

## 2.016 Hydrodynamics

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### Potential Flow Theory

“When a flow is both frictionless and irrotational, pleasant things happen.” –F.M. White, Fluid Mechanics 4th ed.

We can treat external flows around bodies as *inviscid* (i.e. frictionless) and *irrotational* (i.e. the fluid particles are not rotating). This is because the viscous effects are limited to a thin layer next to the body called the boundary layer. In graduate classes like 2.25, you’ll learn how to solve for the inviscid flow and then correct this within the boundary layer by considering viscosity. For now, let’s just learn how to solve for the inviscid flow.

We can define a *potential function*,  $\phi(x, z, t)$ , as a continuous function that satisfies the basic laws of fluid mechanics: *conservation of mass and momentum*, assuming incompressible, inviscid and irrotational flow.

There is a *vector identity* (prove it for yourself!) that states for any scalar,  $\phi$ ,

$$\nabla \times \nabla \phi = 0$$

By definition, for *irrotational flow*,

$$\nabla \times \vec{V} = 0$$

Therefore

$$\vec{V} = \nabla \phi$$

where  $\phi = \phi(x, y, z, t)$  is the velocity *potential function*. Such that the components of velocity in Cartesian coordinates, as functions of space and time, are

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \quad \text{and} \quad w = \frac{\partial \phi}{\partial z} \quad (4.1)$$

## Laplace Equation

The velocity must still satisfy the conservation of mass equation. We can substitute in the relationship between potential and velocity and arrive at the *Laplace Equation*, which we will revisit in our discussion on linear waves.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4.2)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (4.3)$$

$$\text{LaplaceEquation} \Rightarrow \nabla^2 \phi = 0$$

For your reference given below is the Laplace equation in different coordinate systems: Cartesian, cylindrical and spherical.

*Cartesian Coordinates* ( $x, y, z$ )

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = \nabla \phi$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

*Cylindrical Coordinates* ( $r, \theta, z$ )

$$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\vec{V} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z = \frac{\partial \phi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{e}_\theta + \frac{\partial \phi}{\partial z} \hat{e}_z = \nabla \phi$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \underbrace{\frac{1}{r} \frac{\partial \phi}{\partial r}}_{\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right)} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

### Spherical Coordinates ( $r, \theta, \varphi$ )

$$r^2 = x^2 + y^2 + z^2, \quad \theta = \cos^{-1}\left(\frac{z}{r}\right), \quad \text{or } x = r \cos \theta, \quad \varphi = \tan^{-1}\left(\frac{z}{y}\right)$$

$$\vec{V} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_\varphi \hat{e}_\varphi = \frac{\partial \phi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \hat{e}_\varphi = \nabla \phi$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2} = 0$$

$$\underbrace{\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right)}_{\text{}} = 0$$

### Potential Lines

Lines of constant  $\phi$  are called *potential lines* of the flow. In two dimensions

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$$d\phi = u dx + v dy$$

Since  $d\phi = 0$  along a potential line, we have

$$\frac{dy}{dx} = -\frac{u}{v} \tag{4.4}$$

Recall that streamlines are lines everywhere tangent to the velocity,  $\frac{dy}{dx} = \frac{v}{u}$ , so **potential lines are perpendicular to the streamlines**. For inviscid and irrotational flow is indeed quite pleasant to use potential function,  $\phi$ , to represent the velocity field, as it reduced the problem from having three unknowns ( $u, v, w$ ) to only one unknown ( $\phi$ ).

As a point to note here, many texts use stream function instead of potential function as it is slightly more intuitive to consider a line that is everywhere tangent to the velocity. Streamline function is represented by  $\psi$ . Lines of constant  $\psi$  are perpendicular to lines of constant  $\phi$ , except at a stagnation point.

Luckily  $\phi$  and  $\psi$  are related mathematically through the velocity components:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad (4.5)$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad (4.6)$$

Equations (4.5) and (4.6) are known as the Cauchy-Riemann equations which appear in complex variable math (such as 18.075).

