a critical milestone in the educational careers of students. Student entry into the college preparatory mathematics pipeline[5] through Algebra I, an important gatekeeper course, is based on a sometimes arbitrary system of course placement. Students whose parents are well connected, affluent, highly educated and know how to 'work the system' are more likely to be placed in high level mathematics in middle school, regardless of their test scores. Not only are Black and Latino/a students less likely than White and Asian students to enroll in Algebra 'early' (in 8th grade or before), but they are also less likely to enroll in Algebra in 9th grade. Consequently, underserved students are consistently under-represented in the courses that comprise the "advanced" part of the mathematics pipeline in high school: Trigonometry, Pre-Calculus, and Calculus. While gaps in algebra and geometry course-taking having narrowed as states' and districts' graduation requirements have increased, there is still a gap in the participation of students in the highest level courses.

Because Black, Latino/a, and Native American students[6] drop out of mathematics earlier and at rates higher than their White and Asian counterparts, the secondary mathematics classroom has been called "one of the most segregated places in American society" by a former president of the National Council of Teachers of Mathematics. Any casual observer can see this when visiting diverse high schools in the United States. Yet many school administrators and teachers ask me, "Why are there are so few Black and Latino/a students in advanced mathematics courses" without considering their role in this problem. Even after prior mathematics achievement and socioeconomic status are taken into account, Black students, who have similar high school graduation rates to White students, are less likely than their White counterparts to persist in advanced mathematics. It has dire consequences for their test scores and other important educational outcomes. Other enriching experiences--mathematics clubs, competitions, and college programs targeting mathematically talented students—are often not options for underserved since college preparatory mathematics classes are the pools from which these students are drawn.

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[5] The sequence of courses Algebra I, Geometry, Algebra II, Trigonometry, Precalculus, and Calculus.

[6] Additionally, the high national dropout rates for Native American and Latino/a students affect their persistence in mathematics. Nationally, in the 1990s only about 57% of Latino/as and 63% of Native Americans completed high school. White and Black American high school completion rates are comparable, around 87%. Asian American students' completion rates range from 88% (immigrant) to 95% (native-born) (College Board, 1999).

Although increasing the mathematics requirements for all students is perhaps a necessary step, it is not enough. For many years, the number of mathematics courses required to graduate from high schools in most states hovered at around 2 or 3 courses. Although these courses were unspecified, it was clear to some (for example, those who were sons and daughters of college graduates) that one had to take certain types of mathematics courses in order to be strong candidates for college admission. This is even more true today. Although many states and school districts have responded to changes in college admission and the SAT and greater attention to equity in mathematics opportunity by increasing the number of courses required to graduate, it is still true that students and their parents need critical information in order to navigate these mathematics pathways successfully. Even though the states have increased their graduation requirements, largely due to the influence of a key US Department of Education report, *A Nation at Risk*—some are even listing algebra as a requirement for graduation—none are really making transparent the idea that in order to be competitive for college admission and to do well on the SAT there are certain types of math courses that 'count'. The proliferation of non-advanced mathematics courses in secondary school—and who takes them--bears this out. The importance of taking the right courses is paramount: while it is true that a student may be admitted to a decent college without taking calculus, s/he needs to do well enough in courses through pre-calculus to demonstrate that s/he can handle college work. If a student's school has a reputation for rigor, then this is probably sufficient. If a student's school does not have this reputation—as many schools that our most underserved students attend do not—a good grade in precalculus may be viewed with suspicion unless it is aligned with a decent SAT score.

Research around reform efforts to improve equity and increase student achievement also targets the necessity of changing teacher behaviors (largely through curriculum and assessment) in mathematics classrooms. Yet these largely one-dimensional efforts do not critically or emphatically address a crucial dimension of the mathematics classroom: the teacher-student *interaction* in terms of valued norms, behaviors, and instructional practice. Teacher beliefs about mathematics and who can do it, and teacher expectations of certain students' mathematics ability, contribute greatly to the opportunities that teachers provide students in their classroom and those students' responses (as measured by academic behaviors, for example) to the presence or absence of opportunities to learn. For example, there is a great deal of evidence that teachers offer less content, and less rigorous content, to underserved students beginning in kindergarten. Certainly, the curricular and organizational mandates designed to enhance instruction are important to examine through the lens of teaching and learning. Teachers largely enact the curriculum, and students respond to how it is enacted.

It is important to note that the courses that are mandated are not taught by automatons; nor does school curricular and organizational policy happen in a vacuum. The courses that are offered, and the level at which they are offered, reflect school leaders' thinking about the students they teach and of what they think those students are capable. The students sitting in these classrooms are not automatons, either. Their academic and non-academic behaviors--or, as importantly, their *perceived* behaviors--affect how teachers structure and deliver curriculum. In short, student access to quality mathematics depends on what the school adults in a system or school think about their students' capacity for learning mathematics. There is substantial evidence that **many teachers** don't think very highly of the capacity of urban and low income, ethnic minority students' ability to do well in mathematics. When teachers say "Oh, I could do that problem with my advanced kids but not my low kids", or "Just do the first 10 problems—they're the easiest", or "We're not going to cover proofs in this geometry class"[7] they are making critical decisions about the mathematics content that their students will receive. They are making critical decisions about their students!

As the achievement gap between Black, Latino/a, and Native students and Asian-American and White students widens the longer they are in the school, it is clear that teachers have a critical role to play. Working with high school teachers to raise their expectations of students' mathematics abilities is an important step in improving mathematics opportunity and outcomes for underserved students. These expectations affect teachers' instruction and curricular choices in ways that often go undocumented but students note. An example:

Several New York City high school students, participating in a youth activism project, reported that their teacher slept during their mathematics class and read the newspaper. 'We are really concerned about the Regents[8]', one said, 'but we don't know what to do. We talked to the principal but nothing happened'. 'I just don't think she wants to teach us', said another. School district officials present during the session suggested that the students tell their parents, and have the parents complain to the principal, and the superintendent. This session took place in April, nearing the end of the school year.

Examining instructional practices, and the often unspoken statements which give rise to them, is critically important. These practices and beliefs speak volumes about who we think can do mathematics, and only by addressing them can we address educators' roles in perpetuating Black, Latino/a, and Native American underrepresentation among mathematically proficient students. I present

[7] These are all comments I have heard in my years as a mathematics teacher and educator.
[8] The Regents examinations are high-stakes tests that students take in New York State to determine eligibility for the prestigious Regents high school diploma.

another **example** from my own research that **illustrates** how such practices can adversely affect equitable mathematics education.[9]

In one urban school district, students are tested to determine entrance into honors mathematics courses. Although there is a substantial Spanish-speaking population, students who enter the district and speak only Spanish are expected to enroll in a 'transition' general-level mathematics class. Further, they are not allowed to take the entrance exam in Spanish.

This example, and many others shared in my research[10], are rooted in narrow beliefs about the potential for and necessity of high mathematical performance of underserved students in mathematics. It is common practice in many school districts. The implicit assumption that all students who enter school from Spanish-speaking countries need remediation in mathematics does not allow for the possibility that one could be Spanish-speaking and mathematically proficient. It reflects that these students' teachers and administrators think less of them. Not allowing students to test in their language to determine course placement, particularly for mathematics, seems to ensure that these students are consigned to show poor performance in mathematics. For placement purposes, it is perfectly logical to test entering students in their native languages to determine their mathematical abilities, without the confounding element of testing their English also. Most disturbing is that this example, as do many school, administrator, and teacher practices, can have cumulative and long-lasting effects. Although there are many examples of students of color who have persisted in mathematics despite such discouraging experiences, these kinds of obstacles must be removed.

