# PART I. THE REAL NUMBERS

This material assumes that you are already familiar with the real number system and the representation of the real numbers as points on the real line.

## I.1. THE NATURAL NUMBERS AND INDUCTION

Let $\mathbb{N}$ denote the set of natural numbers (positive integers).

**Axiom:** If $S$ is a nonempty subset of $\mathbb{N}$, then $S$ has a least element. That is, there is an element $m \in S$ such that $m \le n$ for all $n \in S$.

**Note:** A set which has the property that each non-empty subset has a least element is said to be *well-ordered*. Thus, the axiom tells us that the natural numbers are well-ordered.

**Mathematical Induction.** Let $S$ be a subset of $\mathbb{N}$. If $S$ has the following properties:

1. $1 \in S$, and
2. $k \in S$ implies $k+1 \in S$,

then $S = \mathbb{N}$.

**Proof:** Suppose $S \neq \mathbb{N}$. Let $T = \mathbb{N} - S$. Then $T \neq \emptyset$. Let $m$ be the least element in $T$. Then $m-1 \notin T$. Therefore, $m-1 \in S$ which implies that $(m-1)+1 = m \in S$, a contradiction.

**Corollary:** Let $S$ be a subset of $\mathbb{N}$ such that

1. $m \in S$.
2. If $k \ge m \in S$, then $k+1 \in S$.

Then, $S = \{n \in \mathbb{N} : n \ge m\}$.

**Example** Prove that $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$ for all $n \in \mathbb{N}$.

*SOLUTION* Let $S$ be the set of integers for which the statement is true.

Since $2^0 = 1 = 2^1 - 1$, $1 \in S$.

Assume that the positive integer $k \in S$. Then

$$2^0 + 2^1 + \cdots + 2^{k-1} + 2^k = \left(2^0 + 2^1 + \cdots + 2^{k-1}\right) + 2^k = 2^k - 1 + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1.$$

Thus, $k+1 \in S$.

We have shown that $1 \in S$ and that $k \in S$ implies $k+1 \in S$. It follows that $S$ contains all the positive integers.

**Exercises 1.1**

1. Prove that $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.
2. Prove that $1^2+2^2+3^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.
3. Let $r$ be a real number $r \neq 1$. Prove that
$$1+r+r^2+r^3+\cdots+r^n = \frac{1-r^{n+1}}{1-r}.$$
for all $n \in \mathbb{N}$
4. Prove that $1+2n \leq 3^n$ for all $n \in \mathbb{N}$.
5. Prove that $\dfrac{1}{\sqrt{1}}+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\cdots+\dfrac{1}{\sqrt{n}} \geq \sqrt{n}$ for all $n \in \mathbb{N}$.
6. Prove that $\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)\cdots\left(1-\dfrac{1}{n^2}\right) = \dfrac{n+1}{2n}$ for all $n \geq 2$.
7. True or False: If $S$ is a non-empty subset of $\mathbb{N}$, then there exists an element $m \in S$ such that $m \geq k$ for all $k \in S$.

## I.2. ORDERED FIELDS

Let $\mathbb{R}$ denote the set of real numbers. The set $\mathbb{R}$, together with the operations of addition $(+)$ and multiplication $(\cdot)$, satisfies the following axioms:

**Addition:**

A1. For all $x,\ y \in \mathbb{R},\ x+y \in \mathbb{R}$ (addition is a closed operation).

A2. For all $x,\ y \in \mathbb{R},\ x+y = y+x$ (addition is commutative)

A3. For all $x,\ y,\ z \in \mathbb{R},\ x+(y+z) = (x+y)+z$ (addition is associative).

A4. There is a unique number $0$ such that $x+0 = 0+x$ for all $x \in \mathbb{R}$. (0 is the *additive identity.*)

A5. For each $x \in \mathbb{R}$, there is a unique number $-x \in \mathbb{R}$ such that $x+(-x) = 0$. ($-x$ is the *additive inverse of* $x$.)

**Multiplication:**

M1. For all $x,\ y \in \mathbb{R},\ x \cdot y \in \mathbb{R}$ (multiplication is a closed operation).

