Gauss' Law

Electric Field Lines

The number of field lines, also known as lines of force, are related to strength of the electric field

More appropriately it is the number of field lines crossing through a given surface that is related to the electric field

Flux

How much of something passes through some surface

- **Number of particles passing through a given surface**

Two ways to define

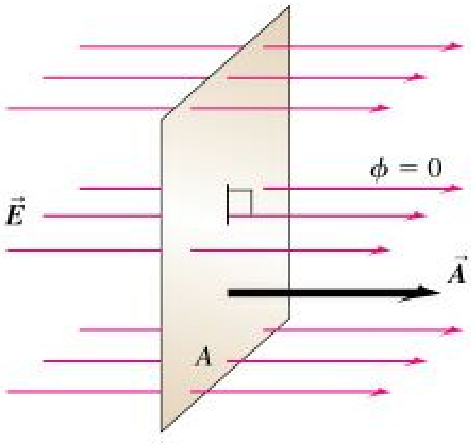
- **Number per unit area (e.g., 10 particles/cm^2)**
- **Number passing through an area of interest**

Electric Flux

The electric flux is defined to be

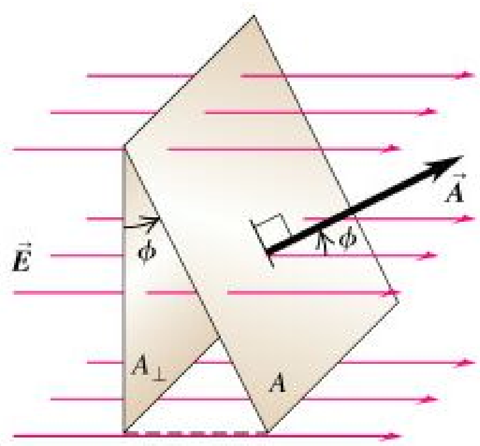
$$\Phi_E = E\,A$$

Where E is the electric field and A is the area



Electric Flux

If surface area is not perpendicular to the electric field we have to slightly change our definition of the flux



$$\Phi_E = E\,A\cos\phi$$

Where φ is the angle between the field and the unit vector that is perpendicular to the surface

Electric Flux

We can see that the relationship between the flux and the electric field and the area vector is just the dot product of two vectors

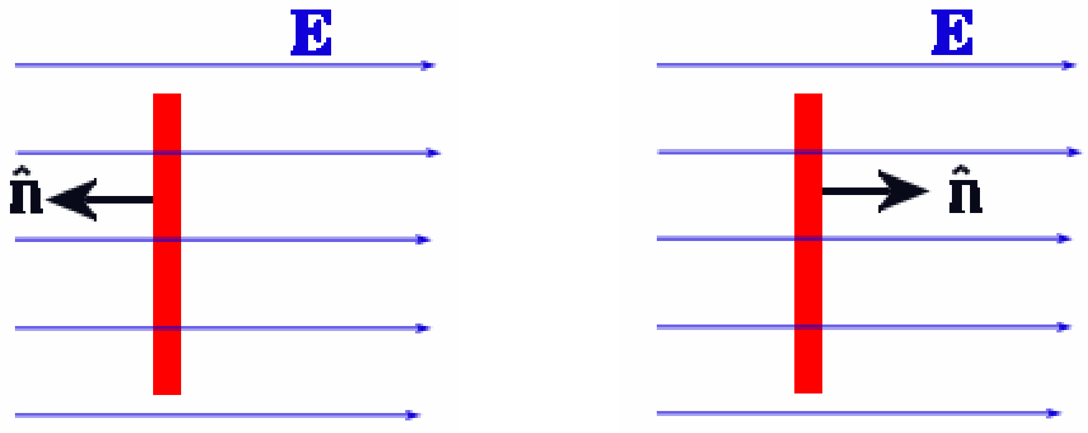
$$\Phi_E = \vec{E} \cdot \vec{A}$$

$$\Phi_E = \vec{E} \cdot A\hat{n}$$

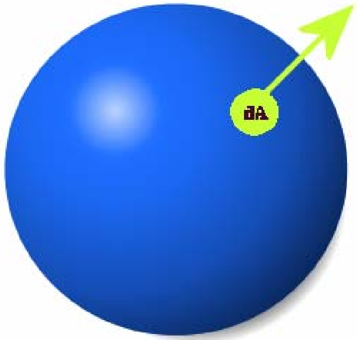
$\hat{n}$ is a unit vector perpendicular to the surface

