Terminology for Bonds and Loans

- *Principal* given to borrower when loan is made
- Simple loan: principal plus interest repaid at one date
- Fixed-payment loan: series of (often equal) repayments
- Bond is issued at some price
- *Face Value* is repayment at maturity date
- *Zero coupon bond* pays only face value at maturity
- *Coupon bond* also makes periodic coupon payments, equal to coupon rate times face value

Compounding

- Assume that the interest rate is 10% p.a.
- What this means is that if you invest \$1 for one year, you have been promised \$1*(1+10/100) or \$1.10 next year
- Investing \$1 for yet another year promises to produce 1.10 *(1+10/100) or \$1.21 in 2-years

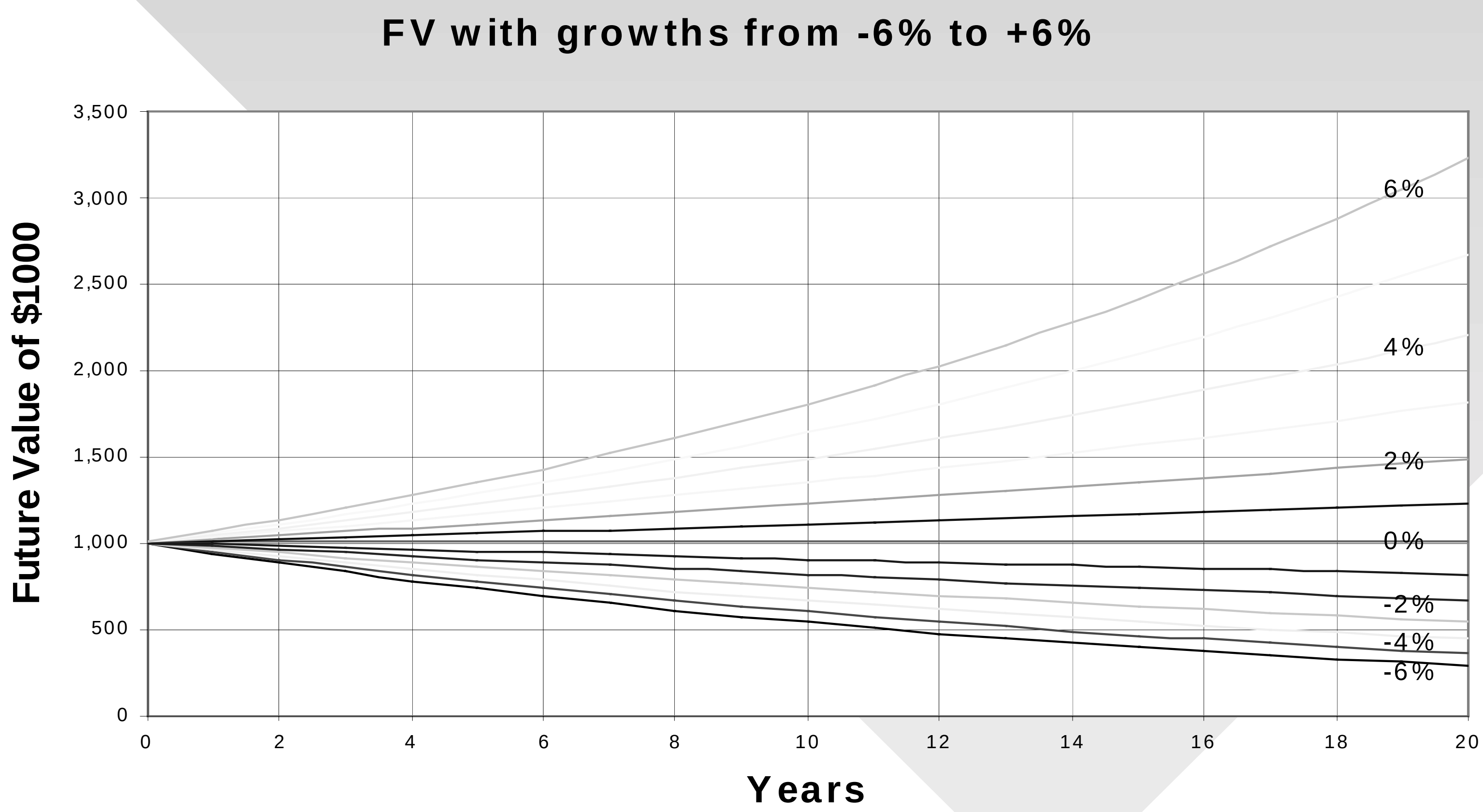
Value of $5 Invested

- More generally, with an investment of $5 at 10% we obtain

1 Year	$5*(1+0.10)	$5.5
2 years	$5.5*(1+0.10)	$6.05
3 years	$6.05*(1+0.10)	$6.655
4 Years	$6.655*(1+0.10)	$7.3205

Future Value of a Lump Sum

$$FV = PV * (1+i)^n$$



Generalizing the method

- Generalizing the method requires some definitions. Let
  - i be the interest rate
  - n be the life of the lump sum investment
  - PV be the present value
  - FV be the future value

Example: Future Value of a Lump Sum

- Your bank offers a CD with an interest rate of 3% for a 5 year investment.
- You wish to invest $1,500 for 5 years, how much will your investment be worth?

$$FV = PV * (1 + i)^n$$
$$= \$1500 * (1 + 0.03)^5$$
$$= \$1738.1111145$$

n	5
i	3%
PV	1,500
FV	?
Result	1738.911111

Present Value of a Lump Sum

$$FV = PV * (1+i)^n$$

Divide both sides by $(1+i)^n$ to obtain:

$$PV = \frac{FV}{(1+i)^n} = FV * (1+i)^{-n}$$

Example: Present Value of a Lump Sum

- You have been offered $40,000 for your printing business, payable in 2 years. Suppose the interest rate is 8%. What is the present value of the offer?