M2. For all $x,\ y \in \mathbb{R},\ x \cdot y = y \cdot x$ (multiplication is commutative)

M3. For all $x,\ y,\ z \in \mathbb{R},\ x \cdot (y \cdot z) = (x \cdot y) \cdot$ (multiplication is associative).

M4. There is a unique number $1$ such that $x \cdot 1 = 1 \cdot x$ for all $x \in \mathbb{R}$. (1 is the *multiplicative identity.*)

M5. For each $x \in \mathbb{R},\ x \neq 0$, there is a unique number $1/x = x^{-1} \in \mathbb{R}$ such that $x \cdot (1/x) = 1$. ($1/x$ is the *multiplicative inverse of* $x$.)

**Distributive Law:**

D. For all $x,\ y,\ z \in \mathbb{R},\ x \cdot (y + z) = x \cdot y + x \cdot z$.

A non-empty set $S$ together with two operations, "addition" and "multiplication" which satisfies A1-A5, M1-M5, and D is called a *field.* The set of real numbers with ordinary addition and multiplication is an example of a field. The set of rational numbers $\mathbb{Q}$, together with ordinary addition and multiplication, is also a field, a sub-field of $\mathbb{R}$. The set of complex numbers $\mathbb{C}$ is another example of a field.

**Order:**

There is a subset $P$ of $\mathbb{R}$ that has the following properties:

a If $x,\ y \in P$, then $x + y \in P$.

b If $x,\ y \in P$, then $x \cdot y \in P$.

c For each $x \in \mathbb{R}$ exactly one of the following holds: $x \in P$, $x = 0$, $-x \in P$.

The set $P$ is the set of positive numbers.

Let $x,\ y \in \mathbb{R}$. Then $x < y$ (read "$x$ is less than $y$") if $y - x \in P$. $x < y$ is equivalent to $y > x$ (read "$y$ is greater than $x$"). $P = \{x \in \mathbb{R} : x > 0\}$. $x \leq y$ means either $x < y$ or $x = y$; $y \geq x$ means either $y > x$ or $y = x$.

The relation "$<$" has the following properties:

O1. For all $x,\ y \in \mathbb{R}$, exactly one of the following holds: $x < y,\ x = y,\ x > y$. (Trichotomy Law)

O2. For all $x,\ y,\ z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$.

O3. For all $x,\ y,\ z \in \mathbb{R}$, if $x < y$, then $x + z < y + z$.

O4. For all $x,\ y,\ z \in \mathbb{R}$, if $x < y$ and $z > 0$, then $x \cdot z < y \cdot z$.

$\{\mathbb{R}, +, \cdot, <\}$ is an *ordered field. Any* mathematical system $\{S, +, \cdot, <\}$ satisfying these 15 axioms is an ordered field. In particular, the set of rational numbers $\mathcal{Q}$, together with ordinary addition, multiplication and "less than", is an ordered field, a *subfield of* $\mathbb{R}$.

**THEOREM 1.** *Let $x,\ y \in \mathbb{R}$. If $x \le y + \epsilon$ for every positive number $\epsilon$, then $x \le y$.*

**Proof:** Suppose that $x > y$ and choose $\epsilon = \dfrac{x-y}{2}$. Then

$$x < y + \epsilon = y + \frac{x-y}{2} = \frac{x+y}{2} < \frac{x+x}{2} = x,$$

a contradiction. Therefore, $x \le y$.

**Definition 1.** *Let $x \in \mathbb{R}$. The absolute value of $x$, denoted $|x|$, is given by*

$$|x| = \begin{cases} x, & \text{if } x \ge 0, \\ -x, & \text{if } x < 0. \end{cases}$$

The properties of absolute value are: for any $x,\ y \in \mathbb{R}$.

(1) $|x| \ge 0$,

(2) $|xy| = |x|\,|y|$,

(3) $|x + y| \le |x| + |y|$.