A Convention

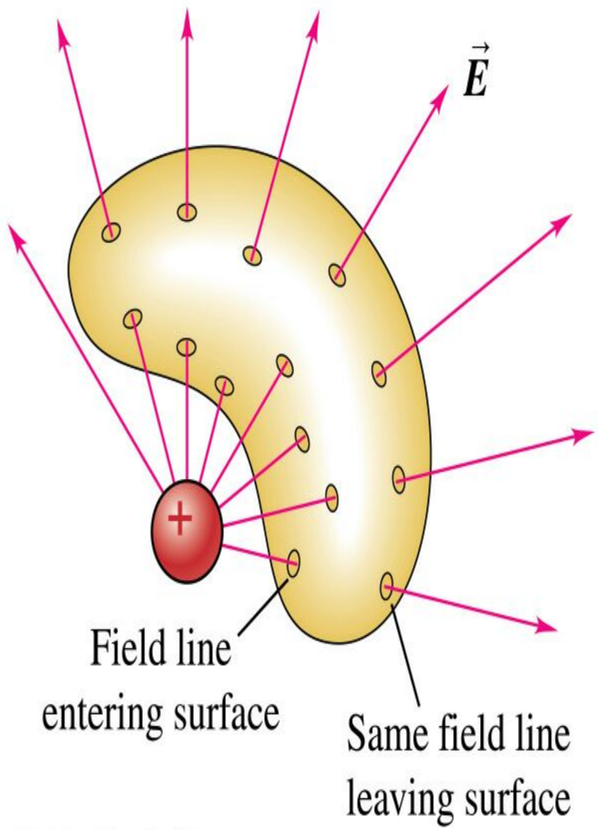
The direction of a unit vector for an open surface is *ambiguous*



For a closed surface, the unit vector is taken as being pointed outward



Electric Flux

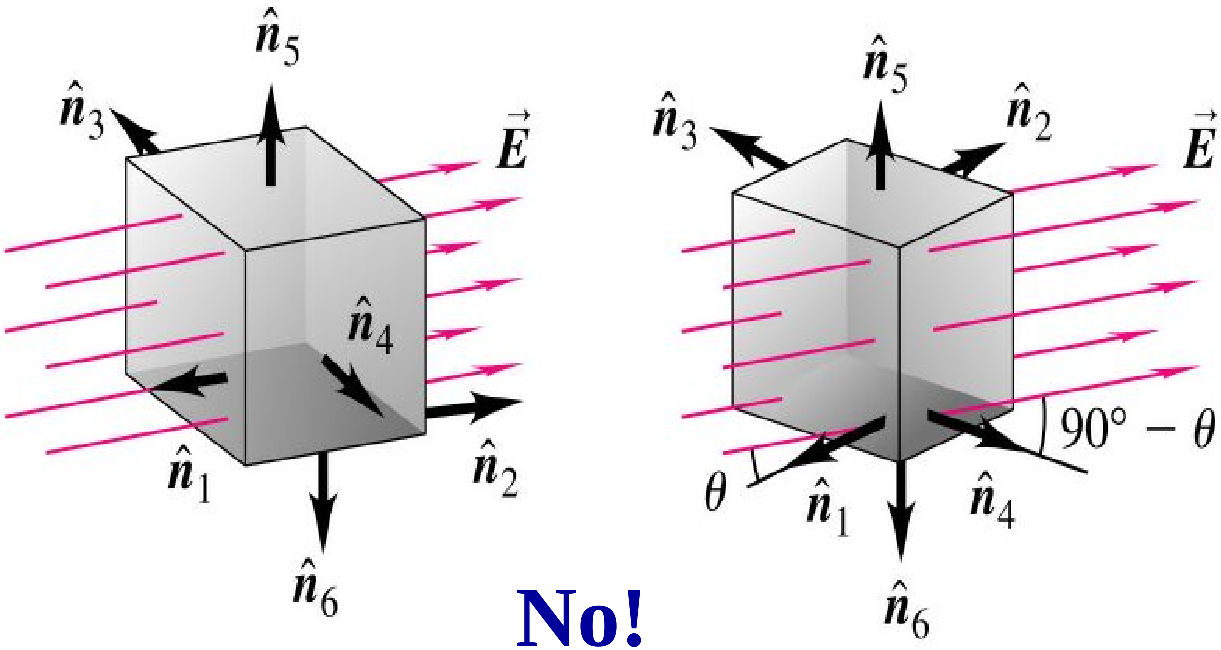


Where flux lines enter the surface, the surface normal and the electric field lines are anti-parallel

Where the flux lines exit the surface they are parallel

Electric Flux

Is there a difference in the net flux through the cube between the two situations?



No!

