PRE-ALGEBRA:
8th Grade

A Common Core State Standards Textbook

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How to Use the Book

The two primary purposes for this book are to be a resource for understanding and a single place for homework assignments. As a resource, you should read the sections in the book that you have a hard time understanding in class. This book won't replace the instruction that you receive from your teacher, but it should supplement that instruction. That means it should help you understand better if you actually read through the examples and think about what is being said.

As a place for homework assignments, this book puts all the homework directly after the explanation of each section. Since you can write in this book directly, you are welcome to do your homework right in this book if you have room to show your work. You will probably end up using a separate sheet of paper to do homework on concepts like solving equations, but most units you'll have room to do the homework in this book.

Included as a homework assignment are unit pre-tests. These pre-tests should be completed at the start of each unit so that your teacher can really zero in on what specific skills you still need help with and what skills you already have mastered. After that you should work on correcting the pre-test which acts like a study guide for the post-test (or end of the unit test). Use the pre-test to help you study.

Please take care of this book as the construction is basic in nature in order to keep the costs down and allow you to write in it. This is your book and only yours. It will not be passed on to students next year. However, if you lose this book, you may be asked to pay for a replacement. Please treat this book gently and with respect.

If along the way, you notice any errors, please let your teacher know so that the error can be corrected for next year's students. We need your help to make this book better and better. Thank you in advance and enjoy!

Unit 1: Exponents

1.1 Operations with Exponents

1.2 Negative Exponents

1.3 Negative Exponent Operations

1.4 Scientific Notation and Appropriate Units

1.5 Scientific Notation Operations

Pre-Test Unit 1: Exponents

No calculator necessary. Please do not use a calculator.

Evaluate, meaning multiply out the exponent, giving your answer as a fraction when necessary.
(5 pts; 2 pts for only simplifying but not evaluating)

1. $\frac{3^{-1}}{3^2}$

2. $(2^3)^{-4} \times 2^8$

3. $(7^{12})(7^{-10})$

4. $\frac{(t^7)(t^4)}{t^5}$

5. $(x^{-2})^{-6}$

6. $(m^5)(m^{-2})$

Determine if the following equations are true. Justify your answer. *(5 pts; 2 pts for answer, 3 pts for justification)*

7. $j^2 \times j^{-7} = j^{-2} \times j^{-3}$

8. $\frac{8^5}{8^0} = (8^3)^2$

Determine the appropriate exponent to make the equation true. *(5 pts; no partial credit)*

9. $(3^{-4})^4 = (3^8)^{\square}$

10. $\frac{b^{-2} \times b^8}{b^5} = \frac{b^{\square}}{b^3}$

Write the following numbers in scientific notation. *(5 pts; 2 pts for correct digits, 3 pts for correct power of ten)*

11. 5,070,000,000

12. 0.000 000 27

Write the following numbers in standard form. *(5 pts; 2 pts for moving the decimal in the correct direction)*

13. 3.4×10^7

14. 9.7×10^{-5}

Choose the best unit of measurement for the following problems. *(5 pts; no partial credit)*

15. A plant grows approximately 3×10^{-4} meters per day. Would this be best expressed using kilometers, meters, or millimeters of growth per day?

Estimate each of the following as a single digit times a power of ten. Then compute each of the following giving your answer in scientific notation. *(5 pts; 2 pts for estimation, 3 pts for scientific notation answer)*

16. $(4 \times 10^{-9})(2 \times 10^{6})$

17. $\frac{2.4\times 10^{8}}{20,000}$

18. $6.3 \times 10^{6} + 300,000$

Answer the following questions giving <u>both</u> an estimated answer (single digit times a power of ten) and a precise answer (scientific notation). *(5 pts; 2 pts for estimation, 3 pts for scientific notation answer)*

19. A town has about $15,000$ people living in it and the mayor wants to send each person $10,000 as a celebration gift because the town won the Federal Lottery for Small Towns. (They'd been buying tickets for years and finally hit the jackpot!) How much money would the town need to give out this celebration gift?

