Math 215 HW #8 Solutions

1. Problem 4.2.4. By applying row operations to produce an upper triangular U, compute

$$\det \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \qquad \text{and} \qquad \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Answer: Focusing on the first matrix, we can subtract twice row 1 from row 2 and add row 1 to row 3 to get

$$\begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}.$$

Next, add twice row 2 to row 4:

$$\begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix}.$$

Finally, add 5/2 times row 3 to row 4:

$$\begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}.$$

Since none of the above row operations changed the determinant and since the determinant of a triangular matrix is the product of the diagonal entries, we see that

$$\det \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} = (1)(-1)(-2)(10) = 20.$$

Turning to the second matrix, we can first add half of row 1 to row 2:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Next, add 2/3 of row 2 to row 3:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Finally, add 3/4 of row 3 to row 4:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}.$$

Therefore, since the row operations didn't change the determinant and since the determinant of a triangular matrix is the product of the diagonal entries,

$$\det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = (2)(3/2)(4/3)(5/4) = \frac{5!}{4!} = 5.$$

NOTE: This second matrix is the same one that came to our attention in Section 1.7 and HW #3, Problem 9.

2. Problem 4.2.6. For each n, how many exchanges will put (row n, row $n-1$, …, row 1) into the normal order (row 1, …, row $n-1$, row n)? Find $\det P$ for the n by n permutation with 1s on the reverse diagonal.

   Answer: Suppose $n = 2m$ is even. Then the following sequence of numbers gives the original ordering of the rows:

   $$2m, 2m-1, \dots, m+1, m, \dots, 2, 1.$$

   Exchanging $2m$ and 1, and then $2m-1$ and 2, …, and then $m+1$ and m yields the correct ordering of rows:

   $$1, 2, \dots, m, m+1, \dots, 2m-1, 2m.$$

   Clearly, we performed $m = n/2$ row exchanges in the above procedure. Thus, for even values of n, we need to perform $n/2$ row exchanges.

   On the other hand, suppose $n = 2m-1$ is odd. Then the original ordering of the rows is

   $$2m-1, 2m-2, \dots, m+1, m, m-1, \dots, 2, 1.$$

   We exchange $2m-1$ and 1, and then $2m-2$ and 2, …, and then $m+1$ and $m-1$. Since m is already in the correct spot, this gives the correct ordering of rows

   $$1, 2, \dots, m-1, m, m+1, \dots, 2m-2, 2m-1.$$

   Clearly, we performed $m-1 = \frac{n-1}{2}$ row exchanges. Thus, for odd values of n, we need to perform $\frac{n-1}{2}$ row exchanges.

   If P is the permutation matrix with 1s on the reverse diagonal, then the rows of P are simply the rows of the identity matrix in precisely the reverse order. Thus, the above reasoning tells us how many row exchanges will transform P into I. Since the determinant of the identity matrix is 1 and since performing a row exchange reverses the sign of the determinant, we have that

   $$\det P = (-1)^{\text{number of row exchanges}} \det I = (-1)^{\text{number of row exchanges}}.$$

Therefore,

$$\det P = \begin{cases} (-1)^{n/2} & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases} = \begin{cases} 1 & \text{if } \frac{n}{4} \text{ has remainder 0 or 1} \\ -1 & \text{if } \frac{n}{4} \text{ has remainder 2 or 3} \end{cases}.$$

3. Problem 4.2.8. Show how rule 6 (det = 0 if a row is zero) comes directly from rules 2 and 3.

   Answer: Suppose A is an $n \times n$ matrix such that the ith row of A is equal to zero. Let B be the matrix which comes from exchanging the first row and the ith row of A. Then, by rule 2,

   $$\det B = -\det A.$$

   Now, the matrix B has all zeros in the first row. Therefore, by rule 3,

   $$\det B = \begin{vmatrix} 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix} = \begin{vmatrix} 0 \cdot 1 & 0 \cdot 1 & \cdots & 0 \cdot 1 \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix} = 0 \begin{vmatrix} 1 & 1 & \cdots & 1 \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix} = 0.$$

   Since $\det B = 0$ and since $\det A = -\det B$, we see that

   $$\det A = -\det B = -0 = 0,$$

   which is rule 6.