Various institutional issues related to equity (funding disparities and teacher shortages) often result in urban school students being taught mathematics by teachers who are less qualified and more inexperienced than those who teach in suburban schools. Thus, urban school students (who are predominantly Black and/or Latino/a) often receive mathematics instruction centered on basic skills and repetition, rather than instruction that provides them with opportunities to learn and exercise higher-order thinking skills. When computers are present in their schools, for example, they may be more likely to be used for basic skills rather than for mathematics exploration or enrichment. Although learning basic skills is necessary, this should not be the upper limit of what is expected from Black, Latino/a, and/or Native students. Regardless of the curriculum in place, teachers make decisions every day that affect what kind of mathematics their

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[9] Elements of this section appeared in Walker, E.N. (2003). Who can do mathematics? In B. Vogeli and A. Karp (Eds.), *Activating mathematical talent,* Monograph Series for Leaders in Mathematics Education. Boston: Houghton Mifflin and National Council of Supervisors of Mathematics.

[10] See, for example, Walker's *Building Mathematics Learning Communities: Improving Outcomes in Urban High Schools,* published by Teachers College Press in 2012.

students receive. If they think their students 'deserve' or are entitled to quality math instruction, or rote repetitive tasks, then that's what their students will receive. This "pedagogy of poverty" (Haberman, 1991) that many teachers practice can hamper their own development as quality mathematics teachers for all students and adversely affect the performance of their students.

Excellence in mathematics instruction requires attention to both *macro* curriculum and organizational issues as well as *micro* teaching and learning issues. Excellent teaching has never been solely defined as the ability to work well with students who have had every advantage. The true test of good teaching should be reflected in a teacher who can—despite students having received poor instruction before; despite what said teacher and others may see as limitations in their home lives, despite what is seen as a lack of motivation—teach students in such a way that they excel in mathematics. There should not be the expectation of a 'magic bullet' curriculum—again, recognizing that teachers and students are active agents in the instructional process. We must not expect that every good teacher teach in from the same textbook in the exact same style. Good teaching for underserved students can be done and has been done: there is a great deal of research that identifies schools and practices that demonstrate the power of equitable and effective math pedagogy and its positive impact on student learning and achievement. We need to highlight schools that do this, examine carefully the curriculum, organization, pedagogy, and instruction that occur, and use this information to improve outcomes for underserved students. We should also note that for years it was believed that girls could not do mathematics: they did not score particularly well on standardized tests and did not take high level courses at the same rates as boys—but with some attention paid to social and cultural issues and expectations, girls now take and achieve in mathematics courses at similar rates to boys.[11]

Teachers themselves are products of societal messages about mathematics and competing schools of thought about how it should be taught. In the United States, there is a prevalent view that people who do well in mathematics do so 'naturally', without effort. Consequently, unlike other disciplines that we believe require hard work—good writing can be developed, for example—our societal emphasis on mathematics as a difficult subject in which we expect few people to do well hampers our development of mathematically gifted, and I would argue, mathematically proficient people. We accept underachievement in mathematics by all as a natural state of affairs unlike the prevailing view in other countries that expect all students to "master a level of mathematical understanding equivalent to that attained by only our best students". In the United States, there

[11] This point is considerably more complicated than outlined here.

is a prevalent idea that mathematics is a completely solitary enterprise, done in the absence of any community. These are dominant stereotypes, as Burton (1999) describes: "a false social stereotype, promoted and reinforced by the media, of the (male) mathematician, locked away in an attic room, scribbling on his (sic) whiteboard and, possibly, solving Fermat's Last Theorem" (p. 127). These notions of mathematics, and who does it, are disseminated to American school students at an early age. Substantial research in mathematics education reveals that both elementary and secondary teachers and students share limited notions of mathematics, and further, narrow ideas about who mathematicians are and the work that they do (Cirillo & Herbel-Eisenmann, 2011; Moreau, Mendick & Epstein, 2009).

We also have to recognize that in the current mathematics reform climate teachers are being required to change their mathematics instructional practice, in some cases drastically. This requires a major paradigm shift on the part of most teachers, because it is wholly different from the ways in which most of them were taught. Quality mathematics has only recently become the supposed school site of democratic practice; until very recently, it was considered to be the purview of the 'elite' students who were exceptional—and at the very least, going to college.

When we consider issues related to curriculum, instruction, and assessment, it is imperative that we note that teachers' beliefs, knowledge, and attitudes about the subject matter and how to teach it are filtered through their beliefs about students and their potential. Further they are rooted in complex historical, political, and cultural contexts. Providing equitable and effective mathematics education for students will require that all of us—researchers, educators, policymakers, teachers, parents, administrators—consider, examine, and address the embedded relationships between what is done in the classroom and our expectations of students, their performance, and their possibilities. Without this work, we will continue to enact piecemeal solutions to a complex problem; and equity will continue to elude underserved students.

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The Dilemma of Advanced Mathematics: Instructional Approaches for Secondary Mathematics Teacher Education

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Mathematics teacher education, particularly at the secondary level, has long since wrestled with the kinds of mathematics courses and content knowledge that should be required of teachers. One of the longstanding debates has been around depth versus breadth. In particular, should a prospective secondary mathematics teacher take a full range of advanced mathematics courses (i.e., similar to other mathematics majors), extending their *breadth* of knowledge, or should they take courses that explicitly focus on the mathematics they will teach, extending their *depth* of knowledge (i.e., secondary mathematics)?

There are compelling arguments on both sides – yet also a dearth of convincing research. For example, some would argue that advanced mathematics is essential for secondary teachers because these ideas underpin and help explain the content of secondary mathematics. This is not untrue; for example, much of the mathematics covered in secondary schools is an instantiation of the algebraic structures studied in abstract algebra (e.g., arithmetic operations with polynomials form a *ring* ($R[x]$,+,×)). Yet there is little evidence that completing these courses influences these future teachers' instruction in a perceivable manner (e.g., Zazkis & Leikin, 2010) or improves their students' subsequent achievement (e.g., Darling-Hammond, 2000; Monk, 1994). Indeed, more recent efforts to conceptualize practice-based approaches to teacher knowledge (e.g., Ball, Thames, & Phelps, 2008; Rowland, Huckstep & Thwaites, 2005) present a tension with respect to advanced mathematics: it represents content that should not end up being explicitly discussed with secondary students – teachers should be teaching algebra, not abstract algebra – but yet it should simultaneously be influential on teachers' instructional approach and the work that they engage in as secondary educators. And yet only studying the mathematics that one will teach is also imprudent. Both mathematicians and mathematics educators still

espouse the value in studying content beyond what one teaches (e.g., CBMS 2001, 2012); indeed, even practice-based frameworks for teacher knowledge include domains that support knowing mathematics beyond what one will teach (e.g., Ball, Thames, and Phelps' (2008) horizon content knowledge; McCrory, et al.'s (2012) advanced mathematics). So with the tension regarding the utility of advanced mathematics in secondary mathematics teacher education unresolved, we turn instead to instruction in such courses.