20. A soccer ball has a volume of about $5,800\ cm^3$ and a baseball $200\ cm^3$. How many times bigger in volume is a soccer ball than a baseball?

1.1 Operations with Exponents

First let's start with a review of what exponents are. Recall that 3^4 means taking four 3's and multiplying them together. So we know that $3^4 = 3 \times 3 \times 3 \times 3 = 81$. You might also recall that in the number 3^4, three is called the base and four is called the exponent. Other reminds include that any number to the zero power is equal to one (so $5^0 = 1$) and any number is equal to itself to the first power (so $5^1 = 5$).

Sometimes it is easier to leave a number written as an exponent. For example, it is much easier to write 5^{20} instead of $95{,}367{,}431{,}640{,}625$. Not only is sometimes simpler to write a number using exponents, but many operations are easier when the numbers are written as exponents.

Multiplying Numbers with the Same Base

Let's examine the problem $3^4 \times 3^4$ and write the answer as an exponent. Yes, we could multiply it out as a standard form number, $81 \times 81 = 6561$, but let's keep it in exponential form to see if it is any easier.

First, let's expand the problem: $3^4 \times 3^4 = (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3)$. Notice that the only operation that is happening here is multiplication and that we are multiplying the same number. That means we can say the following: $3^4 \times 3^4 = (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3) = 3^8$. In short we see that $3^4 \times 3^4 = 3^8$. Do you see a rule that we could generalize from this?

Let's look at another example but this time with a variable.

$$y^7 \times y^4 = (y \times y \times y \times y \times y \times y \times y) \times (y \times y \times y \times y) = y^{11}$$

Can you find a rule that we can use when multiplying two exponent numbers with the same base? Yes, we can add the exponents. In other words, $z^5 \times z^6 = z^{5+6} = z^{11}$ would be a quicker way to show work for this problem. Generalizing this, we have the rule that $x^a \times x^b = x^{a+b}$.

Will this work with numbers without the same base? Let's find out by looking at $5^2 \times 2^3$. Many people think that $5^2 \times 2^3 = 10^5$, but we know that $5^2 \times 2^3 = 25 \times 8 = 200$ and that $10^5 = 100{,}000$. So we see that $5^2 \times 2^3 = 10^5$ is not true. Therefore we know that we can only add the exponents when we have the same base.

In fact, if asked to simplify $4^2 \times 7^2$ we would either have to multiply it out as a regular number or else leave it alone if we wanted it written using exponents.

Dividing Numbers with the Same Base

If multiplying numbers with the same base meant that we could add the exponents, what rule do you think we will discover when dividing numbers with the same exponent? Let's find out by looking at an example.

$$\frac{4^7}{4^5} = \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4 \times 4 \times 4}$$

Note that since only multiplication and division is happening, five of fours in the denominator will "cancel" (they actually become one since four divided by four is one, we just call it "canceling") with five of the fours in the numerator. That means we get the following:

$$\frac{4^7}{4^5} = \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4 \times 4 \times 4} = \frac{4 \times 4}{1} = 4^2$$

Let's look at one more example using variables before generalizing a rule for dividing exponent numbers with the same base.

$$\frac{q^6}{q^2} = \frac{q \times q \times q \times q \times q \times q}{q \times q} = \frac{q \times q \times q \times q}{1} = q^4$$

It looks like our rule is similar to the multiplication of exponent numbers with the same base, but this time we subtract the exponents. This gives us the general rule of $\frac{x^a}{x^b} = x^{a-b}$. For now we will only deal with division cases where the numerator exponent is larger than the denominator, but think ahead to what would happen if the denominator's exponent were larger. What do you think would happen?