$$PV = \frac{FV}{(1+i)^n}$$

$$= \frac{40{,}000}{(1+0.08)^2}$$

$$= 34293.55281$$

$$\cong \$34{,}293.55 \text{ today}$$

Solving Lump Sum Cash Flow for Interest Rate

$$FV = PV * (1+i)^n$$

$$\frac{FV}{PV} = (1+i)^n$$

$$(1+i) = \sqrt[n]{\frac{FV}{PV}}$$

$$i = \sqrt[n]{\frac{FV}{PV}} - 1$$

Calculations for Bonds and Loans

Interest rate i is yield to maturity.

n is time to maturity

- Simple Loans: use lump sum formula; PV = principal; FV = principal plus interest.
- Zero Coupon Bonds: use lump sum formula; PV = price; FV = face value.
- Fixed-Payment Loans: use annuity formula; pmt = loan payment; PV = principal
- Coupon Bonds: combine annuity and lump sum formulas. With

face value F and coupon payment C, the price p is equal to

$$P = \frac{C}{1+i} + \frac{C}{(1+i)^2} + ... + \frac{C}{(1+i)^n} + \frac{C}{(1+i)^n} + \frac{F}{(1+i)^n}$$

Example: Interest Rate on a Lump Sum Investment

- If you invest $15,000 for ten years, you receive $30,000. What is your annual return?
- Other interpretation: if price of 10 year bond with face value 30,000 is 15,000, then interest rate (yield to maturity) on bond is 7.18%

$$i = \sqrt[n]{\frac{FV}{PV}} - 1$$

$$= \sqrt[10]{\frac{30000}{15000}} - 1 = \sqrt[10]{2} - 1 = 2^{\frac{1}{10}} - 1$$

$$= 0.071773463$$

$$= 7.18\% \text{ (to the nearest basis point)}$$

Yield to Maturity: Loans

Yield to maturity = interest rate that equates today's value with present value of all future payments

1. **Simple Loan (*i* = 10%)**

   $\$100 = \$110/(1 + i) \Rightarrow$

$$i = \frac{\$110 - \$100}{\$100} = \frac{\$10}{\$100} = 0.10 = 10\%$$

2. **Fixed Payment Loan (*i* = 12%)**

$$\$1000 = \frac{\$126}{(1+i)} + \frac{\$126}{(1+i)^2} + \frac{\$126}{(1+i)^3} + \ldots + \frac{\$126}{(1+i)^{25}}$$

$$LV = \frac{FP}{(1+i)} + \frac{FP}{(1+i)^2} + \frac{FP}{(1+i)^3} + \ldots + \frac{FP}{(1+i)^n}$$