**Exercises 1.2**

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counterexample.
   (a) If $x,\ y,\ z \in \mathbb{R}$ and $x < y$, then $xz < yz$.
   (b) If $x,\ y \in \mathbb{R}$ and $x < y + \epsilon$ for every positive number $\epsilon$, then $x < y$.
   (c) If $x,\ y \in \mathbb{R}$, then $|x - y| \le |x| + |y|$.
   (d) If $x,\ y \in \mathbb{R}$, then $|\,|x| - |y|\,| \le |x| - |y|$.
2. Prove: If $|x - y| < \epsilon$ for every $\epsilon > 0$, then $x = y$.
3. Suppose that $x_1,\ x_2,\ \ldots,\ x_n$ are real numbers. Prove that

$$|x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n|.$$

## I.3. THE COMPLETENESS AXIOM

$\mathbb{R}$ and $\mathcal{Q}$ are each ordered fields. What distinguishes $\mathbb{R}$ from $\mathcal{Q}$ is the completeness axiom. As you know, $\mathcal{Q}$ is a proper subset of $\mathbb{R}$; i.e., there are real numbers which are not rational numbers. Such numbers are called *irrational numbers.*

**THEOREM 2.** *$\sqrt{2}$ is not a rational number. In general, if $p$ is a prime number, then $\sqrt{p}$ is not a rational number.*

**Proof:** Suppose $2 = p/q$ where $p,\ q \in \mathbb{N}$. Without loss of generality, assume that $p, q$ have no integral factors $> 1$. Now $p^2 = 2q^2$, so $p^2$ is even. $p^2$ even implies $p$ must be even, so $p = 2k$ for some $k \in N$. Consequently, $q^2 = 2k^2$ and so $q$ is even. Thus $p$ and $q$ have the common factor $2$, a contradiction.

Other examples of irrational numbers are $\sqrt{m}$ where $m$ is any rational number which is not a perfect square, $\sqrt[3]{m}$ where $m$ is any rational number which is not a perfect cube, etc. Also, the numbers $\pi$ and $e$ are irrational.

**Definition 2.** *Let $S$ be a subset of $\mathbb{R}$. A number $u \in \mathbb{R}$ is an* **upper bound** *of $S$ if $s \le u$ for all $s \in S$. An element $w \in \mathbb{R}$ is a* **lower bound** *of $S$ if $w \le s$ for all $s \in S$. If an upper bound $u$ for $S$ is an element of $S$, then $u$ is called the* **maximum** *(or* **largest element***) of $S$. Similarly, if a lower bound $w$ for $S$ is an element of $S$, then $w$ is called the* **minimum** *(or* **smallest element***) of $S$.*

**Examples:** Give some examples to illustrate upper bounds, lower bounds, maximum and minimum elements.

**Definition 3.** *A set $S \subseteq \mathbb{R}$ is said to be* **bounded above** *if $S$ has an upper bound; $S$ is* **bounded below** *if it has a lower bound. A subset $S$ of $\mathbb{R}$ is* **bounded** *if it has both an upper bound and a lower bound.*

**Definition 4.** *Let $S \subseteq \mathbb{R}$ be a set that is bounded above. A number $u \in \mathbb{R}$ is called the supremum (least upper bound) of $S$, denoted by $\sup S$, if it satisfies the conditions*

1. *$s \le u$ for all $s \in S$.*
2. *If $v$ is an upper bound for $S$, then $u \le v$.*

**THEOREM 3.** *Let $S \subseteq \mathbb{R}$ be bounded above, and let $u = sup\, S$. Then, given any positive number $\epsilon$, there is an element $s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon \le u$.*

**Proof:** Suppose there exists an $\epsilon > 0$ such that the interval $(u - \epsilon,\, u]$ contains no points of $S$. Then $s \le u - \epsilon$ for all $s \in S$, which implies that $u - \epsilon$ is an upper bound for $S$ which is less than $u$, a contradiction.

**Definition 5.** *: Let $S \subseteq \mathbb{R}$ be a set that is bounded below. A number $u \in \mathbb{R}$ is called the infimum (greatest lower bound ) of $S$ and is denoted by $\inf S$ if it satisfies the conditions*

1. *$u \le s$ for all $s \in S$.*
2. *If $v$ is a lower bound for $S$, then $v \le u$.*

### The Completeness Axiom

**Axiom** Every nonempty subset $S$ of $\mathbb{R}$ that is bounded above has a least upper bound. That is, if $S$ is bounded above, then $\sup S$ exists and is a real number.