A Power to a Power

We can also take exponents themselves to a power. For example, think of the problem $(2^3)^2$. Following our order of operations, we know that we have to do the parentheses first which means we get $(2^3)^2 = 8^2 = 64$. However, what if we wanted to leave our answer as a number to a power? Note the following:

$$(2^3)^2 = (2^3)(2^3) = (2 \times 2 \times 2) \times (2 \times 2 \times 2) = 2^6$$

Again, can you see a rule here? Let's look at an example with a variable to help again.

$$(g^4)^3 = (g^4)(g^4)(g^4) = (g \times g \times g \times g) \times (g \times g \times g \times g) \times (g \times g \times g \times g) = g^{12}$$

For a power to a power when using the same base we get the rule that you can multiply the exponents. This generalizes to $(x^a)^b = x^{ab}$.

Lesson 1.1

Perform the following operations leaving your answer as a number to a power. Remember that the parentheses can mean multiply as well.

1. $5^3 \times 5^7$
2. $(12^9)(12^0)$
3. $\frac{(t^5)(t^4)}{t^2}$
4. $\frac{4^{13}}{4^7} \times 4^{10}$
5. $\frac{f^5}{f}$
6. $\frac{u^{11}}{u^4}$
7. $(5^4)^5$
8. $(b^3)^6 \times (b^2)^9$
9. $(j^{11})^5$

Evaluate, meaning multiply out the exponents.

10. $3^2 \times 3^2$
11. $\frac{(2^{10})(2^2)}{2^9}$
12. $\frac{(5^3)^2}{5^4}$
13. $\frac{4^{12}}{4^{10}}$
14. $(5^3)^1 \times 5^0$
15. $(1^4)^2$

Determine if the following equations are true. Justify your answer.

16. $12^2 \times 12^7 = 12^6 \times 12^3$
17. $\frac{x^8}{x^3} = \frac{x^5}{x}$
18. $(t^5)^2 = (t^2)^5$
19. $(5^{10})^2 = (5^5)^5$
20. $\frac{6^0 \times 6^8}{6^4} = \frac{6^4}{6^0}$
21. $m^5 \times m^5 = (m^{10})^0$
22. $\frac{k^6}{k^2} = k^2 \times k^6$
23. $\frac{(7^4)^2}{7^3} = 7^3 \times 7^2$
24. $\frac{3 \times 3^4}{3^4} = (3^5)^1$

Determine the appropriate exponent to make the equation true.

25. $2^5 \times 2^{\boxed{?}} = 2^3 \times 2^3$
26. $\frac{p^6}{p^2} = \frac{p^7}{p^{\boxed{?}}}$
27. $(3^4)^3 = (3^6)^{\boxed{?}}$
28. $(5^{10})^2 = (5^{\boxed{?}})^5$
29. $\frac{b^2 \times b^8}{b^{\boxed{?}}} = \frac{b^7}{b^3}$
30. $9^{\boxed{?}} \times 9^8 = (9^3)^5$
31. $\frac{h^{\boxed{?}}}{h^2} = h^3 \times h^5$
32. $\frac{(6^{11})^{\boxed{?}}}{6^6} = 6^8 \times 6^8$
33. $\frac{3^{\boxed{?}} \times 3^9}{3^2} = (3^7)^1$

1.2 Negative Exponents

Last time we learned that when we divide exponent numbers with the same base we can subtract the exponents. We only examined problems where the numerator had a higher exponent than the denominator, but what would happen if the denominator had the higher exponent? Let's look.

$$\frac{5^3}{5^5} = \frac{5 \times 5 \times 5}{5 \times 5 \times 5 \times 5 \times 5}$$

Notice that three of the fives will "cancel" (remember that they really become one because five divided by five is one). That means we are left with the following:

$$\frac{5^3}{5^5} = \frac{5 \times 5 \times 5}{5 \times 5 \times 5 \times 5 \times 5} = \frac{1}{5 \times 5} = \frac{1}{5^2}$$

However, by following our rule from last time we know that we can also subtract the exponents which gives us:

$$\frac{5^3}{5^5} = 5^{-2}$$

Since $\frac{5^3}{5^5} = 5^{-2}$ and also $\frac{5^3}{5^5} = \frac{1}{5^2}$, by the transitive property we know that $5^{-2} = \frac{1}{5^2}$. We can now generalize this rule to say the following for any positive integer n:

$$x^{-n} = \frac{1}{x^n}$$

Negative Exponent as the Reciprocal

Another helpful way to think about negative exponents is as the reciprocal. Remember that the reciprocal of an integer is one over that integer because a number times its reciprocal must equal one. So 4^{-2} means the reciprocal of 4^2 which is $\frac{1}{4^2}$ or $\frac{1}{16}$. (Notice that $4^2 \times \frac{1}{4^2} = 1$ proving that we have the reciprocal.)