In this paper, I explicitly address the issue of advanced mathematics courses in secondary mathematics teacher education – which is related to the broader conversation about content knowledge beyond what one is going to teach. In particular, both because the field still values such knowledge for secondary teachers and because secondary teacher certification requirements are frequently linked to completing the equivalent of a mathematics major, this paper considers *instructional approaches* in advanced mathematics courses that have the intent of being meaningful experiences and valuable preparation for secondary teachers. In particular, we address two such instructional approaches: i) an instructional model that connects advanced mathematics not just to the *content* of secondary mathematics but to the *teaching* of secondary mathematics; and ii) instruction that also models good teaching of mathematics. Now, we do mention that advanced mathematics courses designed with a specific audience of secondary teachers are, likely, impractical in many places – there are neither enough students nor instructors to fill such sections; and although I regard the two instructional approaches in this paper as decidedly specific to an audience of teachers, it could be the case that what is useful for prospective mathematics teachers is, in fact, useful for other mathematics students as well.

Connecting advanced mathematics to the *teaching* of secondary mathematics

Recent efforts to describe the mathematical knowledge needed for and used in teaching have drawn on a practice-based conception of knowledge (e.g., Ball, Thames, & Phelps, 2008; McCrory, et al., 2012; Rowland, Huckstep, & Thwaites, 2005). That is, the mathematical knowledge teachers learn should be relevant for and based in teaching – a subset of mathematics that matches the work and practice of teaching and is relevant for activities such as explaining concepts, designing tasks, and questioning, understanding, and accessing students' thinking. As mentioned, situating advanced mathematics in this perspective has some inherent tensions – which is evident in the "provisional" status of the domain of horizon content knowledge in Ball, Thames, and Phelps' (2008) mathematical knowledge for teaching framework. However, we do not view these tensions as unresolvable. Before we elaborate on our instructional model, we briefly differentiate it from another commonly proposed solution: connecting advanced mathematics to the content of secondary mathematics.

Many would regard an instructional approach in advanced mathematics that simply makes more explicit the connections to secondary mathematics as sufficient. Perhaps the first to popularize this idea was Felix Klein, who wrote *Elementary mathematics from an advanced standpoint* (1932). His aim was to demonstrate ways that advanced mathematics is connected to more elementary mathematics. Indeed, the Conference Board of Mathematical Sciences' *Mathematical Education of Teachers II* (CBMS, 2012) advocates a similar position of applying more advanced mathematics to the content that the teacher will be teaching: for example, it would be "quite useful for prospective [secondary] teachers to see how $\boldsymbol{C}$ can be 'built' as a quotient of $\boldsymbol{R}[x]$… [and] Cardano's method, and the algorithm for solving quartics by radicals can all be developed… as a preview to Galois theory" (p. 59). Cuoco (2001) summarizes a principle for redesigning the undergraduate experience of prospective teachers this way: "Make connections to school mathematics" (p. 170). At the heart of this perspective is a desire to make more advanced mathematical study related to what a teacher is going to teach. Now, this can certainly be a useful approach; however, we regard this approach as not going far enough. Indeed, both the general argument – that by the simple merit of some advanced topic (e.g., Galois Theory) being related to the content of school mathematics that such knowledge is important for teachers – and the implicit hope that accompany it – that as a *byproduct* of learning advanced mathematical content teachers will respond differently to instructional situations in the future – are tenuous. We do not presume such a "trickle down" effect to teaching (Figure 1a).

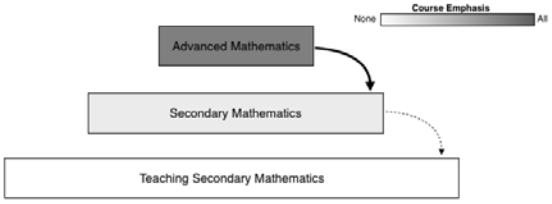

Figure 1a. Implicit model for advanced mathematics courses designed for teachers

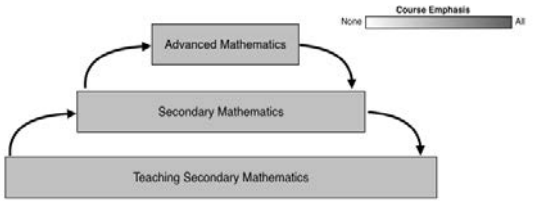

Figure 1b. Our model for advanced mathematics courses designed for teachers

Instead, my colleagues and I (e.g., Wasserman, Fukawa-Connelly, Mejia-Ramos, & Weber, 2016) have proposed an alternative instructional model that connects advanced mathematics to the teaching of secondary mathematics. This instructional approach leverages notions of situated cognition (Powell & Hanna, 2006; Ticknor, 2012) and contends that teachers will best develop resources for teaching if they are *learned in connection to the context of their pedagogical practice*. Our instructional model, illustrated in Figure 1b, is composed of two parts: building up from practice and stepping down to practice. *Building up from*

(teaching) practice involves designing instruction that starts not from advanced mathematics content or secondary content, but from practical school teaching situations – specific things that secondary teachers need to do as part of their professional work. Each situation has specific pedagogical goals as well as mathematical aims. And the specific pedagogical goals and mathematical aims from each school teaching situation are particularly well-suited to being learned in the advanced mathematics course. The second part, *stepping down to (teaching) practice*, then uses these ideas from advanced mathematics as a means to reconsider the related pedagogical situations, clarifying the intended mathematical and pedagogical aims. The advanced mathematics topics are covered with typical formal and rigorous treatment, but the tasks make explicit what connections these have for both secondary mathematics and its teaching. Essentially, by more tightly building from and connecting to professional practice, all three facets – advanced mathematics, secondary mathematics, and pedagogical practice – can be made explicit during instruction in ways that aim to improve the teaching of advanced mathematics in secondary mathematics teacher education.

Algebraic Limit Theorems for Sequences

I provide an example of one such task, specifically designed for a real analysis course for secondary teachers, that aligns with this instructional approach. The algebraic limit theorems for sequences – that two convergent sequences can be added, multiplied, etc., and still maintain convergence – are an essential part of a real analysis course. Indeed, the proofs of convergence in these theorems are based on the typical ε-N definition, which itself can be a challenge. At their core, however, issues about sequence convergence are about error – and we relate these ideas to issues that arise in secondary mathematics teaching of rounding real numbers. In terms of real analysis, the mathematical aim of this task is for students to prove the algebraic limit theorems for sequences; in terms of secondary mathematics, the aim of this task is for students to understand some basic ideas about how operating with rounded numbers accumulates error. In order to specify the pedagogical aim of this lesson, we first conceptualized some related *principles of good mathematics instruction*. The two that are pertinent here are: i) good teaching clarifies mathematical limitations in students' mathematical statements or arguments; and ii) good teaching selects examples that exemplify nuances within and boundaries around a mathematical idea. Following our instructional model, we now present the real analysis task, interweaving discussion about the particular mathematical and pedagogical aims.