Yield to Maturity: Bonds

3. Coupon Bond (Coupon rate = 10% = C/F)

$$P = \frac{\$100}{(1+i)} + \frac{\$100}{(1+i)^2} + \frac{\$100}{(1+i)^3} + \ldots + \frac{\$100}{(1+i)^{10}} + \frac{\$1000}{(1+i)^{10}}$$

$$P = \frac{C}{(1+i)} + \frac{C}{(1+i)^2} + \frac{C}{(1+i)^3} + \ldots + \frac{C}{(1+i)^n} + \frac{F}{(1+i)^n}$$

Consol: Fixed coupon payments of $\$C$ forever

$$P = \frac{C}{i} \qquad i = \frac{C}{P}$$

4. Discount Bond (*P* = \$900, *F* = \$1000), one year

$$\$900 = \frac{\$1000}{(1+i)} \quad \Rightarrow$$

$$i = \frac{\$1000 - \$900}{\$900} = 0.111 = 11.1\%$$

$$i = \frac{F - P}{P}$$

Relationship Between Price and Yield to Maturity

Table 1 Yields to Maturity on a 10%-Coupon-Rate Bond Maturing in Ten Years (Face Value = $1,000)

Price of Bond ($)	Yield to Maturity (%)
1,200	7.13
1,100	8.48
1,000	10.00
900	11.75
800	13.81

Three Interesting Facts in Table 1

1. When bond is at par, yield equals coupon rate
2. Price and yield are negatively related
3. Yield greater than coupon rate when bond price is below par value

Bond Yields and Bond Prices

- yield to maturity on a coupon bond cannot be calculated analytically; need computer or financial calculator
- YTM high if bond price is low
- YTM equals coupon rate if bond price equals face value (bond trades at par)
- YTM higher (lower) than coupon rate if bond price lower (higher) than face value
- current yield (coupon rate/price) is approximation to YTM; works well if long maturity, bond trades close to par.

The Problem

How much will I have available in my retirement account if I deposit $2,000 per year into an IRA that pays on average 16% per year if I plan to retire in 40 years time?

Annuity = Stream of Cash Flows, such that

- the first cash flow will occur exactly one period form now
- all subsequent cash flows are separated by exactly one period
- all periods are of equal length
- the interest rate is constant
- all cash flows have the same value

Annuity Formula Notation

- PV = the present value of the annuity
- i = interest rate to be earned over the life of the annuity
- n = the number of payments
- pmt = the periodic payment (cash flow)

Derivation of PV of Annuity Formula

$$PV = \frac{pmt}{(1+i)^1} + \frac{pmt}{(1+i)^2} +$$

$$\frac{pmt}{(1+i)^3} + \Lambda + \frac{pmt}{(1+i)^{n-1}} + \frac{pmt}{(1+i)^n}$$

PV of Annuity Formula

$$PV = \frac{pmt * \{1 - \frac{1}{(1+i)^n}\}}{i}$$

$$= \frac{pmt}{i} * \left(1 - \frac{1}{(1+i)^n}\right)$$

PV Annuity Formula: Payment

$$PV = \frac{pmt}{i} * \left(1 - \frac{1}{(1+i)^n} \right)$$

$$= \frac{pmt}{i} * \left(1 - (1+i)^{-n} \right)$$

$$pmt = \frac{PV * i}{\left(1 - (1+i)^{-n} \right)}$$

Derivation of FV of Annuity Formula: Algebra

$$PV = \frac{pmt}{i} * \left(1 - \frac{1}{(1+i)^n}\right) \quad \text{(reg. annuity)}$$

$$\text{FV} = PV * (1+i)^n \quad \text{(lump sum)}$$

$$\text{FV} = \frac{pmt}{i} * \left(1 - \frac{1}{(1+i)^n}\right) * (1+i)^n$$

$$= \frac{pmt}{i} * \left((1+i)^n - 1\right)$$

FV Annuity Formula: Payment

$$FV = \frac{pmt}{i} * \left((1+i)^n - 1 \right)$$

$$pmt = \frac{FV * i}{\left((1+i)^n - 1 \right)}$$

Data for Problem

- I = 16%
- n = 40
- Pmt = $2,000
- FV = ?

Solution

$$F = \frac{pmt}{i}\left(\left(1+i\right)^{n}-1\right)$$

$$= \frac{2000}{0.16}\left(\left(1+0.16\right)^{40}-1\right)$$

$$= \$4,721,514.481$$