One last note is that except for scientific notation, we never leave negative exponents in a solution. We also take the reciprocal so that our exponent is positive. Let's look at a few more examples. Notice that we can evaluate the integer powers, but the variables to a power we have to leave the exponent.

$3^{-3} = \frac{1}{3^3} = \frac{1}{27}$ $\qquad 2^{-4} = \frac{1}{2^4} = \frac{1}{16}$ $\qquad 10^{-5} = \frac{1}{10^5} = \frac{1}{100{,}000}$ $\qquad 13^{-1} = \frac{1}{13^1} = \frac{1}{13}$

$q^{-3} = \frac{1}{q^3}$ $\qquad w^{-7} = \frac{1}{w^7}$ $\qquad g^{-11} = \frac{1}{g^{11}}$ $\qquad j^{-1} = \frac{1}{j^1} = \frac{1}{j}$

Lesson 1.2

Evaluate the following negative exponents giving your answer as a fraction.

1. 5^{-3} 2. 2^{-2} 3. 3^{-2} 4. 7^{-2} 5. 4^{-3} 6. 10^{-3}

7. 10^{-2} 8. 1^{-14} 9. 6^{-2} 10. 2^{-4} 11. 9^{-1} 12. 5^{-2}

13. 10^{-4} 14. 8^{-1} 15. 3^{-4} 16. 6^{-1} 17. 4^{-2} 18. 11^{-1}

Simplify the negative exponents giving your answer as a fraction.

19. a^{-3} 20. b^{-2} 21. c^{-5} 22. d^{-6} 23. f^{-11} 24. g^{-13}

25. h^{-1} 26. j^{-4} 27. k^{-20} 28. m^{-9} 29. n^{-7} 30. p^{-10}

1.3 Negative Exponents Operations

Now that we know negative exponents mean reciprocal, we can perform operations with negative exponents just like we did with positive exponents. Consider the following example of the multiplication rule. Notice that we still added the exponents, but just need to write our answer as a fraction if we have a negative exponent left after multiplication.

$$(5^3)(5^{-5}) = 5^{3\,+\,-5} = 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$$

$$(4^7)(4^{-5}) = 4^{7\,+\,-5} = 4^2 = 16$$

Now let's look at a division example. Remember that we found we can subtract the exponents as long as we have the same base.

$$\frac{5^2}{5^{-2}} = 5^{2\,-\,-2} = 5^4 = 625$$

$$\frac{4^{-1}}{4^3} = 4^{-1\,-\,3} = 4^{-4} = \frac{1}{4^4} = \frac{1}{256}$$

Finally we can see that the power to a power rule still works with negative exponents. We simply multiply the exponents.

$$(2^3)^{-2} = 2^{-6} = \frac{1}{2^6} = \frac{1}{64}$$

$$(3^{-2})^{-2} = 3^4 = 81$$

Lesson 1.3

Evaluate the following exponents operations giving your answer as a fraction where necessary.

1. $5^3 \times 5^{-4}$

2. $(12^9)(12^{-7})$

3. $\frac{(t^{-5})(t^4)}{t^2}$

4. $\frac{4^3}{4^{-7}} \times 4^{-10}$

5. $\frac{f^5}{f^{-1}}$

6. $(y^{-4})^{-5}$

7. $(2^3)^{-6} \times (2^2)^7$

8. $12^2 \times 12^{-4}$

9. $\frac{(k^{-3})^2}{k^4}$

10. $\frac{4^{-2}}{4}$

11. $(5^{-3})^2 \times 5^9$

12. $(0^{-4})^{10}$

Determine if the following equations are true. Justify your answer.