Building up from teaching practice. Rather than begin with the real analysis content of the algebraic limit theorems, our instructional model incorporates a classroom teaching situation. In this classroom situation, secondary students have been given a problem about the perimeter and area of a rectangle (depicted in Figure 2). One student approaches the teacher: "My calculator says

$\sqrt{75} = 8.66025404$ and $\sqrt{362} = 19.02629759$. Is it okay if I round them to 8.66 and 19.02?" Looking at her answer key, the teacher notes that she, herself, rounded the final answer to 55.37 units for perimeter and 164.77 units^2 for area. She wonders: "How accurate would someone need to round the square roots so that their final answer was accurate to the hundredths, like my answer key?" The situation ends, leaving the teacher's response to the student unspecified. The prospective secondary mathematics teachers (PSMTs) in the real analysis course are then prompted to consider, first, their initial pedagogical response and, second, to explore some of the mathematics in this particular problem. Notably, using a truncating (rather than rounding) procedure, the student in the classroom situation would need to use an approximation accurate to the thousandths (8.660 and 19.026) to get a perimeter accurate to the hundredths (55.37), and an approximation accurate to the ten-thousandths (8.6602 and 19.0262) to get an area accurate to the hundredths (164.77). PSMTs then explore some of the mathematics of how error accumulates in perimeter and area problems. Namely, if a_k is an approximation of $\sqrt{75}$, and b_k is an approximation of $\sqrt{362}$, both rounded to the same decimal, then for some error, e, we have:

$\sqrt{75} - e < a_k < \sqrt{75} + e$ and $\sqrt{362} - e < b_k < \sqrt{362} + e$. Presuming e to be small enough so that all values are positive (and relatively small enough so that e^2 becomes negligible), we come to the following expressions: $2(\sqrt{75} + \sqrt{362}) - 4e < 2(a_k + b_k) < 2(\sqrt{75} + \sqrt{362}) + 4e$, and

$\sqrt{75} \cdot \sqrt{362} - e(\sqrt{75} + \sqrt{362}) + e^2 < a_k b_k < \sqrt{75} \cdot \sqrt{362} + e(\sqrt{75} + \sqrt{362}) + e^2$. That is, the error in the perimeter estimate is, at most, 4 times the original error e, and the error in the area estimate is, at most, a little less than 28 times $(\sqrt{75} + \sqrt{362})$ the original error e. Notably, in the perimeter estimate which sums the estimates, the error at most sums for each of the four side lengths; however, in the area estimate which takes the product, the error gets scaled by a factor that depends on the original values of a and b.

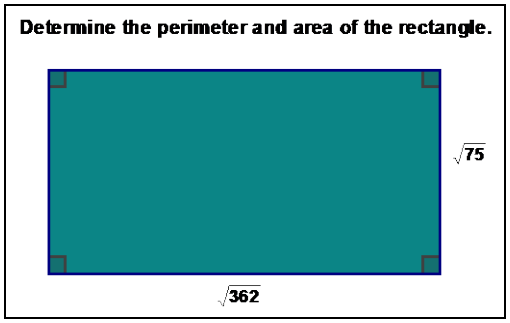

Figure 2. Problem in the classroom teaching situation

Real Analysis. At this point, the real analysis instruction with the PSMTs delves into proofs of the algebraic limit theorems. In particular, we include the two most relevant proofs in Table 1 (which presume $a_n \to a$ and $b_n \to b$).

Table 1. *Proofs of the sum and product algebraic limit theorems*

$a_n + b_n \to a + b$	$a_n b_n \to ab$
Proof. Let $\varepsilon > 0$. For all n, $\lvert(a_n + b_n) - (a + b)\rvert =$ $\lvert(a_n - a) + (b_n - b)\rvert \le \lvert a_n - a\rvert + \lvert b_n - b\rvert$. Since $a_n \to a$, there exists an N_1 such that for all $n \ge N_1$, $\lvert a_n - a\rvert < \frac{\varepsilon}{2}$. Since $b_n \to b$, there exists an N_2 such that for all $n \ge N_2$, $\lvert b_n - b\rvert < \frac{\varepsilon}{2}$. Therefore, for $n \ge \max[N_1, N_2]$, $\lvert(a_n + b_n) - (a + b)\rvert \le \lvert a_n - a\rvert + \lvert b_n - b\rvert < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$. So for any $\varepsilon > 0$, the sequence $(a_n + b_n)$ is within ε of $a + b$ (for $n \ge \max[N_1, N_2]$).	Proof. Let $\varepsilon > 0$. For all n, $\lvert a_n b_n - ab\rvert = \lvert a_n b_n - ab_n + ab_n - ab\rvert \le \lvert a_n b_n - ab_n\rvert + \lvert ab_n - ab\rvert =$ $\lvert b_n\rvert\lvert a_n - a\rvert + \lvert a\rvert\lvert b_n - b\rvert$. Since $b_n \to b$, there exists an N_2 such that for all $n \ge N_2$, $\lvert b_n - b\rvert < \frac{\varepsilon}{2\lvert a\rvert}$. Since every convergent sequence is bounded, there is an M such that for all n, $\lvert b_n\rvert \le M$. Since $a_n \to a$, there exists an N_1 such that for all $n \ge N_1$, $\lvert a_n - a\rvert < \frac{\varepsilon}{2\lvert M\rvert}$. Let $N = \max[N_1, N_2]$. Then, for all $n \ge N$, $\lvert a_n b_n - ab\rvert \le \lvert M\rvert\lvert a_n - a\rvert + \lvert a\rvert\lvert b_n - b\rvert <$ $\lvert M\rvert\frac{\varepsilon}{2\lvert M\rvert} + \lvert a\rvert\frac{\varepsilon}{2\lvert a\rvert} = \varepsilon$.

Stepping down to teaching practice. Based on some of the conclusions about how error accumulated in the classroom teaching scenario, PSMTs are next asked to consider some more general conclusions that can be seen in the proofs about how error grows. In particular, it can be seen from the proofs that when adding two approximations (a_k and b_k), the error, at most, adds (the statement that $|(a_n+b_n)-(a+b)| \le |a_n-a| + |b_n-b|$ is one indication of this); similarly, the proofs also indicate that when taking the product of two approximations (a_k and b_k), the error gets scaled by a factor of the sum of the values, a and b (the statement that $|a_n b_n - ab)| \le |b_n|\,|a_n-a| + |a|\,|b_n-b|$ is one indication of this). In terms of re-considering the original classroom scenario, it now becomes clearer that if the original error for both approximated numbers is less than e, that in perimeter problems the error would be no more than $4e$, while in area problems the error would be no more than $e(a+b)$. If one would like the estimate for perimeter to be accurate to the hundredths place (.01), then the estimates for the side lengths should at least be within one-fourth of this (.0025) – or, accurate to the thousandths place; whereas for area to be accurate to the hundredths place (.01), then the estimates for the side lengths should at least be within about one-twenty eighth of this (~.000361) – or, accurate to the ten-thousandths place. In other words, these general rules for understanding how error accumulates – which the proofs of the algebraic limit theorems for sequences help convey – becomes useful knowledge for the teacher to be able to respond to questions from

students about rounding. Notably, the teacher acquires a sense that regardless of the initial values, the potential error in perimeter (or adding) problems remains a constant, whereas in area (or multiplying) problems, the potential error depends on the values of the initial side lengths.