4. Problem 4.2.10. If Q is an orthogonal matrix, so that $Q^T Q = I$, prove that $\det Q$ equals $+1$ or -1. What kind of box is formed from the rows (or columns) of Q?

   Answer: By rule 10, we know that $\det(Q^T) = \det Q$. Therefore, using rules 1 and 9,

   $$1 = \det I = \det(Q^T Q) = \det(Q^T)\det Q = (\det Q)^2.$$

   Hence,

   $$\det Q = \pm\sqrt{1} = \pm 1.$$

   We see that the columns of Q form a box of volume 1. In fact, they form a cubical box.

5. Problem 4.2.14. True or false, with reason if true and counterexample if false.

   (a) If A and B are identical except that $b_{11} = 2a_{11}$, then $\det B = 2\det A$.
   Answer: False. Suppose

   $$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

   Then $\det A = 0$ and $\det B = 2 - 1 = 1 \neq 2\det A$.

   (b) The determinant is the product of the pivots.
   Answer: False. Let

   $$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

   Then $\det A = 0 - 1 = -1$, but the two pivots are 1 and 1, so the product of the pivots is 1. (The issue here is that we have to do a row exchange before we try elimination and the row exchange changes the sign of the determinant)

(c) If A is invertible and B is singular, then $A + B$ is invertible.

Answer: False. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then A, being the identity matrix, is invertible, while B, since it has a row of all zeros, is definitely singular. However,

$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is singular since it has a zero row.

(d) If A is invertible and B is singular, then AB is singular.

Answer: True. Since B is singular, $\det B = 0$. Therefore,

$$\det(AB) = \det A \det B = \det A \cdot 0 = 0.$$

Since $\det(AB) = 0$ only if AB is singular, we can conclude that AB is singular.

(e) The determinant of $AB - BA$ is zero.

Answer: False. Let

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 3 \\ 5 & 0 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 0 & 3 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}.$$

Therefore,

$$AB - BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

which has determinant equal to -1.

6. Problem 4.2.26. If a_{ij} is i times j, show that $\det A = 0$. (Exception when $A = [1]$).

Proof. Notice that the first row of A is

$$[1\ 2\ 3\ 4\ \cdots\ n]$$

and the second row of A is

$$[2\ 4\ 6\ 8\ \cdots\ 2n].$$

Thus, the first two rows of A are linearly dependent, meaning that A is singular since elimination will produce a row of all zeros in the second row. Thus, the determinant of A must be zero. (In fact, every row is a multiple of the first row, so A is about as far as a non-zero matrix can be from being non-singular). □

7. Problem 4.3.6. Suppose A_n is the n by n tridiagonal matrix with 1s on the three diagonals:

$$A_1 = [1], \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \ldots$$

Let D_n be the determinant of A_n; we want to find it.

(a) Expand in cofactors along the first row to show that $D_n = D_{n-1} - D_{n-2}$.

Proof. We want to find the determinant of

$$A_n = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Doing a cofactor expansion along the first row, D_n will be equal to 1 times the determinant of the matrix given by deleting the first row and first column minus 1 times the determinant of the matrix given by deleting the first row and second column.