The set of real numbers $\mathbb{R}$ is a complete, ordered, field. The set of rational numbers $\mathcal{Q}$, although an ordered field, is not complete. For example, the set $T = \{r \in \mathcal{Q} : r < \sqrt{2}\}$ is bounded above, but $T$ does not have a rational least upper bound.

**The Archimedean Property**

**THEOREM 4.** *(The Archimedean Property) The set $\mathbb{N}$ of natural numbers is unbounded above.*

**Proof:** Suppose $\mathbb{N}$ is bounded above. Let $m = \sup \mathbb{N}$. By Theorem 3 there exists a positive integer $k$ such that $m - 1 < k \le m$. But then $k + 1$ is a positive integer and $k + 1 > m$, a contradiction.

**THEOREM 5.** *The following are equivalent:*

*(a) The Archimedean Property.*

*(b) For each $z \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > z$.*

*(c) For each $x > 0$ and for each $y \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $nx > y$.*

*(d) For each $x > 0$, there exists an $n \in \mathbb{N}$ such that $0 < 1/n < x$.*

**Proof:** (a) $\Rightarrow$ (b). Suppose there exists a real number $r$ such $n \le r$ for all $n \in \mathbb{N}$. Then $\mathbb{N}$ is bounded above by $r$, contradicting (a).

(b) $\Rightarrow$ (c). Let $z = y/x$. Then, by (b), there exists $n \in \mathbb{N}$ such that $n > y/x$ which implies $nx > y$.

(c) $\Rightarrow$ (d). By (c) there exists $n \in \mathbb{N}$ such that $n > 1/x$ which implies $1/n < x$. Since $n > 0$, $1/n > 0$. Thus, $0 < 1/n < x$.

(d) $\Rightarrow$ (a). Suppose $\mathbb{N}$ is bounded above. Let $m = \sup \mathbb{N}$. Then $n \le m$ for all $n \in \mathbb{N}$ which implies $1/m \le 1/n$ for all $n \in \mathbb{N}$ contradicting (d).

**THEOREM 6.** *There exists a real number $x$ such that $x^2 = 2$. In general, if $p$ is a prime number, then there exists a real number $y$ such that $y^2 = p$.*

There is a "technical" proof which could be given here, but this result is an easy consequence of the Intermediate-Value Theorem in Part III.

**The Density of the Rational Numbers and the Irrational Numbers**

**Lemma:** Let $y$ be a positive number. Then there exists an $m \in \mathbb{N}$ such that $m - 1 \le y < m$.

**Proof:** Let $K = \{n \in \mathbb{N} : n > y\}$. By Theorem 5 (b), $K$ is not empty. By the well-ordering axiom, $K$ has a least element $m$. It follows that $m - 1 \le y < m$.

**THEOREM 7.** *If $x$ and $y$ are real numbers, $x < y$, then there exists a rational number $r$ such that $x < r < y$.*

**Proof:** Assume first that $x > 0$. There exists a positive integer $n$ such that $n > 1/(y-x)$, which implies $nx+1 < ny$. By the lemma, there exists a positive integer $m$ such that $m-1 \leq nx < m$. Therefore, $m \leq nx+1 < ny$. We now have

$$nx < m < ny \quad \text{which implies} \quad x < \frac{m}{n} < y.$$

Take $r = m/n$.

For $x \leq 0$, choose a positive integer $k$ such that $x+k > 0$ and apply the result above to find a rational number $q$ such that $x+k < q < y+k$. Then $r = q-k$ satisfies $x < r < y$.

