Orthogonal Functions: The Legendre, Laguerre, and Hermite Polynomials

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Outline

Overview

When discussed in $\mathbb{R}^2$, vectors are said to be orthogonal when the dot product is equal to 0.

$$\hat{w} \cdot \hat{v} = w_1 v_1 + w_2 v_2 = 0.$$

Overview

Definition

We define an inner product $(y_1|y_2) = \int_a^b y_1(x)\overline{y_2(x)}dx$ where $y_1, y_2 \in C^2[a,b]$.

Definition

Two functions are said to be **orthogonal** if $(y_1|y_2) = 0$.

Definition

A linear operator L is **self-adjoint** if $(Ly_1|y_2) = (y_1|Ly_2)$ for all y_1, y_2.

Trigonometric Functions and Fourier Series

- Orthogonality of the Sine and Cosine Functions
- Expansion of the Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

Legendre Polynomials

Legendre Polynomials are usually derived from differential equations of the following form:

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

We solve this equation using the standard power series method.

Legendre Polynomials

Suppose y is analytic. Then we have

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$y'(x) = \sum_{k=0}^{\infty} a_{k+1}(k+1)x^k$$

$$y''(x) = \sum_{k=0}^{\infty} a_{k+2}(k+1)(k+2)x^k$$

Recursion Formula

After implementing the power series method, the following recursion relation is obtained.

$$a_{k+2}(k+2)(k+1) - a_k(k)(k-1) - 2a_k(k) - n(n+1)a_k = 0$$

$$a_{k+2} = \frac{a_k[k(k+1) - n(n+1)]}{(k+2)(k+1)}$$

Using this equation, we get the coefficients for the Legendre polynomial solutions.

Legendre Polynomials

$$L_0(x) = 1$$

$$L_1(x) = x$$

$$L_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$L_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$L_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$L_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Legendre Graph

Figure: Legendre Graph

Sturm-Liouville

A Sturm-Liouville equation is a second-order linear differential equation of the form

$$(p(x)y')' + q(x)y + \lambda r(x)y = 0$$

$$p(x)y'' + p'(x)y' + q(x)y + \lambda r(x)y = 0$$

which allows us to find solutions that form an orthogonal system.

Sturm-Liouville cont.

We can define a linear operator by

$$Ly = (p(x)y')' + q(x)y$$

which gives the equation

$$Ly + \lambda r(x)y = 0.$$

Self-adjointness

To obtain orthogonality, we want L to be self-adjoint.

$$(Ly_1|y_2) = (y_1|Ly_2)$$

which implies

$$0 = (Ly_1|y_2) - (y_1|Ly_2)$$

$$= ((py_1')' + qy_1|y_2) - (y_1|(py_2')' + qy_2)$$

$$= \int_a^b (p'y_1'\overline{y_2} + py_1''\overline{y_2} + qy_1\overline{y_2} - y_1 p'\overline{y_2'} - y_1 p\overline{y_2''} - y_1\overline{q_1 y_2})dx$$

Self-adjointness

$$= \int_a^b (p' y_1' \overline{y_2} + p y_1'' \overline{y_2} - y_1 p' \overline{y_2'} - y_1 p \overline{y_2''}) dx$$

$$= \int_a^b [p(y_1' \overline{y_2} - \overline{y_2'} y_1)]' dx$$

$$= p(b)(y_1'(b)\overline{y_2}(b) - \overline{y_2'}(b) y_1(b)) - p(a)(y_1(a)\overline{y_2}(a) - \overline{y_2'}(a) y_1(a))$$

Orthogonality Theorem

Theorem

If (y_1, λ_1) *and* (y_2, λ_2) *are eigenpairs and* $\lambda_1 \neq \lambda_2$ *then* $(y_1|y_2)_r = 0$.

Proof.

$$(Ly_1|y_2) = (y_1|Ly_2)$$

$$(-\lambda_1 ry_1|y_2) = (y_1|-\lambda_2 ry_2)$$

$$\lambda_1 \int_a^b y_1 \overline{y_2} r dx = \lambda_2 \int_a^b y_1 \overline{y_2} r dx$$

$$\lambda_1 (y_1|y_2)_r = \lambda_2 (y_1|y_2)_r$$

$$(y_1|y_2)_r = 0$$

Legendre Polynomials - Orthogonality

Recall the Legendre differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

So

$$Ly = ((1 - x^2)y')'$$

$$\lambda = n(n+1)$$

$$r(x) = 1.$$

We want L to be self-adjoint, so we must determine necessary boundary conditions.