Deleting the first row and first column of A_n just leaves a copy of A_{n-1}, the determinant of which is D_{n-1}. Thus,

$$D_n = 1 \cdot D_{n-1} - 1 \cdot \det(\text{matrix left when deleting first row and second column}). \quad (1)$$

Deleting the first row and second column yields the matrix

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \quad (2)$$

Notice that if we delete the first row and first column of this matrix, we're left with a copy of A_{n-2} (the determinant of which is D_{n-2}), whereas when we delete the first row and second column we get a matrix with all zeros in the first column (which must have determinant zero). Thus, the determinant of the matrix from (2) is, using cofactor expansion, equal to

$$1 \cdot D_{n-2} - 1 \cdot 0.$$

Therefore, combining this with (1), we see that

$$D_n = 1 \cdot D_{n-1} - 1 \cdot (1 \cdot D_{n-2} - 1 \cdot 0)$$

or, equivalently,

$$D_n = D_{n-1} - D_{n-2}.$$

□

(b) Starting from $D_1 = 1$ and $D_2 = 0$, find $D_3, D_4, \dots, D_8$. By noticing how these numbers cycle around (with what period?) find D_{1000}.

Answer: Since $D_1 = 1$ and $D_2 = 0$, we have, using the result from part (a), that

$$\begin{aligned} D_3 &= D_2 - D_1 = 0 - 1 = -1 \\ D_4 &= D_3 - D_2 = -1 - 0 = -1 \\ D_5 &= D_4 - D_3 = -1 - (-1) = 0 \\ D_6 &= D_5 - D_4 = 0 - (-1) = 1 \\ D_7 &= D_6 - D_5 = 1 - 0 = 1 \\ D_8 &= D_7 - D_6 = 1 - 1 = 0 \\ &\vdots \end{aligned}$$

Since each term depends only on the two preceding terms and since $D_8 = D_2$ and $D_7 = D_1$, the above pattern will repeat indefinitely. Thus, the D's have a period of $7 - 1 = 6$, so $D_{1+6m} = D_1$ for each m and, more generally, $D_{k+6m} = D_k$ for any m, where $k \in \{1, 2, 3, 4, 5, 6\}$. Therefore,

$$D_{1000} = D_{4+6\cdot 166} = D_4 = -1.$$

8. Problem 4.3.8. Compute the determinants of A_2, A_3, A_4. Can you predict A_n?

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Use row operations to produce zeros, or use cofactors of row 1.

Answer: Using the formula for determinants of 2×2 matrices, we see that

$$\det(A_2) = 0 \cdot 0 - 1 \cdot 1 = -1.$$

Then, expanding in cofactors along the first row,

$$\begin{aligned} \det(A_3) &= 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ &= 0 - 1(-1) + 1(1) \\ &= 2. \end{aligned}$$

Again, doing a cofactor expansion along the first row,

$$\begin{aligned} \det(A_4) &= 0 \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \\ &= 0 - 1 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} + 1 \cdot (-1) \cdot \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 1 \cdot (-1)^2 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \qquad (3) \end{aligned}$$

using Property 2 of the determinant (which says that exchanging rows changes the sign of the determinant). Now,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1(-1) - 1(-1) + 1(1) = 1,$$

so, plugging this into (3), we see that

$$\det(A_4) = 0 - 1(1) + 1(-1)(1) - 1(1)(1) = -3.$$

In general, it will turn out that

$$\det(A_n) = (-1)^{n-1}(n-1).$$

9. Problem 4.3.14. Compute the determinants of A, B, C. Are their columns independent?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Answer: First, compute the determinant of A using cofactors:

$$\begin{aligned} \det A &= 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 1(-1) - 1(1) + 0(1) \\ &= -2. \end{aligned}$$

Since $\det A \neq 0$, the matrix A is invertible and thus the columns of A are necessarily linearly independent.

Next, compute the determinant of B using cofactors:

$$\begin{aligned} \det B &= 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) \\ &= -3 + 12 - 9 \\ &= 0. \end{aligned}$$

Thus, since $\det B = 0$, the matrix B is not invertible and so its columns are not linearly independent.

Turning attention to the matrix C, note that, since the columns of B are linearly dependent, the last three columns of C must also be linearly dependent, meaning that $\det C = 0$.