**THEOREM 8.** *If $x$ and $y$ are real numbers, $x < y$, then there exists an irrational number $z$ such that $x < z < y$.*

**Exercises 1.3**

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.
   (a) If a non-empty subset of $\mathbb{R}$ has a infimum, then it is bounded.
   (b) Every non-empty bounded subset of $\mathbb{R}$ has a maximum and a minimum.
   (c) If $v$ is an upper bound for $S$ $u < v$, then $u$ is not an upper bound for $S$.
   (d) If $w = \inf S$ and $z < w$, then $z$ is a lower bound for $S$.
   (e) Every nonempty subset of $\mathbb{N}$ has a minimum.
   (f) Every nonempty subset of $\mathbb{N}$ has a maximum.
2. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.
   (a) If $x$ and $y$ are irrational, then $xy$ is irrational.
   (b) Between any two distinct rational numbers, there is an irrational number.
   (c) Between any two distinct irrational numbers, there is a rational number.
   (d) The rational and irrational numbers alternate.
3. Let $S \subseteq \mathbb{R}$ be non-empty and bounded above and let $u = \sup S$. Prove that $u \in S$ if and only if $u = \max S$.
4. (a) Let $S \subseteq \mathbb{R}$ be non-empty and bounded above and let $u = \sup S$. Prove that $u$ is unique.
   (b) Prove that if each of $m$ and $n$ is a maximum of $S$, then $m = n$.
5. Let $S \subseteq \mathbb{R}$ and suppose that $v = \inf S$. Prove that for any positive number $\epsilon$, there is an element $s_\epsilon \in S$ such that $v \leq s_\epsilon < v + \epsilon$.
6. Prove that if $x$ and $y$ are real numbers with $x < y$, then there are infinitely rational numbers in the interval $[x, y]$.

## I.4. TOPOLOGY OF THE REALS

**Definition 6.** *Let* $S \subseteq \mathbb{R}$. *The set* $S^c = \{x \in \mathbb{R} : x \notin S\}$ *is called the* **complement** *of* $S$.

**Definition 7.** *Let* $x \in \mathbb{R}$ *and let* $\epsilon > 0$. *An* **$\epsilon$-neighborhood** *of* $x$ *(often shortened to "neighborhood of* $x$*") is the set*

$$N(x, \epsilon) = \{y \in \mathbb{R} : |y - x| < \epsilon\}.$$

*The number* $\epsilon$ *is called the radius of* $N(x, \epsilon)$.

Note that an $\epsilon$-neighborhood of a point $x$ is the open interval $(x - \epsilon, x + \epsilon)$ centered at $x$ with radius $\epsilon$.

**Definition 8.** *Let* $x \in \mathbb{R}$ *and let* $\epsilon > 0$. *A* **deleted $\epsilon$-neighborhood** *of* $x$ *(often shortened to "deleted neighborhood of* $x$*") is the set*

$$N^*(x, \epsilon) = \{y \in \mathbb{R} : 0 < |y - x| < \epsilon\}.$$

A deleted $\epsilon$-neighborhood of $x$ is an $\epsilon$-neighborhood of $x$ with the point $x$ removed;

$$N^*(x, \epsilon) = (x - \epsilon, x) \cup (x, x + \epsilon).$$

**Definition 9.** *Let* $S \subseteq \mathbb{R}$. *A point* $x \in S$ *is an* **interior point** *of* $S$ *if there exists a neighborhood* $N$ *of* $x$ *such that* $N \subseteq S$. *The set of all interior points of* $S$ *is denoted by* $\text{int}\, S$ *and is called the* **interior of** $S$.

**Examples** Make up some examples to illustrate "interior point" and "interior of $S$."

**Definition 10.** *Let* $S \subseteq \mathbb{R}$. *A point* $z \in \mathbb{R}$ *is a* **boundary point** *of* $S$ *if* $N \cap S \neq \emptyset$ *and* $N \cap S^c \neq \emptyset$ *for every neighborhood* $N$ *of* $z$. *The set of all boundary points of* $S$ *is denoted by* bd $S$ *and is called the* **boundary of** $S$.

**Examples** Make up some examples to illustrate "boundary point" and "boundary of $S$."

### Open Sets and Closed Sets

**Definition 11.** *Let* $S \subseteq \mathbb{R}$. $S$ *is* **open** *if every point of* $S$ *is an interior point. That is,* $S$ *is open if and only if* $S = \text{int}\, S$. $S$ *is* **closed** *if and only if* $S^c$ *is open.*

**Examples:** A neighborhood $N$ of a point $x$ is an open set; an open interval $(a, b)$ is an open set; $\mathbb{R}$ is an open set. A closed interval $[a, b]$ is a closed set.