13. $12^{-2} \times 12^7 = 12^{-8} \times 12^3$

14. $\frac{x^{-5}}{x^{-3}} = \frac{x^5}{x^7}$

15. $(t^{-5})^2 = (t^{-2})^5$

16. $(5^{10})^2 = (5^{-5})^{-4}$

17. $\frac{6^{-6} \times 6^8}{6^4} = \frac{6^{-2}}{6^0}$

18. $m^7 \times m^7 = (m^{-7})^2$

19. $\frac{k^{-6}}{k^2} = k^2 \times k^{-10}$

20. $\frac{(7^{-4})^2}{7^3} = 7 \times 7^{12}$

21. $\frac{3 \times 3^4}{3^{10}} = (3^5)^{-1}$

Determine the appropriate exponent to make the equation true.

22. $2^5 \times 2^{\square} = 2^{-6} \times 2^3$

23. $\frac{p^6}{p^{-2}} = \frac{p^{\square}}{p^2}$

24. $(3^{-4})^3 = (3^{-2})^{\square}$

25. $(5^{12})^{-2} = (5^3)^{\square}$

26. $\frac{b^{-2} \times b^8}{b^5} = \frac{b^{\square}}{b^3}$

27. $9^2 \times 9^{-8} = (9^{\square})^3$

28. $\frac{h^{-2}}{h^{\square}} = h^3 \times h^{-5}$

29. $\frac{(6^2)^{\square}}{6^6} = 6^{-8} \times 6^8$

30. $\frac{3^{-4}}{3^{\square} \times 3^9} = (3^7)^{-1}$

1.4 Scientific Notation and Appropriate Units

We've been dealing with very large or very small numbers by rounding them to a single digit times a power of ten. Scientists have long had to deal with these very large or very small numbers from the distances in outer space to the length of atoms. Because of the frequency that they deal with these numbers, they have adopted a similar method using powers of ten that we call scientific notation.

Scientific notation is a number written such that there is a single digit followed by a decimal and any other significant digits multiplied by some power of ten. For example, the number 5.43×10^{-14} is in scientific notation while the number 15.2×10^{23} is not in scientific notation because it has two digits before the decimal.

Writing Very Large Numbers in Scientific Notation

The distance from our Sun to the outer edge of our solar system is approximately $17{,}600{,}000{,}000$ miles. Rather than write out that long number every time we need it, we would normally write it in scientific notation. Previously we rounded the number so that there was a single digit times a power of ten. To be more accurate, we are not going to round but instead use the number as given. So we need a single digit followed by a decimal and the rest of the significant digits. (Significant digits are the non-zero digits. We don't need all the zeros at the end of a very large number.)

$$17{,}600{,}000{,}000 = 1.76 \times 10^{?}$$

The question now is what power to put for the ten. Remember that powers of ten actually affect place value since our number system is based on tens. Since the one is in the 10^{10} place, we know we should use that power. Therefore we get the following:

$$17{,}600{,}000{,}000 = 1.76 \times 10^{10}$$

Let's look at a few more quick examples of converting a standard form number into scientific notation. Notice that in the last example we didn't need a decimal since there were no significant digits after the first digit.

$$20{,}500{,}000{,}000{,}000{,}000 = 2.05 \times 10^{16}$$

$$38{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000 = 3.8 \times 10^{25}$$

$$4{,}000{,}000 = 4 \times 10^{6}$$

To convert a number from scientific notation to standard form, we simply need to apply this process backwards. Whatever the power of ten is, that's the place value of where we should begin writing the number in standard form. For example, 9.12×10^{7} means that the 9 should be in the 10^{7} place (some people think of this as the decimal "moving" 7 places). That will give us $91{,}200{,}000$ which we can verify with a place value chart.