It is important to remember to properly take into account the various dot products

Electric Flux

The equation we have for flux is fine for simple situations

the electric field is uniform and

the surface area is plane

What happens when either one or the other or both is not true

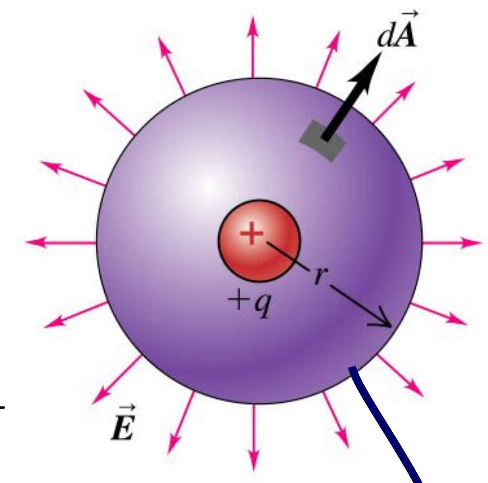
Electric Flux

- We proceed as we did in the transition from discrete charges to a continuous distribution of charges
- We break the surface area into small pieces and then calculate the flux through each piece and then sum them
- In the limit of infinitesimal areas this just becomes an integral

$$\Phi_E = \int \vec{E} \cdot d\vec{A}$$

Electric Flux of a Point Charge

What is electric flux that comes from a point charge?



We start from $\Phi_E = \int \vec{E} \cdot d\vec{A}$

The electric field is given by $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}$

The problem has spherical symmetry, we therefore use a sphere as the Gaussian surface

Since E is radial, its dot product with the differential area vector, which is also radial, is always one

Also E is the same at every point on the surface of the sphere

Electric Flux of a Point Charge

For these reasons, E can be pulled out from the integral and what remains is

$$\Phi_E = E\int dA$$

The integral over the surface area of the sphere yields

$$A = 4\pi\, r^2$$

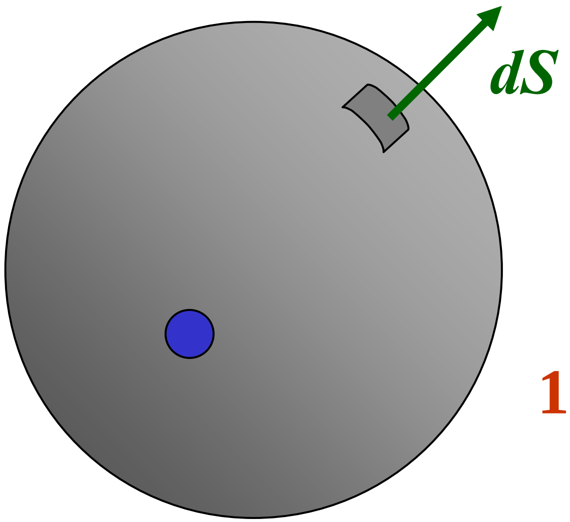
Pulling all this together then yields

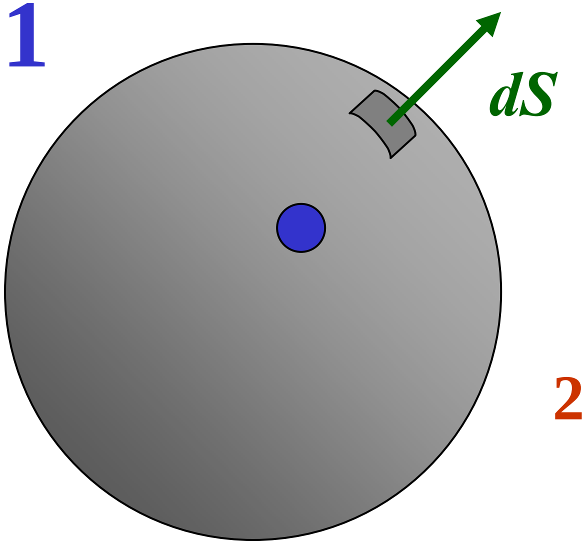
$$\Phi_E = EA; \quad \Phi_E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}4\pi\, r^2$$

$$\Phi_E = \frac{q}{\varepsilon_0}$$

Notice that this is *independent* of the radius of the sphere

Example 1





A positive charge is contained inside a spherical shell. How does the differential electric flux, $d\Phi_E$, through the surface element dS change when the charge is moved from position 1 to position 2?