**THEOREM 9.** *Let* $S \subseteq \mathbb{R}$. $S$ *is closed if and only if* bd $S \subseteq S$.

**Proof:** Suppose $S$ is closed. Let $x \in S^c$. Since $S^c$ is open, there is a neighborhood $N$ of $x$ such that $N \cap S = \emptyset$. Therefore $x$ is not a boundary point of $S$. Therefore bd $S \subseteq S$.

Now suppose bd $S \subseteq S$. Let $x \in S^c$. Then $x \notin S$ and $x \notin \text{bd}\, S$. Therefore there is a neighborhood $N$ of $x$ such that $N \cap S = \emptyset$. This implies that $S^c$ is open and $S$ is closed.

**THEOREM 10.** *(a) The union of any collection of open sets is open.*

*(b) The intersection of any finite collection of open sets is open.*

**Proof:** (a) Let $\mathcal{G}$ be a collection of open sets and let $x \in \cup_{G\in\mathcal{G}} G$. Then $x \in G$ for some $G \in \mathcal{G}$. Since $G$ is open, there is a neighborhood $N$ of $x$ such that $N \subseteq G$. Since $N \subseteq G$, $N \subseteq \cup_{G\in\mathcal{G}} G$. Therefore $\cup_{G\in\mathcal{G}} G$ is open.

(b) Let $G_1, G_2, \ldots, G_n$ be a (finite) collection of open sets, and let $x \in \cap G_i$. Since $x \in G_1$ and $G_1$ is open, there is an $\epsilon_1$-neighborhood $N_1$ of $x$ such that $N_1 \subseteq G_1$; since $x \in G_2$ and $G_2$ is open, there is an $\epsilon_2$-neighborhood $N_2$ of $x$ such that $N_2 \subseteq G_2$; $\ldots$; since $x \in G_n$ and $G_n$ is open, there is an $\epsilon_n$-neighborhood $N_n$ of $x$ such that $N_n \subseteq G_n$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ and let $N = N(x, \epsilon)$. Now, $N \subseteq N_i$ for $i = 1, 2, \ldots, n$ which implies $N \subseteq G_i$ for $i = 1, 2, \ldots, n$, and so $N \subseteq \cap G_i$.

**Note:** The restriction "finite" in Theorem 10(b) is necessary. For example, the intersection of the (infinite) collection of open sets $\left(-\frac{1}{n}, 1+\frac{1}{n}\right)$, $n = 1, 2, 3, \ldots$ is the closed interval $[0, 1]$.

**COROLLARY** *(a) The intersection of any collection of closed sets is closed.*

*(b) The union of any finite collection of closed sets is closed.*

The Corollary follows directly from the Theorem by means of **De Morgan's Laws :**

Let $\{S_\alpha\}$, $\alpha \in A$ be a collection of sets. Then

1. $(\cup_{\alpha\in A} S_\alpha)^c = \cap_{\alpha\in A} S_\alpha^c$
2. $(\cap_{\alpha\in A} S_\alpha)^c = \cup_{\alpha\in A} S_\alpha^c$

### Accumulation Points

**Definition 12.** *Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is an* **accumulation point** *of $S$ if every deleted neighborhood $N$ of $x$ contains a point of $S$. The set of accumulation points of $S$ is denoted by $S'$. If $x \in S$ and $x$ is not an accumulation point of $S$, then $x$ is an* **isolated point** *of $S$.*

**Examples** Make up some examples to illustrate "accumulation point" and "isolated point."

**Note:** An accumulation point of $S$ may or may not be a point of $S$.

**Definition 13.** *Let $S \subseteq \mathbb{R}$. The* **closure** *of $S$, denoted by $\overline{S}$, is the set*

$$\overline{S} = S \cup S'.$$

**THEOREM 11.** *Let $S \subseteq \mathbb{R}$. Then*

*(a) $S$ is closed if and only if $S$ contains all its accumulation points.*

*(b) $\overline{S}$ is a closed set.*

**Proof:** (a) Suppose $S$ is closed. Let $x$ be an accumulation point of $S$ and suppose $x \notin S$. Then $x \in S^c$, an open set. Therefore, there is a neighborhood $N$ of $x$ such that $N \subseteq S^c$. Now, $N \cap S = \emptyset$ which implies $x$ is not an accumulation point of $S$, a contradiction.