10. Problem 4.3.28. The n by n determinant C_n has 1s above and below the main diagonal:

$$C_1 = |0| \quad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \quad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

(a) What are the determinants C_1, C_2, C_3, C_4?
Answer: Clearly, $\det C_1 = |0| = 0$. Next,

$$\det C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 \cdot 0 - 1 \cdot 1 = -1.$$

Now, expanding in cofactors,

$$\begin{aligned} \det C_3 &= 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 0(-1) - 1(0) + 0(1) \\ &= 0. \end{aligned}$$

Finally, we also determine $\det C_4$ by expanding in cofactors:

$$\begin{aligned} \det C_4 &= 0 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= -1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\ &= -1 \cdot \left(1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \right) \\ &= -1(1 \cdot \det C_2) \\ &= 1. \end{aligned}$$

(b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2}. Find C_{10}.
Answer: Just as in the $n = 4$ case, doing a cofactor expansion along the first row yields only one non-zero term, namely

$$\det C_n = -1 \cdot \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}$$

Deleting the first row and second column yields a matrix with all zeros in the first column, which necessarily has determinant zero. Therefore, using a cofactor expansion, the above is equal to

$$\det C_n = -1\,(1 \cdot \det C_{n-2} - 1 \cdot 0) = -\det C_{n-2}.$$

Thus, we have that $\det C_n = -\det C_{n-2}$. Hence,

$$\det C_{10} = -\det C_8 = \det C_6 = -\det C_4 = -1.$$

11. Let the numbers S_n be the determinants defined in Problem 4.3.31.

(a) For any $n > 2$ prove that $S_n = 3S_{n-1} - S_{n-2}$.

Proof. We can compute S_n using a cofactor expansion:

$$\begin{aligned}
S_n &= \begin{vmatrix} 3 & 1 & 0 & \cdots & 0 \\ 1 & 3 & 1 & \cdots & 0 \\ 0 & 1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 3 \end{vmatrix} \\
&= 3 \cdot \begin{vmatrix} 3 & 1 & \cdots & 0 \\ 1 & 3 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & \cdots & 0 \\ 0 & 3 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 3 \end{vmatrix} \\
&= 3S_{n-1} - 1 \cdot \begin{vmatrix} 1 & 1 & \cdots & 0 \\ 0 & 3 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 3 \end{vmatrix}.
\end{aligned}$$

Doing a cofactor expansion of this new determinant gives $1 \cdot S_{n-2}$ plus 1 times the determinant of a matrix with all zeros in the first column. Thus, the second term in the above expression is just $1 \cdot S_{n-2}$, so we can conclude that

$$S_n = 3S_{n-1} - S_{n-2}.$$

□

(b) For any k let F_k denote the kth Fibonacci number (recall that the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$ is defined by $F_k = F_{k-1} + F_{k-2}$). Prove that $F_{2n+2} = 3F_{2n} - F_{2n-2}$.

Proof. By definition of the Fibonacci sequence, we know that

$$\begin{aligned}
F_{2n+2} &= F_{2n+1} + F_{2n} \\
F_{2n+1} &= F_{2n} + F_{2n-1} \\
F_{2n} &= F_{2n-1} + F_{2n-2}.
\end{aligned}$$

From the third line, we have that $F_{2n-1} = F_{2n} - F_{2n-2}$. Therefore, substituting the second line into the first and using this expression for F_{2n-1}, we have that

$$\begin{aligned}
F_{2n+2} &= F_{2n+1} + F_{2n} \\
&= (F_{2n} + F_{2n-1}) + F_{2n} \\
&= (F_{2n} + (F_{2n} - F_{2n-2})) + F_{2n} \\
&= 3F_{2n} - F_{2n-2}.
\end{aligned}$$

□

(c) Show that $S_n = F_{2n+2}$ for each n.

Proof. I will prove this using the principle of mathematical induction. Let P_k be the statement that $S_k = F_{2k+2}$.