## Bernoulli Equation

The Bernoulli equation is the most widely used equation in fluid mechanics, and assumes frictionless flow with no work or heat transfer. However, flow may or may not be irrotational. When flow is irrotational it reduces nicely using the potential function in place of the velocity vector. The potential function can be substituted into equation 3.32 resulting in the unsteady Bernoulli Equation.

$$\rho \left\{ \frac{\partial}{\partial t} \nabla \phi + \frac{1}{2} (\nabla \phi)^2 \right\} + \nabla p + \rho g \nabla z = 0 \quad (4.7)$$

or

$$\nabla \left\{ \rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho V^2 + p + \rho g z \right\} = 0. \quad (4.8)$$

$$\text{Unsteady Bernoulli} \Rightarrow \rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho V^2 + p + \rho g z = c(t) \quad (4.9)$$

## Summary

	Potential	Stream Function
Definition	$\mathbf{V} = \nabla\phi$	$\mathbf{V} = \nabla \times \psi$
Continuity ( $\nabla \cdot \mathbf{V} = 0$ )	$\nabla^2\phi = 0$	Automatically Satisfied
Irrotationality ( $\nabla \times \mathbf{V} = 0$ )	Automatically Satisfied	$\nabla \times (\nabla \times \psi) = \nabla(\nabla \cdot \psi) - \nabla^2\psi = 0$
In 2D : $w = 0, \frac{\partial}{\partial z} = 0$		
	$\nabla^2\phi = 0$ for continuity	$\psi \equiv \psi_z, \nabla^2\psi = 0$ for irrotationality
Cauchy-Riemann Equations for $\phi$ and $\psi$ from complex analysis: $\Phi = \phi + i\psi$ , where $\phi$ is real part and $\psi$ is the imaginary part		
Cartesian ( $x, y$ )	$u = \frac{\partial\phi}{\partial x}$ $v = \frac{\partial\phi}{\partial y}$	$u = \frac{\partial\psi}{\partial y}$ $v = -\frac{\partial\psi}{\partial x}$
Polar ( $r, \theta$ )	$u = \frac{\partial\phi}{\partial r}$ $v = \frac{1}{r} \frac{\partial\phi}{\partial\theta}$	$u = \frac{1}{r} \frac{\partial\psi}{\partial\theta}$ $v = -\frac{\partial\psi}{\partial r}$

For irrotational flow use:  $\phi$

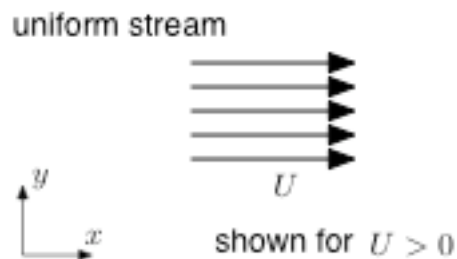
For incompressible flow use:  $\psi$

For incompressible **and** irrotational flow use:  $\phi$  and  $\psi$

## Potential flows

Potential functions  $\phi$  (and stream functions,  $\psi$ ) can be defined for various simple flows. These potential functions can also be superimposed with other potential functions to create more complex flows.

### Uniform, Free Stream Flow (1D)



$$\vec{V} = U\hat{i} + 0\hat{j} + 0\hat{k} \quad (4.10)$$

$$u = U = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \quad (4.11)$$

$$v = 0 = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \quad (4.12)$$

We can integrate these expressions, ignoring the constant of integration which ultimately does not affect the velocity field, resulting in  $\phi$  and  $\psi$

$$\phi = Ux \quad \text{and} \quad \psi = Uy \quad (4.13)$$

Therefore we see that streamlines are horizontal straight lines for all values of  $y$  (tangent everywhere to the velocity!) and that equipotential lines are vertical straight lines perpendicular to the streamlines (and the velocity!) as anticipated.

- **2D Uniform Flow:**  $\mathbf{V} = (U, V, 0)$ ;  $\phi = Ux + Vy$ ;  $\psi = Uy - Vx$
- **3D Uniform Flow:**  $\mathbf{V} = (U, V, W)$ ;  $\phi = Ux + Vy + Wz$ ; no stream function in 3D

## Line Source or Sink

Consider the  $z$ -axis (into the page) as a porous hose with fluid radiating outwards or being drawn in through the pores. Fluid is flowing at a rate  $Q$  (positive or outwards for a source, negative or inwards for a sink) for the entire length of hose,  $b$ . For simplicity take a unit length into the page ( $b = 1$ ) essentially considering this as 2D flow.

source ( $Q > 0$ ) or sink ( $Q < 0$ )



shown for  $Q > 0$

Polar coordinates come in quite handy here. The source is located at the origin of the coordinate system. From the sketch above you can see that there is no circumferential velocity, but only radial velocity. Thus the velocity vector is

$$\vec{V} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z = u_r \hat{e}_r + 0 \hat{e}_\theta + 0 \hat{e}_z \quad (4.14)$$

$$u_r = \frac{Q}{2\pi r} = \frac{m}{r} = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad (4.15)$$

and

$$u_\theta = 0 = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} \quad (4.16)$$