As a part of stepping down to practice, other similar pedagogical situations are discussed. For example, a student sets up and solves the equation $\sin(59^\circ) = x/4\sqrt{5}$, by using her calculator and doing the following: $0.85 = x/8.94$, so $x = 0.85 \cdot 8.94 = 7.599$. The teacher tells the student not to round until the end; the student objects to this, however, noting that their answer in this problem is, in fact, very close to the actual answer. PSMTs are then asked to consider, how might one change the original equation to help exemplify to the student some of the potential issues with rounding in this manner. Notably, the pedagogical situations are intended to help students develop particular pedagogical aptitudes. In this example, namely, of being able to recognize that the student's method or argument in this case *can* be problematic (even if in the current problem it is not all that much of an issue), and also being able to craft examples that help *exemplify* to the student some of the potential issues. Indeed, the key to being able to accomplish these pedagogical objectives – of recognizing and exemplifying the potential issue in the student's approach – lies in understanding how error accumulates when multiplying two rounded numbers. Since the error in the product of two estimates grows as the sum of the original two values, perhaps the easiest way to highlight the issue is to make one of the numbers larger – e.g., use $360\sqrt{5}$ instead of $4\sqrt{5}$ in the equation. (In this case, the error in the answer using the student's approach ends up being close to 6 units off – a fairly sizeable amount given initial the rounding of each number to the hundredths.) In general, we argue that our instructional model has helped to situate the real analysis content of algebraic limit theorems in a classroom-based example, and in particular ways whereby both mathematical and pedagogical aims are mutually developed and reinforced.

Modeling good teaching of mathematics

As a second instructional approach in advanced mathematics courses, in addition to our model of building up from and stepping down to teaching practice, we explore modeling good teaching of mathematics. In other words, the ways in which advanced mathematics instructors conduct their classes should model and resemble, broadly, good instructional practices in mathematics education. An age-old adage is that "we teach how we were taught." In this respect, providing PSMTs with an opportunity to *learn* mathematics in ways that model good instruction sets them up to *teach* in ways that leverage such good instructional practices. Now, to clarify, there are many good instructional approaches, and I cannot and will not address them all – in fact, I will only address one in this paper: the use of technology. Technology can be a powerful

tool in education; however, it is not a panacea – there is nothing magical about it or intrinsically good within it. Rather, technology becomes powerful as educated teachers succeed in using it as a tool to enhance student learning and promote student reasoning. In this paper, I elaborate on only one example of the use of technology – dynamic technology – and provide an example, again, from a real analysis course.

Dynamic technology in mathematics – such as Geometer's Sketchpad, GeoGebra, Fathom, TinkerPlots, etc. – has had a profound effect on mathematics education. Generally, the research on such software indicates that using such technologies can enhance learning (e.g., Battista, 2007; Sträβer, 2002). The use of the term "dynamic technology" is intended to convey that: i) manipulation is direct (i.e., a user points at a surface and drags it); ii) motion is continuous (i.e., change happens continuously in real time); and iii) the environment is immersive (i.e., the interface is minimally intrusive so that users feel as though they are interacting with the objects) (e.g., Finzer & Jackiw, 1998). In terms of secondary mathematics, dynamic geometry software, for instance, allows students to construct a rectangle, and, by dragging one corner, produce thousands of different rectangles, providing a live interaction between the student and the two-dimensional object. The interface allows students to see, amongst other things, what properties hold for these thousands of cases; in this situation, students may notice that the lengths of the two diagonals of the rectangle are always congruent. The dynamic nature of the technology allows students to use inductive reasoning to help form and justify conclusions about two-dimensional figures, which can also provide insight into particular aspects that are influential for maintaining these properties, fostering intuition about how to go about deductively proving the claim.

Before I move into an example of modeling the use of dynamic technology in real analysis instruction, I elaborate on a few other specific orientations about the intent of modeling instruction with dynamic technology. First, an advanced mathematics course is not the opportunity to teach PSMTs about a particular technology, helping them learn how to use specific features, etc.; rather, it is meant as an opportunity for them, as *students*, to experience learning mathematics from the use of dynamic technology. Second, the mathematics that the students are learning in this context is advanced mathematics – not secondary mathematics. This means that the use of dynamic technology must be adapted in a way befitting of the content to be learned. In the context of real analysis, which is a rigorous proof-based mathematics course about real numbers, real-valued functions, etc., the use of the software must be productive for and enhance their learning of this content. Specifically, we will discuss what I term the use of *dynamic proof visualizations*. Third, my particular perspective about dynamic technology is that, pedagogically, they are most powerful when students (not instructors) are intended to interact with them. That is, although

dynamic technologies could be used simply as a visual accompaniment to an instructor's lecture, my intent is to model instruction that allows students to interact with them on their own. Of course, proper guidance and scaffolding by an instructor is always an important component of helping students be mathematically productive in their explorations.

Dynamic Proof Visualizations in Real Analysis

Real analysis deals with the nature of real numbers and real-valued functions, which inherently contain notions of infinity. For example, we use (infinite) sequences to help us understand the behavior of real numbers; arbitrarily small ε values are indicative of (infinitely) small domains of functions; and infinite processes are frequently employed in the construction of various proofs. Indeed, one might say that real analysis takes "static" objects – such as real numbers or functions – and makes them "dynamic," by conceptualizing them as infinite sequences, or the coordination of infinite sequences. Thus, the use of dynamic technology to help visualize concepts may be particularly productive in real analysis. In addition, however, much of the aim of a real analysis course is proof-related (in fact, this is true for many advanced mathematics courses). As such, I elaborate on one particular use of dynamic technology: *dynamic proof visualizations*.

Although visuals, at times, can be a barrier to proof (i.e., reliance on pictures that make additional presumptions about the given context), they are also frequently essential to providing mathematical insight. One of the inherent difficulties with proof lies in being able to link the *general* claim being made to the (various) *specific* examples that both instantiate and substantiate (or refute) the claim. Dynamic technology provides an opportunity not only to visualize the general and the specific, but to interact with them. So, in the context of the real analysis course, *dynamic proof visualizations* were constructed using the tools of GeoGebra. A few things guided this process. First, the aim of a dynamic proof visualization is not just convincing a student about the truth or falsehood of a *claim*, but rather intentionally mimicking some of a proof's processes, arguments, ideas, etc., in a manner intended to foster insight and comprehension about the *proof of a claim*. Second, because the goal is intended to tie in with proof, the "givens" of a claim are both explicit within the pre-constructed environment and can be altered without disrupting the argument. Third, the particular processes or arguments in a proof can be repeated and rerun on the specific example, providing an opportunity to visualize, dynamically, a specific instantiation of the process or argument while also exploring the generality of the claim. In particular, changing the "givens" allows students to repeat the argument and explore some of the nuances of a proof, especially why and in what ways the argument may fail with or without the given constraints. Last, as mentioned previously, the pre-constructed environments have been intentionally created to allow students to interact with them.