$d\Phi_E$ a) increases b) decreases c) doesn't change

Example 1 - continued

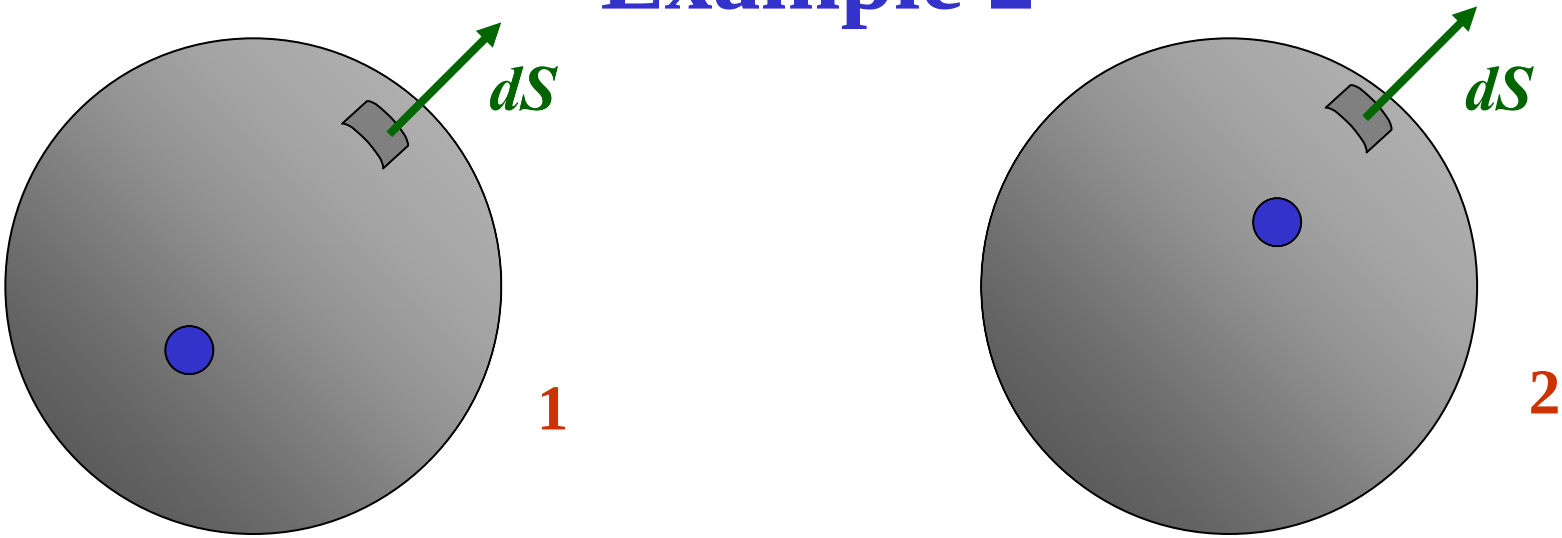
$d\Phi_E$ a) increases b) decreases c) doesn't change

The total flux of a charge is constant, with the density of flux lines being higher the closer you are to the charge

Therefore as you move the charge closer to the surface element, the density of flux lines increases

Multiplying this higher density by the same value for the area of *dS* gives us that the incremental flux also increases

Example 2



A positive charge is contained inside a spherical shell. How does the total flux, Φ_E, through the entire surface change when the charge is moved from position 1 to position 2?

a) Φ_E increases **b) Φ_E decreases** **c) Φ_E doesn't change**

As we previously calculated, the total flux from a point charge depends only upon the charge

Gauss' Law

The result for a single charge can be extended to systems consisting of more than one charge

One repeats the calculation for each of the charges enclosed by the surface and then sum the individual fluxes

$$\Phi_E = \frac{1}{\varepsilon_0} \sum_i q_i$$

Gauss' Law relates the flux through a closed surface to charge within that surface

Gauss' Law

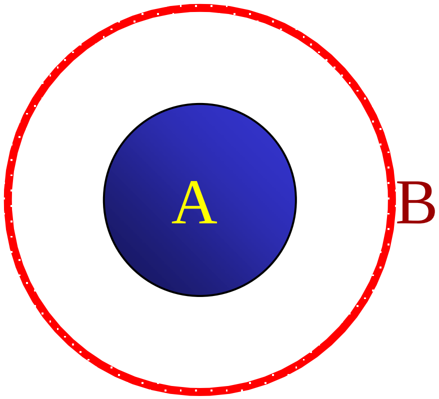
Gauss' Law states that

The net flux through any closed surface equals the net (total) charge inside that surface divided by ε_0

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{net}}{\varepsilon_0}$$

Note that the integral is over a *closed* surface

Example 3



A blue sphere A is contained within a red spherical shell B. There is a charge Q_A on the blue sphere and charge Q_B on the red spherical shell.

The electric field in the region between the spheres is completely independent of Q_B the charge on the red spherical shell.