Integrating the velocity we can solve for  $\phi$  and  $\psi$

$$\phi = m \ln r \quad \text{and} \quad \psi = m\theta \quad (4.17)$$

where  $m = \frac{Q}{2\pi}$ . Note that  $\phi$  satisfies the Laplace equation *except* at the origin:

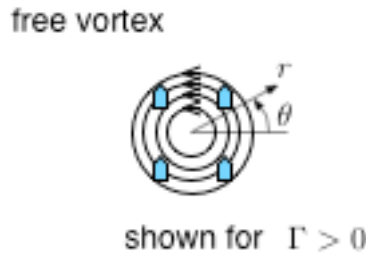
$r = \sqrt{x^2 + y^2} = 0$ , so we consider the origin a singularity (mathematically speaking) and exclude it from the flow.

- The net outward volume flux can be found by integrating in a closed contour around the origin of the source (sink):

$$\int_C \mathbf{V} \cdot \hat{n} \, dS = \iint_S \nabla \cdot \mathbf{V} \, dS = \int_0^{2\pi} u_r r \, d\theta = Q$$

## Irrotational Vortex (Free Vortex)

A free or *potential vortex* is a flow with circular paths around a central point such that the velocity distribution still satisfies the irrotational condition (i.e. the fluid particles do not themselves rotate but instead simply move on a circular path). See [figure 2](#).



**Figure 2:** Potential vortex with flow in circular patterns around the center. Here there is no radial velocity and the individual particles do not rotate about their own centers.

It is easier to consider a cylindrical coordinate system than a Cartesian coordinate system with velocity vector  $\mathbf{V} = (u_r, u_\theta, u_z)$  when discussing point vortices in a local reference frame. For a 2D vortex,  $u_z = 0$ . Referring to [figure 2](#), it is clear that there is also no radial velocity. Thus,

$$\vec{V} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z = 0 \hat{e}_r + u_\theta \hat{e}_\theta + 0 \hat{e}_z \quad (4.18)$$

where

$$u_r = 0 = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad (4.19)$$

and

$$u_\theta = ? = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}. \quad (4.20)$$

Let us derive  $u_\theta$ . Since the flow is considered irrotational, all components of the vorticity vector must be zero. The vorticity in cylindrical coordinates is

$$\nabla \times \mathbf{V} = \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \mathbf{e}_r + \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \mathbf{e}_\theta + \left( \frac{1}{r} \frac{\partial r u_\theta}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \mathbf{e}_z = 0, \quad (4.21)$$

where



$$u_r^2 = u_x^2 + u_y^2, \quad (4.22)$$

$$u_\theta = u_x \cos \theta, \quad (4.23)$$

and

$$u_z = 0 \text{ (for 2D flow).}$$

Since the vortex is 2D, the z-component of velocity and all derivatives with respect to z are zero. Thus to satisfy irrotationality for a 2D potential vortex we are only left with the z-component of vorticity ( $\mathbf{e}_z$ )

$$\frac{\partial ru_\theta}{\partial r} - \frac{\partial u_r}{\partial \theta} = 0 \quad (4.24)$$

Since the vortex is axially symmetric all derivatives with respect to  $\theta$  must be zero. Thus,

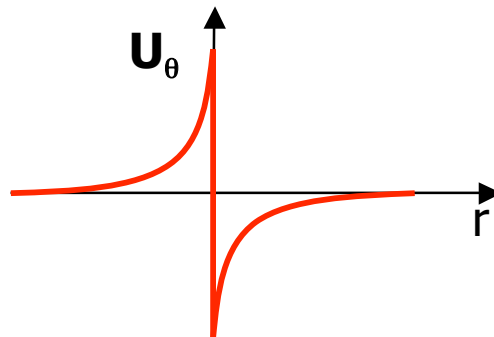
$$\frac{\partial(ru_\theta)}{\partial r} = \frac{\partial u_r}{\partial \theta} = 0 \quad (4.25)$$

From this equation it follows that  $ru_\theta$  must be a constant and the velocity distribution for a potential vortex is

$$u_\theta = \frac{K}{r}, \quad u_r = 0, \quad u_z = 0 \quad (4.26)$$

By convention we set the constant equal to  $\frac{\Gamma}{2\pi}$ , where  $\Gamma$  is the *circulation*. Therefore

$$u_\theta = \frac{\Gamma}{2\pi r} \quad (4.27)$$



**Figure 3:** Plot of velocity as a function of radius from the vortex center. At the core of the potential vortex the velocity blows up to infinity and is thus considered a singularity.