Now suppose that $S$ contains all its accumulation points. Let $x \in S^c$. Then $x \notin S$ and $x$ is not an accumulation point of $S$. Therefore, there is a neighborhood $N$ of $x$ such that $N \cap S = \emptyset$ so $N \subseteq S^c$. This implies that $S^c$ is open and $S$ is closed.

(b) Let $y$ be an accumulation point of $\overline{S}$ and let $N$ be a neighborhood of $y$. Then $N$ contains a point $x \in \overline{S}$, $x \neq y$. Therefore $x$ is in $S$ or $x$ is an accumulation point of $S$. Suppose $x$ is an accumulation point of $S$. Since $N$ is an open set, there is a neighborhood $N_x$ of $x$ such that $N_x \subseteq N$, and since $x$ is an accumulation point of $S$, $N_x$ contains a point $z$ of $S$, $z \neq x$ (we also choose $N_x$ such that $z \neq y$). Thus, each neighborhood of $N$ of $y$ contains a point of $S$. Therefore $y$ is an accumulation point of $S$ so $y \in \overline{S}$. Since $\overline{S}$ contains all its accumulation points, it is closed.

**Exercises 1.4**

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.
   (a) int $S \cap$ bd $S = \emptyset$.
   (b) bd $S \subseteq S$.
   (c) $S$ is closed if and only if $S =$ bd $S$.
   (d) If $x \in S$, then $x \in$ int $S$ or $x \in$ bd $S$.
   (e) bd $S =$ bd $S^c$.
   (f) bd $S \subseteq S^c$.
2. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.
   (a) A neighborhood is an open set.
   (b) The union of any collection of open sets is open.
   (c) The union of any collection of closed sets is closed.
   (d) The intersection of any collection of open sets is open.
   (e) The intersection of any collection of closed sets is closed.
   (f) $\mathbb{R}$ is neither open nor closed.
3. Classify each set as open, closed, neither, or both.
   (a) $\mathbb{N}$
   (b) $\mathcal{Q}$
   (c) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

(d) $\displaystyle\bigcap_{n=1}^{\infty}\left(0,\frac{1}{n}\right)$

(e) $\{x : x^2 > 0\}$

(f) $\{x : |x-2| \le 3\}$

4. Let $S$ be a bounded infinite set and let $u = \sup S$.

   (a) Prove that if $u \notin S$, then $u \in S'$.

   (b) True or false: if $u \in S$, then $u \notin S'$?

5. Prove that if $x$ is an accumulation point of $S$, then every neighborhood of $x$ contains infinitely many points of $S$.

## I.5. COMPACT SETS

**Definition 14.** *Let $S \subseteq \mathbb{R}$. A collection $\mathcal{G}$ of open sets such that $S \subseteq \bigcup_{G\in\mathcal{G}} G$ is called an* **open cover** *of $S$. A subcollection $\mathcal{F}$ of $\mathcal{G}$ which also covers $S$ is called a* **subcover** *of $S$.*

**Examples** Make up some examples to illustrate "open cover" and "subcover."

**Definition 15.** *A set $S \subseteq \mathbb{R}$ is* **compact** *if and only if every open cover $\mathcal{G}$ of $S$ contains a finite subcover. That is, $S$ is compact if for every open cover $\mathcal{G}$ of $S$ there is a finite collection of open sets $G_1,\ G_2, \ldots\ G_n$ in $\mathcal{G}$ such that $S \subseteq \bigcup_{k=1}^{n} G_k$.*

**Examples** The interval $(0,1]$ is not compact; the open intervals $(1/n,\,1+1/n),\ n = 1,2,3,\ldots$ form an open cover with no finite subcover. The interval $[0,\infty)$ is not compact; the open intervals $(-1/n,\,n),\ n = 1,2,3,\ldots$ form an open cover with no finite subcover.