True

False

Surfaces

Choose surface appropriate to problem

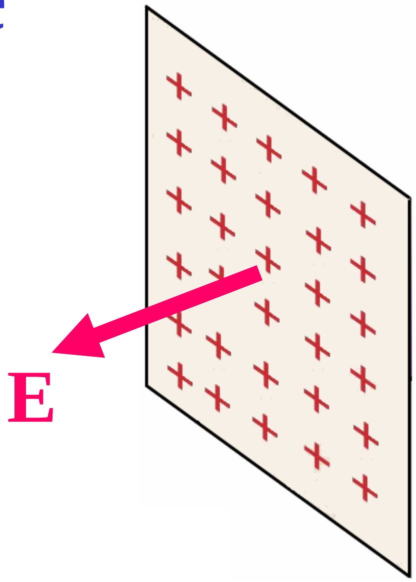
It does not have to be a sphere

Exploit symmetries, if any

Example 4
Thin Infinite Sheet of Charge

A given sheet has a charge density given by σ C/m^2

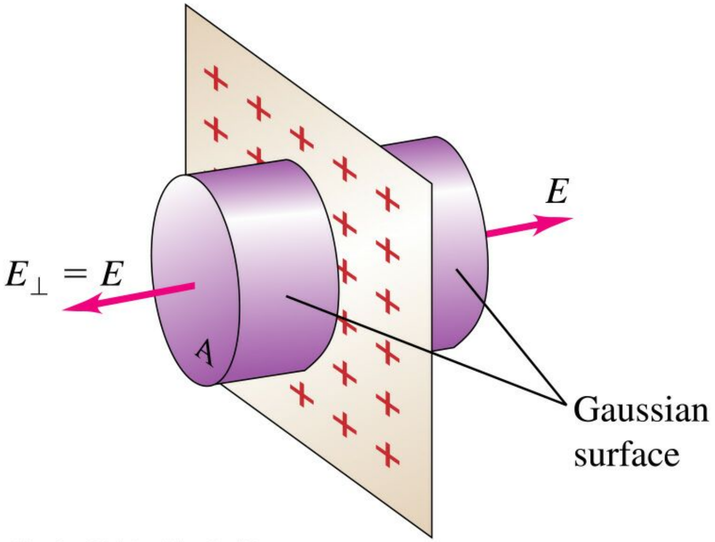
By symmetry, E is perpendicular to the sheet



Use a surface that exploits this fact

A cylinder

A Gaussian pillbox



Thin Infinite Sheet of Charge

$$\oint \vec{E} \cdot d\vec{A} = \frac{\sigma A}{\varepsilon_0}$$

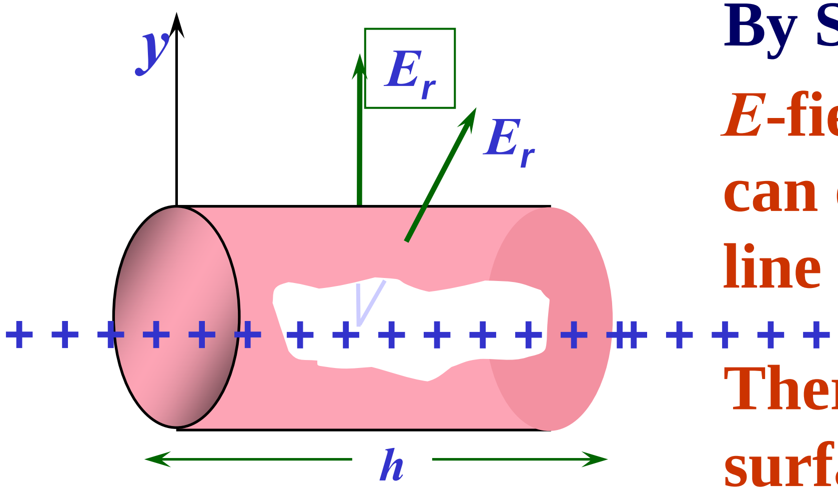
$$E_\perp A_{left} + EA_{curved} + E_\perp A_{right} = \frac{\sigma A}{\varepsilon_0}$$

But E and A_{curved} are perpendicular to each other so their dot product is zero and the middle term on the left disappears

$$2E_\perp A = \frac{\sigma A}{\varepsilon_0}$$

$$E = \frac{\sigma}{2\varepsilon_0}$$

Example 5
Infinite Line having a Charge Density λ



By Symmetry

E-field must be ⊥ to line of charge and can only depend on distance from the line

Therefore, choose the Gaussian surface to be a cylinder of radius *r* and length *h* aligned with the *x*-axis

Apply Gauss' Law:

On the ends, $\vec{E} \bullet d\vec{S} = 0$ since $E_{//}$ is zero

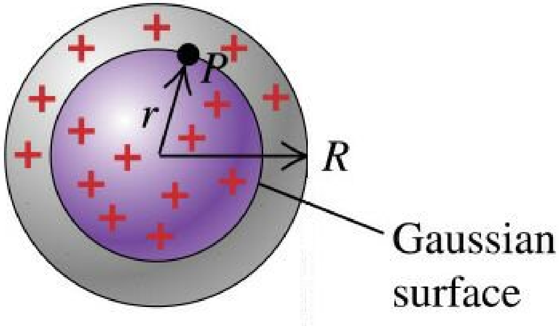
On the barrel, $\oint \vec{E} \bullet d\vec{S} = 2\pi rhE$ and $q = \lambda h$

Equating these and rearranging yields $E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$

This is the same result as using the integral formulation

Example 6
Solid Uniformly Charged Sphere

A charge Q is uniformly distributed throughout the volume of an insulating sphere of radius R.



What is the electric field for r < R?

Calculate average charge density

$$\textbf{Charge Density}: \rho = \frac{\mathbf{Q}}{\mathbf{4}\pi\,\mathbf{R}^{\mathbf{3}}/3}$$

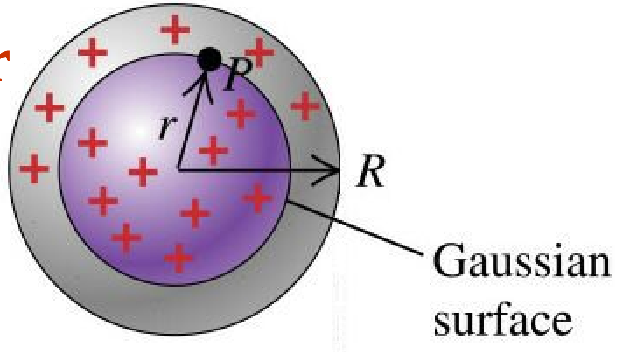
Now select a Gaussian sphere of radius r within this larger sphere

Charge within this sphere is given by

$$Q_{encl} = \rho V_{encl} = \left(\frac{Q}{4\pi R^3/3}\right)\left(\frac{4}{3}\pi r^3\right) = Q\frac{r^3}{R^3}$$

Example 6
Solid Uniformly Charged Sphere

Electric Field is everywhere perpendicular to surface, i.e. parallel to surface normal



Gauss' Law then gives

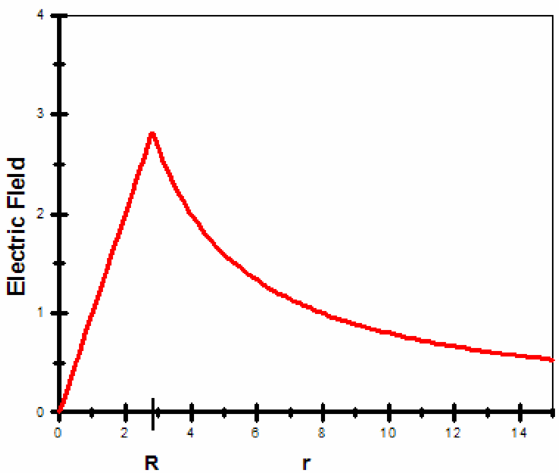
$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$$

$$E\,4\pi r^2 = \frac{Q}{\varepsilon_0}\frac{r^3}{R^3}$$

$$E = \frac{Q}{4\pi\varepsilon_0}\frac{r}{R^3}$$

Field increases linearly within sphere

Outside of sphere electric field is given by that of a point charge of value Q



Charges on Conductors

Given a solid conductor, on which is placed an excess charge

then in the static limit

The excess charge will reside on the surface of the conductor

and

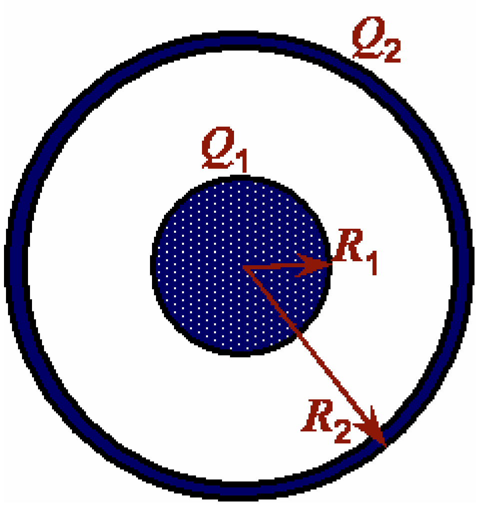
Everywhere the electric field due to this excess charge will be perpendicular to the surface