Algebraic Limit Theorems for Sequences

In what follows, I provide an example of one such dynamic proof visualization, specifically designed for a real analysis course for secondary teachers, that aligns with this instructional approach of modeling good mathematics instruction with dynamic technology. Notably, I will use the same mathematical context as the previous example: algebraic limit theorems for sequences. As mentioned, proofs of the algebraic properties of limits for sequences are essential in a real analysis course. In what follows, we elaborate on a dynamic proof visualization of one of these – the sum property for limits (see Table 1). In an analysis class, this theorem of course depends on a definition for sequence convergence, something equivalent to: *A sequence (a_n) converges to a real number a if, for every $\varepsilon > 0$, there exists a natural number N such that whenever $n \geq N$ it follows that $|a_n - a| < \varepsilon$.* Presuming students to be both familiar with this definition of convergence and also knowledgeable about what it means visually, the proof of the algebraic limit theorem still demands an ability to coordinate and separate multiple uses of this definition on three distinct sequences. First, the proof requires understanding the given information, particularly as it relates to an ability to choose any particular epsilon value, $\varepsilon > 0$, and be guaranteed that at some point the sequence will permanently enter that ε-neighborhood. In the familiar style of an "ε-N challenge," the target ε value is only linked to the sequence (a_n+b_n), but otherwise simply represents some specific number that can be manipulated. Second, in connection to the previous discussion, the use of the triangle inequality in the proof, $\left|\left(a_n+b_n\right)-(a+b)\right| \leq \left|a_n-a\right|+\left|b_n-b\right|$, conveys a statement about how the error for a sum of two sequences is no worse than the sum of the errors for the two individual sequences. Third, in addition to having to negotiate various meanings and uses of ε in the definition of convergence, the proof also requires coordinating the meaning for different N values (i.e., N_1 and N_2) – which can get reasonably large – as well as their relationship to the desired value of N. In all of these potentially difficult aspects of the proof, the dynamic proof visualization fosters additional insight.

An initial look at the dynamic proof visualization. Figure 3 depicts the pre-constructed GeoGebra file for the dynamic proof visualization. The file has been created so as to mimic the so-called ε-N challenge and response associated with using the definition to prove convergence of the sequence (a_n+b_n). First, the specific givens have been included – the sequences (a_n) and (b_n), which in this sketch are limited to explicit formulas in terms of n, and their respective limits a and b. Each of these can be modified. The first few terms of each sequence and the differences of their last terms from their respective limit values are depicted, so as to be able to tie into the meaning of the triangle inequality in the proof. Also, note an initial random value for ε, which can be changed by either using the slider or, better yet, generating a new random value. The ε–neighborhood is shown as associated with the sequence (a_n+b_n). Students are also prompted to

consider a scalar for ε, which will determine the ε_1- and ε_2-neighborhoods for the sequences (a_n) and (b_n) respectively – the simplest version is ½, but other scalars might also be acceptable. (Initially, this scalar is undefined, but once it is designated, the two ε_1- and ε_2-neighborhoods become visualized – Figure 4a.) Second, a brief description of the claim and the goal are described, and sliders for n_1 and n_2 allow students to dynamically visualize the first terms of each sequence. This provides a dynamic visual for the inherent movement within sequences as they converge (or do not converge) to a specific value. Following the proof, students need to consider a general way, based on what they know about (a_n) and (b_n), to locate a sufficient N for the given ε-challenge for the sequence (a_n+b_n). For the scaled ε_1- and ε_2-neighborhoods, students can physically drag the sliders until the remainder of (a_n) after a term, a_{N1}, is within the ε_1-neighborhood of a and the remainder of (b_n) after a term, b_{N2}, is within the ε_2-neighborhood of b. Initially students can also independently manipulate n; however, for a proof, one would like to determine a sufficient value of N in any situation, based on the values of N_1 and N_2 generated previously – this ends up being, max[N_1, N_2]. By inputting this into the sketch, the slider for n is removed and the first N terms of (a_n+b_n) are automatically generated based on the values of N_1 and N_2 (Figure 4a). By updating ε for a few specific cases, and manually adjusting N_1 and N_2 (the given information guarantees their existence), students can verify that their term for N is, indeed, sufficient. In fact, if using (a_n) and (b_n) for which students are familiar, one could also determine and input a value of N_1 as a function of ε_1, and N_2 as a function of ε_2, which would then generalize the proof and update everything accordingly for any newly generated ε-challenge (Figure 4b).