**THEOREM 12.** *If $S \subseteq \mathbb{R}$ is non-empty, closed and bounded, then $S$ has a maximum and a minimum.*

**Proof:** Since $S$ is bounded, $S$ has a least upper bound $m$ and a greatest lower bound $k$. Since $m$ is the least upper bound for $S$, given any $\epsilon > 0$, $m-\epsilon$ is not an upper bound for $S$. If $m \notin S$, then there exists an $x \in S$ such that $m - \epsilon < x < m$ which implies that $m$ is an accumulation point of $S$. Since $S$ is closed, $m \in S$ and $m = \max S$. A similar argument holds for $k$.

**THEOREM 13.** *(Heine-Borel Theorem) A subset $S$ of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

**Proof:** Suppose $S$ is compact. Let $\mathcal{G}$ be the collection of open intervals $I_n = (-n,n),\ n = 1,2,\ldots$. Then $\mathcal{G}$ is an open cover of $S$. Since $S$ is compact, $G$ contains a finite subcover $I_{n_1},\ I_{n_2}, \ldots,\ I_{n_k}$. Let $m = \max n_i$. Then $S \subseteq\ I_m$, and for all $x \in S$, $|x| \le m$. Therefore $S$ is bounded.

To show that $S$ is closed, we must show that $S$ contains all its accumulation points. Suppose that $p$ is an accumulation point of $S$ and suppose $p \notin S$. For each positive integer $n$, let $G_n = [p-1/n, p+1/n]^c$. Since the complement of a closed interval is an open set, $G_n$ is an open for all $n$. Since $p \notin S$, $S \subseteq \cup G_n$. That is, the sets $G_n$, $n = 1, 2, 3 \ldots$ form an open cover of $S$. Since $S$ is compact, this open cover has a finite subcover, $G_{n_1}$, $G_{n_2}, \ldots,$ $G_{n_k}$. Let $m = \max n_i$. Then the neighborhood $N$ of $p$ of radius $1/m$ contains no points of $S$ contradicting the assumption that $p$ is an accumulation point of $S$.

We omit the proof that $S$ closed and bounded implies that $S$ is compact.

**THEOREM 14.** *(Bolzano-Weierstrass Theorem) If $S \subseteq \mathbb{R}$ is a bounded infinite set, then $S$ has at least one accumulation point.*

**Proof:** Let $S$ be a bounded infinite set and suppose that $S$ has no accumulation points. Then $S$ is closed (vacuously), and $S$ is compact. For each $x \in S$, let $N_x$ be a neighborhood of $x$ such that $S \cap N_x = \{x\}$. The set of neighborhoods $N_x$, $x \in S$ is an open cover of $S$. Since $S$ is compact, this open cover has a finite subcover $N_{x_1}$, $N_{x_2}$, $\ldots$, $N_{x_k}$. But $S \cap [N_{x_1} \cup N_{x_2} \cup \ \ldots \cup N_{x_k}] = \{x_1, x_2, \ldots, x_k\}$ which implies that $S$ is finite, a contradiction.

**Exercises 1.5**

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.
   (a) Every finite set is compact.
   (b) No infinite set is compact.
   (c) If a set is compact, then it has a maximum and a minimum.
   (d) If a set has a maximum and a minimum, then it is compact.
   (e) If $S \subseteq \mathbb{R}$ is compact, then there is at least one point $x \in \mathbb{R}$ such that $x$ is an accumulation point of $S$.
   (f) If $S \subseteq \mathbb{R}$ is compact and $x$ is an accumulation point of $S$, then $x \in S$.
2. Show that each of the following subsets $S$ of $\mathbb{R}$ is not compact by giving an open cover of $S$ that has no finite subcover.
   (a) $S = [0, 1)$
   (b) $S = \mathbb{N}$
   (c) $S = \{1/n : n \in \mathbb{N}\}$
3. Prove that the intersection of any collection of compact sets is compact.
4. Prove that if $S \subseteq \mathbb{R}$ is compact and $T$ is a closed subset of $S$, then $T$ is compact.