and

The electric field within the conductor will everywhere be zero

Example 7

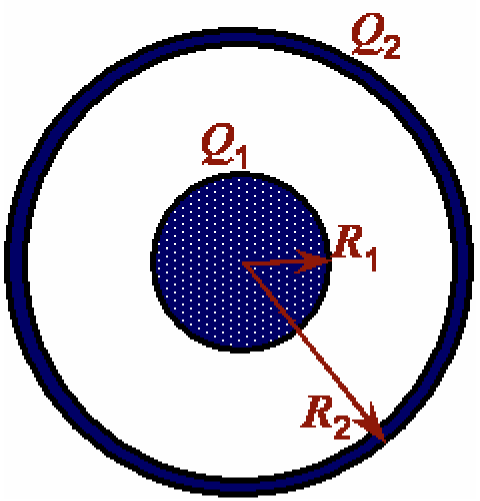
A solid *conducting* sphere is concentric with a thin *conducting* shell, as shown



The inner sphere carries a charge Q_1, and the spherical shell carries a charge Q_2, such that $Q_2 = -3\,Q_1$

1. ***How is the charge distributed on the sphere?***
2. ***How is the charge distributed on the spherical shell?***
3. ***What is the electric field at $r < R_1$? Between R_1 and R_2? At $r > R_2$?***
4. ***What happens when you connect the two spheres with a wire? (What are the charges?)***

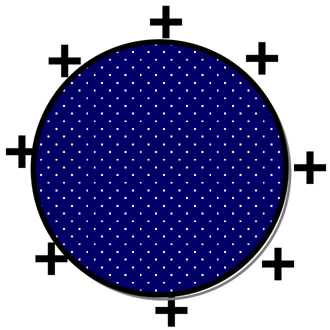
1. **How is the charge distributed on the sphere?**



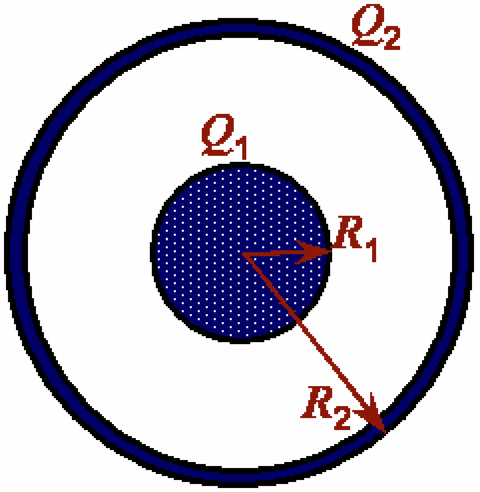
Remember that the electric field inside a conductor in a static situation is zero

By Gauss's Law, there can be no net charge inside the conductor

The charge, Q_1, must reside on the outside surface of the sphere



2. **How is the charge distributed on the spherical shell?**



The electric field inside the conducting shell is zero.

There can be no net charge inside the conductor

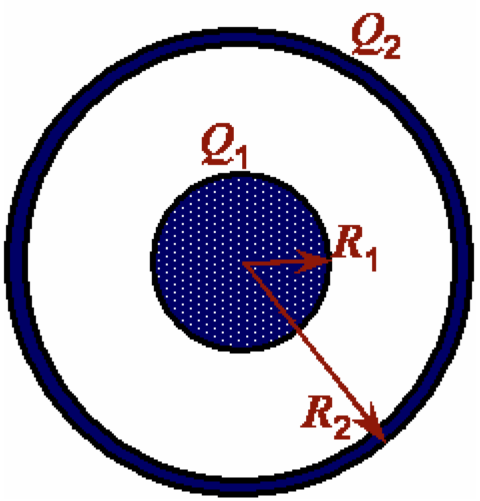
Using Gauss' Law it can be shown that the inner surface of the shell must carry a net charge of -Q_1

The outer surface must carry the charge +Q_1 + Q_2, so that the net charge on the shell equals Q_2

The charges are distributed uniformly over the inner and outer surfaces of the shell, hence

$$\sigma_{inner} = -\frac{Q_1}{4\pi R_2^2} \quad \text{and} \quad \sigma_{outer} = \frac{Q_2 + Q_1}{4\pi R_2^2} = \frac{-2Q_1}{4\pi R_2^2}$$

3. **What is the Electric Field at $r < R_1$? Between R_1 and R_2? At $r > R_2$?**



The electric field <u>inside</u> a conductor is zero.