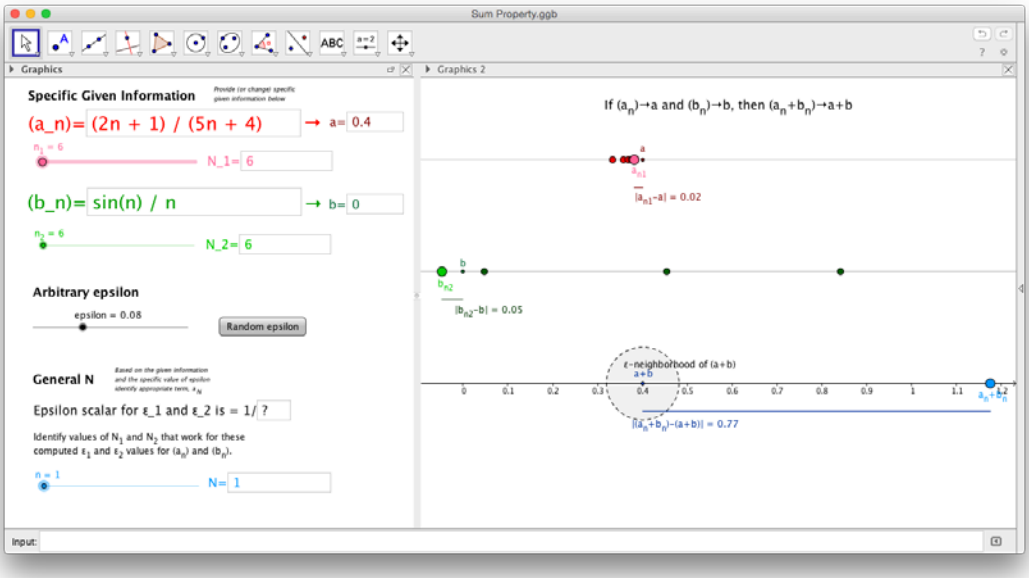

Figure 3. Pre-constructed GeoGebra file for Dynamic proof visualization

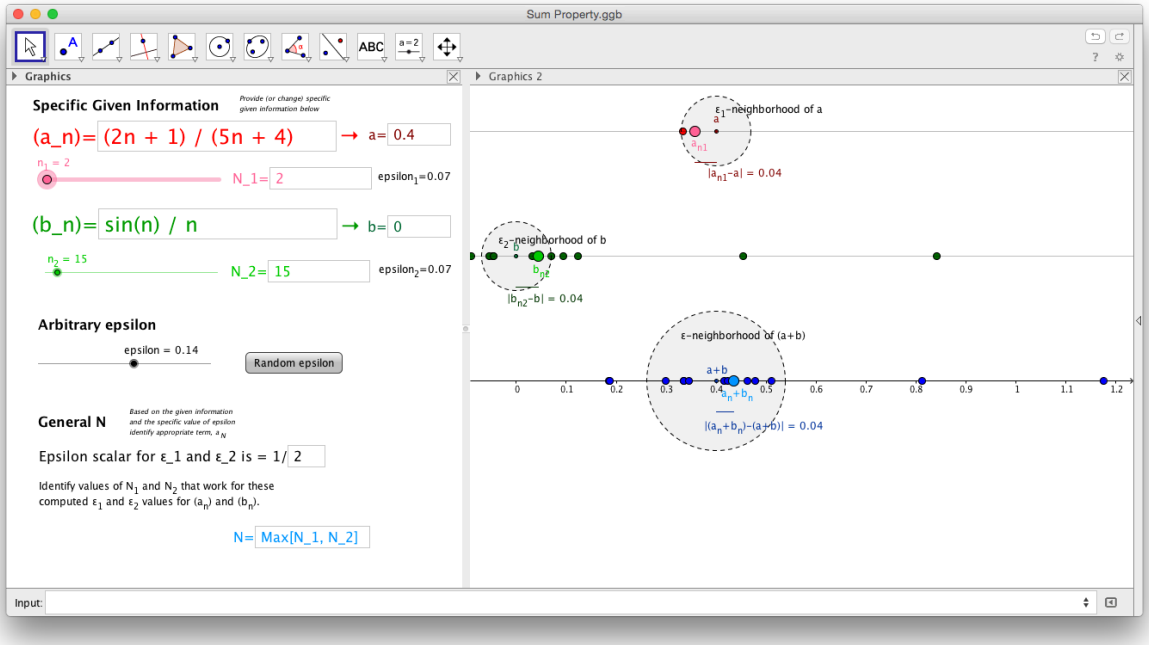

Figure 4a. Setting N as the max of N_1 and N_2.

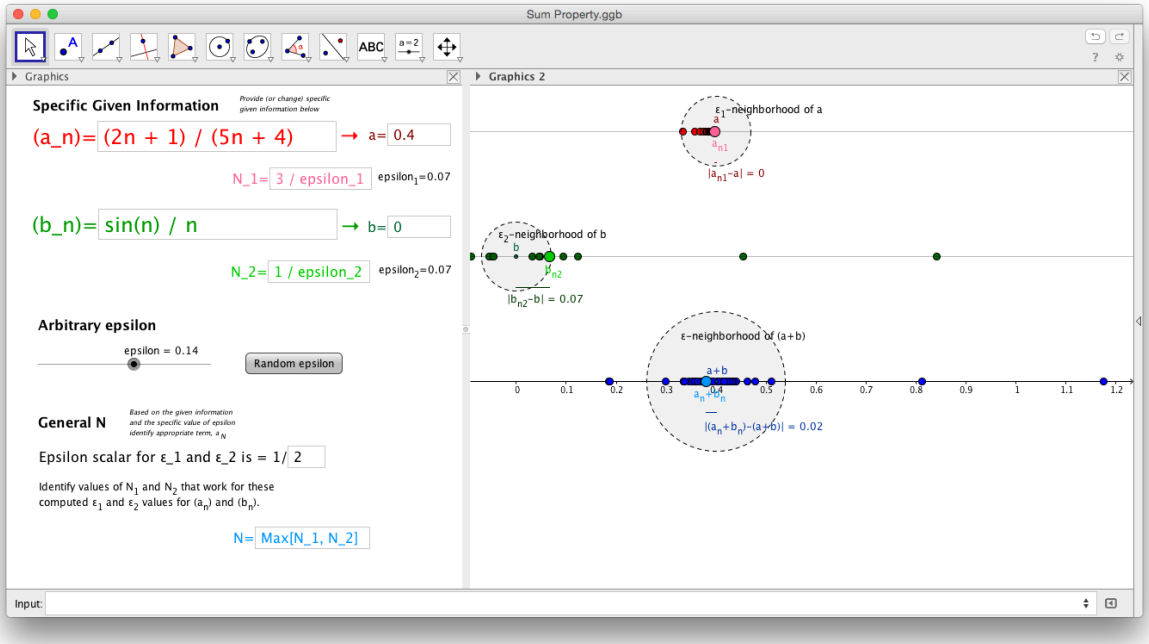

Figure 4b. Setting N_1 and N_2 based on the specific sequence convergence

Extensions to consider the use of a dynamic proof visualization. As mentioned previously, one of the productive things about a dynamic reconstruction of

proving the sum of two convergent sequences converges by the ε-N definition is that it affords exploring some notable nuances that exist in such arguments. For example, what happens if you change the scalar? Although the typical proof uses a scalar of $\varepsilon/2$ for (a_n) and (b_n), this need not be the case. One could, for example, use $\varepsilon/3$ instead. Essentially, this makes the ε_1- and ε_2-neighborhoods even smaller, increasing the value(s) of N_1 and N_2. This only further increases the value of N, which will of course also be sufficient for the ε-neighborhood of (a_n+b_n). (See Figure 5a.) Indeed, the line in the proof only changes slightly, from "$|(a_n+b_n) - (a+b)| < \varepsilon/2+ \varepsilon/2 = \varepsilon$" to "$|(a_n+b_n) - (a+b)| < \varepsilon/3+ \varepsilon/3= 2/3\varepsilon < \varepsilon$." Interestingly, students can also gather further insight by considering scalar values that would *not* result in a valid proof – for example, not changing the value of ε at all (scaling by $\varepsilon/1$). Evident in Figure 5b, both the N_1 and N_2 values are an appropriate response to the (unscaled) ε-challenge ($\varepsilon_1 = \varepsilon_2 = \varepsilon$), but the maximum of these two values is still not a sufficient N value for the sequence (a_n+b_n) – the term (a_N+b_N) is still outside the ε-neighborhood of $a+b$.