You will notice (see [figure 3](#)) that the velocity at the center of the vortex goes to infinity (as  $r \rightarrow 0$ ) indicating that the potential vortex core represents a singularity point. This is not true in a real, or viscous, fluid. Viscosity prevents the fluid velocity from becoming infinite at the vortex core and causes the core rotate as a solid body. The flow in this core region is no longer considered irrotational. Outside of the viscous core potential flow can be considered acceptable.

Integrating the velocity we can solve for  $\phi$  and  $\psi$

$$\phi = K\theta \quad \text{and} \quad \psi = -K \ln r \quad (4.28)$$

where  $K$  is the strength of the vortex. By convention we consider a vortex in terms of its *circulation*,  $\Gamma$ , where  $\Gamma = 2\pi K$  is positive in the clockwise direction and represents the strength of the vortex, such that

$$\phi = \frac{\Gamma}{2\pi}\theta \quad \text{and} \quad \psi = -\frac{\Gamma}{2\pi} \ln r. \quad (4.29)$$

Note that, using the potential or stream function, we can confirm that the velocity field resulting from these functions has no radial component and only a circumferential velocity component.

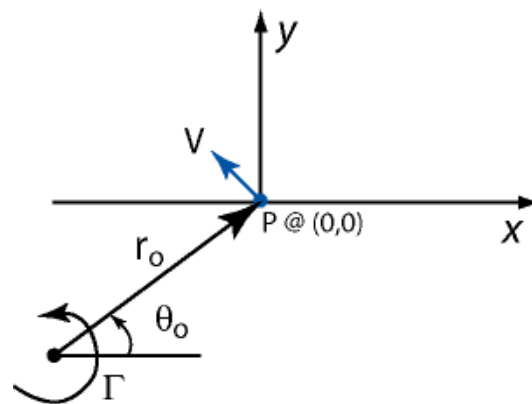
The circulation can be found mathematically as the line integral of the tangential component of velocity taken about a closed curve,  $C$ , in the flow field. The equation for circulation is expressed as

$$\Gamma = \int_C \mathbf{V} \cdot d\mathbf{s}$$

where the integral is taken in a counterclockwise direction about the contour,  $C$ , and  $d\mathbf{s}$  is a differential length along the contour. No singularities can lie directly on the contour. The origin (center) of the potential vortex is considered as a singularity point in the flow since the velocity goes to infinity at this point. If the contour encircles the potential vortex origin, the circulation will be non-zero. If the contour does not encircle any singularities, however, the circulation will be zero.

To determine the velocity at some point  $P$  away from a point vortex (figure 4), we need to first know the velocity field due to the individual vortex, in the reference frame of the vortex. Equation **Error! Reference source not found.** can be used to determine the tangential velocity at some distance  $r_o$  from the vortex. It was given up front that  $u_r = 0$  everywhere.

Since the velocity at some distance  $r_o$  from the body is constant on a circle, centered on the vortex origin, the angle  $\theta_o$  is not crucial for determining the magnitude of the tangential velocity. It is necessary, however, to know  $\theta_o$  in order to resolve the direction of the velocity vector at point  $P$ . The velocity vector can then be transformed into Cartesian coordinates at point  $P$  using equations (4.22) and (4.23).



**Figure 4:** Velocity vector at point  $P$  due to a potential vortex, with strength  $\Gamma$ , located some distance  $r_o$  away.

## Linear Superposition

All three of the simple potential functions, presented above, satisfy the Laplace equation. Since Laplace equation is a linear equation we are able to superimpose two potential functions together to describe a complex flow field. Laplace's equation is