$r < R_1$: This is inside the conducting sphere, therefore $\vec{E} = 0$

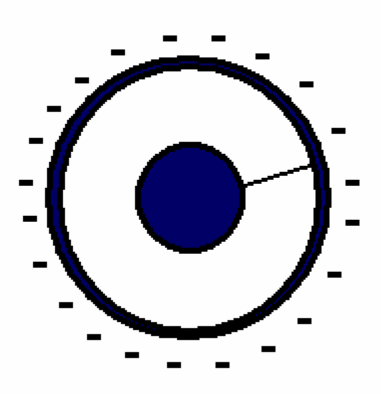
Between R_1 and R_2 : $R_1 < r < R_2$
Charge enclosed within a Gaussian sphere = Q_1

$$\vec{E} = k\frac{Q_1}{r^2}\hat{r}$$

$r > R_2$
Charge enclosed within a Gaussian sphere = $Q_1 + Q_2$

$$\vec{E} = k\frac{Q_1 + Q_2}{r^2}\hat{r} = k\frac{Q_1 - 3Q_1}{r^2}\hat{r} = -k\frac{2Q_1}{r^2}\hat{r}$$

4. **What happens when you connect the two spheres with a wire? (What are the charges?)**



After electrostatic equilibrium is reached,

there is no charge on the inner sphere,

and none on the inner surface of the shell

The charge $Q_1 + Q_2$ resides on the outer surface

Also, for $r < R_2$ $\quad \vec{E} = 0$

and for $r > R_2$ $\quad \vec{E} = -k\dfrac{2Q_1}{r^2}\hat{r}$

Example 8

An uncharged spherical conductor has a weirdly shaped cavity carved out of it. Inside the cavity is a charge -q.

i) How much charge is on the cavity wall?

(a) Less than q (b) Exactly q (c) More than q

ii) How is the charge distributed on the cavity wall?

(a) Uniformly

(b) More charge closer to –q

(c) Less charge closer to -q

iii) How is the charge distributed on the outside of the sphere?

(a) Uniformly

(b) More charge near the cavity

(c) Less charge near the cavity

Example 8

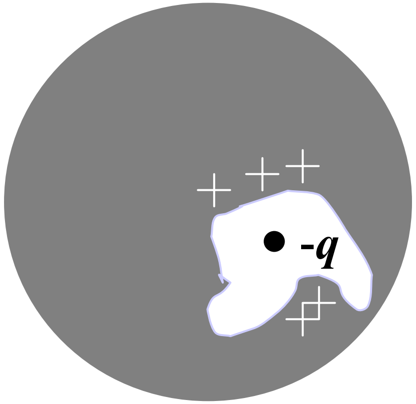
An uncharged spherical conductor has a weirdly shaped cavity carved out of it. Inside the cavity is a charge -*q*.

i) How much charge is on the cavity wall?

(a) Less than< *q* $\boxed{\text{(b) Exactly } q}$ (c) More than *q*

By Gauss' Law, since *E* = 0 inside the conductor, the total charge on the inner wall must be *q* (and therefore -*q* must be on the outside surface of the conductor, since it has no net charge).

Example 8



An uncharged spherical conductor has a weirdly shaped cavity carved out of it. Inside the cavity is a charge -*q*.

ii) How is the charge distributed on the cavity wall?

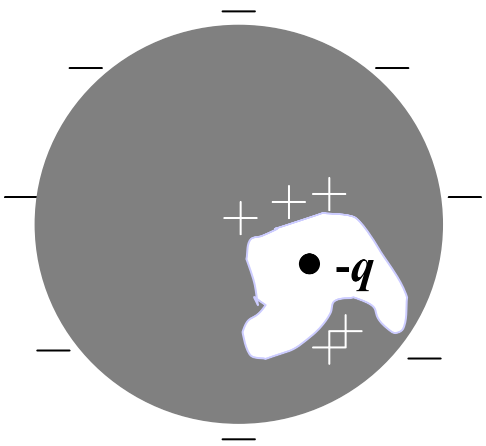
(a) Uniformly

(b) More charge closer to –*q*

(c) Less charge closer to -*q*

The induced charge will distribute itself non-uniformly to exactly cancel $\vec{E}$ everywhere in the conductor. The surface charge density will be higher near the -*q* charge.

Example 8



An uncharged spherical conductor has a weirdly shaped cavity carved out of it. Inside the cavity is a charge -*q*.

iii) How is the charge distributed on the outside of the sphere?

(a) Uniformly

(b) More charge near the cavity

(c) Less charge near the cavity

The charge will be uniformly distributed (because the outer surface is symmetric). Outside the conductor, the *E* field <u>always points directly to the center of the sphere</u>, regardless of the cavity or charge or its location.

Note: this is why your radio, cell phone, etc. won't work inside a metal building!