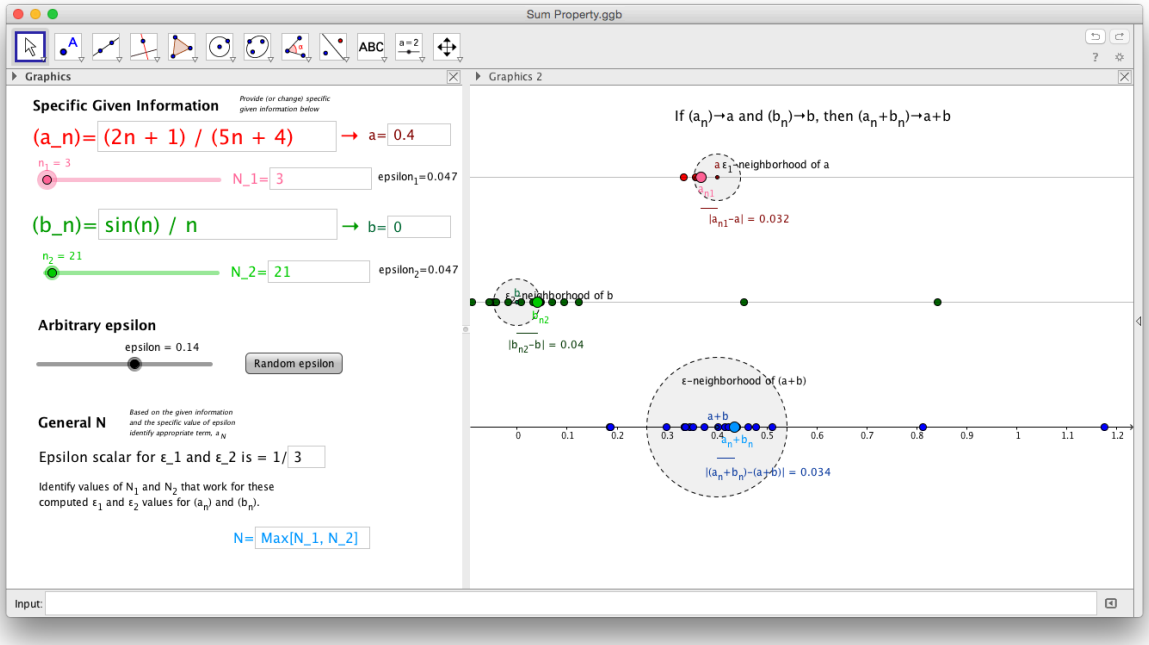

Figure 5a. Setting the scalar to another appropriate value like $\varepsilon/3$

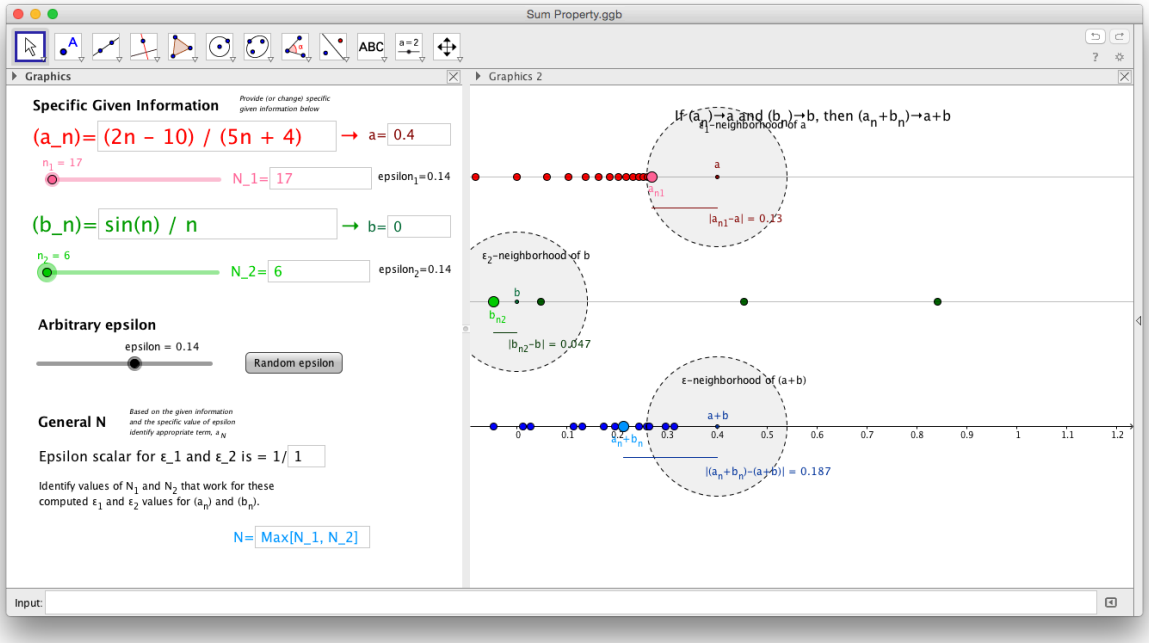

Figure 5b. Setting the scalar to an inappropriate value like $\varepsilon/1$

The dynamic proof visualizations, along with an ability to dynamically interact and vary particular aspects of the setup, has led to important insights about the algebraic limit theorem (for the sum of two sequences) and its proof. It has been utilized to help clarify some of the different aspects of the definition of convergence – namely, the various εs and Ns – and their uses and coordination in the proof. The dynamic nature of a reconstructed proof is powerful for precisely this reason – that one can visualize the coordination amongst all the various sequences under discussion, which can help foster deeper insight into the process of and logic underlying such a proof. For PSMTs, experiencing learning with the use of dynamic technology in ways the foster their own mathematical development can have powerful implications on their future teaching.

Discussion and Conclusion

Although there are multiple unresolved tensions around the inclusion of advanced mathematics courses in secondary mathematics teacher education, this paper has focused on discussing productive instructional practices in such courses (rather than arguing about the merits or deficiencies of the content) – in particular, ones that have some potential for being meaningful experiences and valuable preparation for secondary teachers. Although these may not resolve some of the broader debates, the two instructional approaches described in the paper provide some possibilities for instructors of advanced mathematics with regard to secondary mathematics teacher education.

The instructional model – of building up from and stepping down to teaching practice – is intended to situate the advanced mathematics being learned in contexts whereby such knowledge might be useful in a PSMT's professional practice. Importantly, this model pushes the boundaries – rather that simply connecting the advanced mathematics to the *content* of secondary mathematics, these tasks intend to be insightful for the *teaching* of secondary mathematics. Notably, this includes making particular pedagogical ideas part of the explicit aims of the course. That is, the advanced mathematics course must include not just mathematical aims but also the goal of developing good instructional practices – in reference to the real analysis example in the paper, PSMTs should be able to: i) clarify the mathematical limitations in students' mathematical statements or arguments; and ii) select examples that exemplify nuances within and boundaries around a mathematical idea. Notably, these pedagogical aims were also directly connected to the mathematical aims: it was the proof of the algebraic limit theorems that provided insight into issues of rounding and fostered the ability of PSMTs to exemplify the potential limitations.

In coordination with this instructional approach, which includes situating the real analysis content in relation to a pedagogical situation, we also described one particular way that instruction in real analysis can model good pedagogical use of technology. Indeed, the use of dynamic proof visualizations to teach real analysis content was intentionally intended to mirror some specific practices of using such technology for mathematics instruction more broadly – that the dynamic nature of the interactions is particularly helpful for learning the desired content, and that students (not just instructors) are intended to interact with the technology. This modeling also provides yet another opportunity to be explicit with PSMTs about some of the pedagogical ideas and approaches for using dynamic technology. Notably, this approach can be considered as complementary to the building up from and stepping down to teaching practice idea. If the pedagogical situations frame and bookend the real analysis content, providing a connection to secondary teaching, the use of dynamic technology can be considered a way of modeling good instruction for the real analysis content that is in the middle. That is, although the teaching of the algebraic limit theorems of sequences in the middle of the first task could take on many forms, the notion of modeling good instructional approaches is specific to the way in which the instructor teaches the advanced mathematics content. And in this regard, although it is only one potential way of modeling good instruction, the use of dynamic proof visualizations was intended to not only be productive for learning the desired content, but also was meant as another reference point for PSMT's own thinking about mathematics instruction – based on their experience as a student of learning with dynamic technology.

In conjunction with one another – both situating the advanced mathematics content within pedagogical contexts and also teaching the advanced mathematics

in ways that model good instruction – these two instructional approaches can be leveraged to make the teaching of advanced mathematics course more relevant to secondary mathematics teachers. Indeed, they appear to complement each other, both focusing on different parts of real analysis instruction but in ways that also can make conversations about teaching explicit. While more work certainly needs to be done to improve secondary mathematics teacher education – particularly in the realm of understanding and defining important mathematical knowledge and skills associated with good teaching, which includes considering the role that more advanced mathematics can play in the teaching of more elementary mathematics – the two instructional approaches in this paper provide a perspective on teaching that, in the mean time, at least aims to capitalize on some opportunities in the teaching of advanced mathematics courses to further promote good secondary mathematics instruction and provide meaningful experiences and valuable preparation for secondary teachers.

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