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (4.30)$$

Let  $\phi = \phi_1 + \phi_2$  where  $\nabla^2 \phi_1 = 0$  and  $\nabla^2 \phi_2 = 0$ . Laplace's equation for the total potential is

$$\nabla^2 \phi = \frac{\partial^2 (\phi_1 + \phi_2)}{\partial x^2} + \frac{\partial^2 (\phi_1 + \phi_2)}{\partial y^2} + \frac{\partial^2 (\phi_1 + \phi_2)}{\partial z^2}. \quad (4.31)$$

$$\nabla^2 \phi = \left( \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial x^2} \right) + \left( \frac{\partial^2 \phi_1}{\partial y^2} + \frac{\partial^2 \phi_2}{\partial y^2} \right) + \left( \frac{\partial^2 \phi_1}{\partial z^2} + \frac{\partial^2 \phi_2}{\partial z^2} \right) \quad (4.32)$$

$$\nabla^2 \phi = \left( \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} + \frac{\partial^2 \phi_1}{\partial z^2} \right) + \left( \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} + \frac{\partial^2 \phi_2}{\partial z^2} \right) \quad (4.33)$$

$$\therefore \nabla^2 \phi = \nabla^2 \phi_1 + \nabla^2 \phi_2 = 0 + 0 = 0 \quad (4.34)$$

Therefore the combined potential also satisfies continuity (Laplace's Equation)!

### Example: Combined source and sink

Take a source, strength  $+m$ , located at  $(x,y) = (-a,0)$  and a sink, strength  $-m$ , located at  $(x,y) = (+a,0)$ .

$$\phi = \phi_{\text{source}} + \phi_{\text{sink}} = \frac{1}{2} m \left[ \ln \left( (x+a)^2 + y^2 \right) - \ln \left( (x-a)^2 + y^2 \right) \right] \quad (4.35)$$

This is presented in cartesian coordinates for simplicity. Recall  $r^2 = x^2 + y^2$  so that

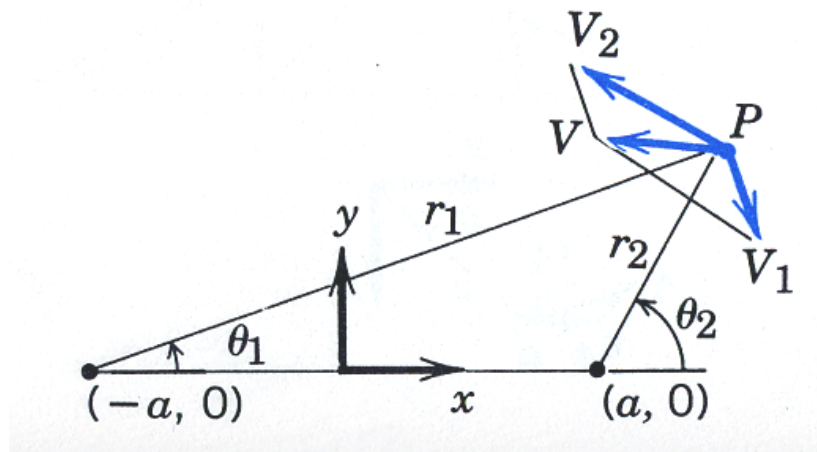
$$m \ln r = m \ln \left[ (x^2 + y^2)^{1/2} \right] = \frac{1}{2} m \ln (x^2 + y^2) \quad (4.36)$$

$$\phi = \frac{1}{2} m \ln \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \quad (4.37)$$

This is analogous to the electro-potential patterns of a magnet with poles at  $(\pm a, 0)$ .

### Example: Multiple Point Vortices

Since we are able to represent a vortex with a simple potential velocity function, we can readily investigate the effect of multiple vortices in close proximity to each other. This can be done simply by a linear superposition of potential functions. Take for example two vortices, with circulation  $\Gamma_1$  and  $\Gamma_2$ , placed at  $\pm a$  along the x-axis (see figure 5). The velocity at point  $P$  can be found as the vector sum of the two velocity components  $\mathbf{V}_1$  and  $\mathbf{V}_2$ , corresponding to the velocity generated independently at point  $P$  by vortex 1 and vortex 2, respectively.



**Figure 5:** Formulation of the combined velocity field from two vortices in close proximity to each other. Vortex 1 is located at point  $(x, y) = (-a, 0)$  and vortex 2 at point  $(x, y) = (+a, 0)$ .

The total velocity potential function is simply a sum of the potentials for the two individual vortices

$$\phi_T = \phi_{v1} + \phi_{v2} = \phi_1 + \phi_2 = \frac{\Gamma_1}{2\pi} \theta_1 + \frac{\Gamma_2}{2\pi} \theta_2$$

with  $\theta_1$  and  $\theta_2$  taken as shown in figure 5. One vortex in close proximity to another vortex tends to induce a velocity on its neighbor, causing the free vortex to move.