The base case of induction is to prove that P_1 and P_2 are true; i.e. that $S_1 = F_4$ and $S_2 = F_6$. However, both are clearly true, as

$$S_1 = 3 = F_4 \quad \text{and} \quad S_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8 = F_6.$$

For the inductive step, we want to show that P_k and P_{k-1} being true implies P_{k+1} is true for any k. To see this, suppose P_k and P_{k-1} are true, meaning that

$$S_k = F_{2k+2} \quad \text{and} \quad S_{k-1} = F_{2k}.$$

Then, using part (a),

$$S_{k+1} = 3S_k - S_{k-1} = 3F_{2k+2} - F_{2k}.$$

However, by part (b), the right-hand side of the equation is equal to F_{2k+4}, so we see that $S_{k+1} = F_{2k+4}$, which is to say that P_{k+1} is true.

Therefore, since we've shown that P_1 and P_2 are true and we've shown that P_k and P_{k-1} being true implies P_{k+1} is true, so, by induction, we can conclude that P_n is true for all n. In other words,

$$S_n = F_{2n+2}$$

for all n. □

12. (**Bonus Problem**) Problem 3.5.12. Compute $F_8 c$ by the three steps of the Fast Fourier Transform if $c = (1, 0, 1, 0, 1, 0, 1, 0)$. Repeat the computation with $c = (0, 1, 0, 1, 0, 1, 0, 1)$.

Answer: Note, first of all, that

$$c' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence,

$$y' = F_4 c' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$y'' = F_4 c'' = F_4 \vec{0} = \vec{0}.$$

Therefore,

$$\begin{aligned}
y_1 &= y_1' + w_8 y_1'' = 4 + 0 = 4 \\
y_2 &= y_2' + w_8^2 y_2'' = 0 + 0 = 0 \\
y_3 &= y_3' + w_8^3 y_3'' = 0 + 0 = 0 \\
y_4 &= y_4' + w_8^4 y_4'' = 0 + 0 = 0 \\
y_5 &= y_1' - w_8^5 y_1'' = 4 - 0 = 4 \\
y_6 &= y_2' - w_8^6 y_2'' = 0 - 0 = 0 \\
y_3 &= y_3' - w_8^7 y_3'' = 0 - 0 = 0 \\
y_4 &= y_4' - w_8^8 y_4'' = 0 - 0 = 0,
\end{aligned}$$

where w_8 is an eighth root of 1. Therefore,

$$y = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Turning to $c = (0, 1, 0, 1, 0, 1, 0, 1)$, we see that

$$c' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence,

$$y' = F_4 c' = F_4 \vec{0} = \vec{0}$$

and

$$\begin{aligned}
y'' = F_4 c'' &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Therefore, since $w_8 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, we see that

$$\begin{aligned}
y_1 &= y_1' + w_8 y_1'' = 0 + \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)4 = 2\sqrt{2} + 2\sqrt{2}i.\\
y_2 &= y_2' + w_8^2 y_2'' = 0 + 0 = 0\\
y_3 &= y_3' + w_8^3 y_3'' = 0 + 0 = 0\\
y_4 &= y_4' + w_8^4 y_4'' = 0 + 0 = 0\\
y_5 &= y_1' - w_8^5 y_1'' = 0 - \left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)4 = 2\sqrt{2} + 2\sqrt{2}i\\
y_6 &= y_2' - w_8^6 y_2'' = 0 - 0 = 0\\
y_3 &= y_3' - w_8^7 y_3'' = 0 - 0 = 0\\
y_4 &= y_4' - w_8^8 y_4'' = 0 - 0 = 0.
\end{aligned}$$

Then

$$y = \begin{bmatrix} 2\sqrt{2} + 2\sqrt{2}i \\ 0 \\ 0 \\ 0 \\ 2\sqrt{2} + 2\sqrt{2}i \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$