10^{13}	10^{12}	10^{11}	10^{10}	10^{9}	10^{8}	10^{7}	10^{6}	10^{5}	10^{4}	10^{3}	10^{2}	10^{1}	10^{0}
						9	1	2	0	0	0	0	0

Writing Very Small Numbers in Scientific Notation

As we would guess, writing very small numbers in scientific notation works much the same way except that we will have negative exponents. Consider the following example:

$$0.000\ 000\ 000\ 000\ 79 = 7.9 \times 10^{-13}$$

Notice that we used the leftmost non-zero digit as our single digit and then followed it with a decimal the rest of the significant digits. The power was negative thirteen because the 7 is in the 10^{-13} place. Again, some people think of it as the decimal "moving" 13 places.

Let's look at a few more quick examples.

$$0.000\ 000\ 005\ 04 = 5.04 \times 10^{-9}$$

$$0.000\ 000\ 000\ 000\ 000\ 000\ 000\ 000\ 83 = 8.3 \times 10^{-25}$$

$$0.007 = 7 \times 10^{-3}$$

We can also convert small numbers written in scientific notation back to standard form by working backwards. For example, the number 6.1×10^{-5} means that in standard form the 6 will be in the 10^{-5} place giving us the number 0.000061 in standard form. Let's look at this in the place value chart just to be sure.

10^{0}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	10^{-13}
0.	0	0	0	0	6	1							

Interpreting Technology

Calculators usually do have a scientific notation function. For example, type 2.56×10^{25} into your calculator and hit equals. Your calculator probably has one of three displays. Some calculators might give you an answer that looks like this: $\boxed{\quad 2.56\ ^{25}}$. While this looks like 2.56^{25}, or two and fifty-six hundredths to the twenty-fifth power, the little exponent off to the side actually means that it is in scientific notation.

Other calculators will display the following for the same problem: $\boxed{\quad 2.56\ \ {}^{25}_{EE}}$ or $\boxed{\quad 2.56\ \ _{E25}}$. The double "EE" or a single "E" also stands for scientific notation.

Finally, other calculators will display a much more intuitive answer such as: $\boxed{\quad 2.56\ \ {}^{25}_{\times 10}}$.

All of these different displays mean the same thing: 2.56×10^{25}. If you perform an operation on the calculator where the answer is too many digits (either large or small) to display on the calculator, it will automatically put the answer in scientific notation.

The Common Sense Approach to Units

With very large or very small numbers it is important to know what units we are dealing with. For example, having 5.2×10^7 dollars would be more than ten times the amount of money needed to retire while most financial experts would say that 5.2×10^7 cents would only be about half of what is necessary for retirement.

As another example, consider the fact that Mars is on average about 7.8×10^7 km from Earth. It would not make sense to report this distance in millimeters because of how huge that number really is.

When deciding on which unit of measurement to use in any given situation, just use common sense. If we're talking about the distance between towns, miles will work better than inches. If we're talking about how quickly your toenails grow, meters per day would (hopefully) be inappropriate while millimeters per day might do nicely.

Lesson 1.4

Estimate each number as a single digit times a power of ten. Then rewrite each number in scientific notation.

1. $1{,}234{,}000$

2. $190{,}000$

3. $99{,}000{,}000$

4. $5{,}390{,}000{,}000$

5. $10{,}900{,}000{,}000$

6. $7{,}800{,}000{,}000$

7. $0.000\ 000\ 42$

8. $0.000\ 019$

9. $0.000\ 000\ 016\ 87$

10. $0.000\ 000\ 000\ 321$

11. $0.000\ 001\ 987$

12. $0.000\ 000\ 008\ 5$

13. The Earth has an approximate mass of 5,980,000,000,000,000,000,000,000 kg.

14. A quarter (meaning the coin) is 0.000 000 25 of a million dollars.

15. The mass of a dust particle is 0.000 000 000 753 kg.

16. The speed of light is 299,792,458 m/sec.

Rewrite each number in standard form.