Sturm-Liouville Problem - Legendre

For any two functions $f, g \in C[-1, 1]$, by the general theory, we get

$$\begin{aligned}&\int_{-1}^{1} Lf(x)g(x) - f(x)Lg(x)dx \\ &= \int_{-1}^{1} ((1 - x^2)f')'g(x) - f(x)((1 - x^2)g')'dx \\ &= [(1 - x^2)(f'g - g'f)]_{-1}^{1} \\ &= 0.\end{aligned}$$

Legendre Polynomials - Orthogonality

Because $(1 - x^2) = 0$ when $x = -1, 1$ we know that L is self-adjoint on $C[-1, 1]$.Hence we know that the Legendre polynomials are orthogonal by the orthogonality theorem stated earlier.

Hermite Polynomials

For a Hermite Polynomial, we begin with the differential equation

$$y'' - 2xy' + 2ny = 0$$

Hermite Orthogonality

First, we need to arrange the differential equation so it can be written in the form

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0.$$

We must find some $r(x)$ by which we will multiply the equation. For the Hermite differential equation, we use $r(x) = e^{-x^2}$ to get

$$\begin{aligned} (e^{-x^2}y')' + 2ne^{-x^2}y &= 0 \\ \implies e^{-x^2}y'' - 2xe^{-x^2}y' + 2ne^{-x^2}y &= 0 \end{aligned}$$

Hermite Orthogonality

Sturm-Liouville problems can be written in the form

$$Ly + \lambda r(x)y = 0.$$

In our case, $Ly = (e^{-x^2}y')'$ and $\lambda r(x) = 2ne^{-x^2}y$.

$$0 = (Lf|g) - (f|Lg) = \int_{-\infty}^{\infty} Lf(x)g(x) - f(x)Lg(x)dx$$

Hermite Orthogonality

So we get from the general theory that

$$\int_{-\infty}^{\infty} (e^{-x^2} f'(x))' g(x) - f(x)(e^{-x^2} g'(x))' dx$$

$$= \int_{-\infty}^{\infty} [(e^{-x^2})(f'(x)g(x) - g'(x)f(x))]' dx$$

Hermite Orthogonality

With further manipulation we obtain

$$\lim_{a\to-\infty}[(e^{-x^2})(f'(x)g(x)-g'(x)f(x))]_a^0$$
$$+\lim_{b\to\infty}[(e^{-x^2})(f'(x)g(x)-g'(x)f(x))]_0^b$$

Hermite Orthogonality

We want

$$\lim_{x \to \pm\infty} e^{-x^2} f(x) g'(x) = 0$$

for all $f, g \in BC^2(-\infty, \infty)$. So we impose the following conditions on the space of functions we consider

$$\lim_{x \to \pm\infty} e^{-x^2/2} h(x) = 0$$

and

$$\lim_{x \to \pm\infty} e^{-x^2/2} h'(x) = 0$$

for all $h \in C^2(-\infty, \infty)$.

Conclusion

- Let $\phi_1(x)$,$\phi_2(x)$,...,$\phi_n(x)$,... be an system of orthogonal, real functions on the interval $[a, b]$.
- Let $f(x)$ be a function defined on the interval [a,b].
- Assume that $\int_a^b \phi_n^2(x) \neq 0$.
- Suppose that $f(x)$ can be represented as a series of the above orthogonal system. That is $f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) + \cdots$

Conclusion

- Multiplying $f(x)$ by $\phi_n(x)$ to get
  $f(x)\phi_n(x) = c_0\phi_0(x)\phi_n(x) + c_1\phi_1(x)\phi_n(x) + c_2\phi_2(x)\phi_n(x) + \cdots + c_n\phi_n^2(x) + c_{n+1}\phi_{n+1}(x)\phi_{n+1}(x) + \cdots$
- $\int_a^b f(x)\phi_n(x)dx = c_n \int_a^b \phi_n^2(x)dx$
- Therefore $c_n = \frac{\int_a^b f(x)\phi_n(x)dx}{\int_a^b \phi_n^2(x)dx}$ are called the Fourier coefficients of $f(x)$ with respect to the orthogonal system.
- The corresponding Fourier series is called the Fourier series of f(x) with respect to the orthogonal system.
- We may test whether this series converges or diverges.