17. 2.3×10^{13}

18. 6.07×10^{7}

19. 5×10^{23}

20. 1.8×10^{3}

21. 2.3×10^{-11}

22. 6.07×10^{-9}

23. 5×10^{-5}

24. 1.8×10^{-16}

Choose the most appropriate unit of measurement for the given situation.

25. The amount of lava coming from a volcano: fluid ounces per hour, cups per hour, or gallons per hour

26. The speed human hair grows: inches per year, feet per year, or yards per year

27. The growth of a tree: inches per hour, inches per year, yards per year

28. Speed of a swimming dolphin: centimeters per hour, meters per hour, kilometers per hour

29. The rate of water flow from a shower head: fluid ounces per minute, cups per minute, gallons per minute

30. A cell phone measures 2.3×10^{-5} kilometers in thickness. Would this be best expressed using kilometers, meters, or centimeters?

31. The average pace for a biker is 3.2×10^{6} centimeters per hour. Would this be best expressed using kilometers, meters, or centimeters?

32. A bullet travels 3.4×10^{5} millimeters per second. Would this be best expressed using millimeters, centimeters, or meters per second?

1.5 Scientific Notation Operations

Now that we understand what scientific notation is, we can begin performing operations with these numbers just as scientists have to in their research.

Multiplication and Division with Scientific Notation

Let's say the debt in the United States is now near 1.8×10^{13} dollars and that there are about $300{,}000{,}000$ people in the United States. If every person paid their fair share of the debt, approximately how much would that be per person? To solve this problem we need to divide the debt between all the people. Let's start by writing the standard form number in scientific notation.

$$300{,}000{,}000 = 3 \times 10^{8}$$

Now we can begin to divide those scientific notation numbers by writing the problem in fraction form:

$$\frac{1.8 \times 10^{13}}{3 \times 10^{8}}$$

How do we actually divide these? Since the only operations happening are multiplication and division, we can divide the whole numbers and then divide powers of ten. Why does this work? Because we can actually split this into two different fractions and simply each fraction as follows:

$$\frac{1.8 \times 10^{13}}{3 \times 10^{8}} = \frac{1.8}{3} \times \frac{10^{13}}{10^{8}} = 0.6 \times 10^{5}$$

The problem now is that our answer is not in scientific notation. We need to be a single non-zero digit before the decimal. We could multiply the six tenths by ten to make it six, but remember with any expression we can't just change values. The only thing we can multiply or divide by is one. So if multiply by ten, we have to divide by ten so that there is no change in the actual value of the number, like so:

$$0.6 \times 10^{5} = 0.6 \times (10 \div 10) \times 10^{5} = (0.6 \times 10) \times (10^{5} \div 10) = 6 \times 10^{4}$$

While this appears confusing, it really just keeping everything balanced. Since we made the 0.6 bigger by a factor of 10, we have to make the 10^{5} smaller by a factor of 10 to balance it.

Let's look at another example. There are approximately 6×10^{9} people in the world. If each person made 2.5×10^{4} dollars in a year, how much money was made worldwide? In this case we need to multiply the numbers, rearranging using the commutative property as follows:

$$(6 \times 10^{9})(2.5 \times 10^{4}) = 6 \times 2.5 \times 10^{9} \times 10^{4} = 15 \times 10^{13}$$

Again, we need to get our answer in scientific notation. In this case, the 15 needs to be a 1.5, meaning it needs to be smaller a single factor of 10. Therefore we will have to make the 10^{13} bigger by a factor of ten making it 10^{14}. This means our final answer is 1.5×10^{14} dollars produced in that year.

Addition and Subtraction with Scientific Notation

When adding or subtracting numbers in scientific notation, we have to remember that place value matters. For example when adding 123 and 56, we have to add the 6 and 3 because they are both in the ones place. In the same way, we have to add the 2 and 5 because they are both in the tens place.

Since scientific notation makes every number a single digit followed by a decimal and a power of ten, the place value gets hidden. This means that we can only add or subtract scientific notation numbers if they have the same power of ten (since the power of ten controls the place value). If they don't have the same power of ten, we will have to rewrite one of the numbers in a way such that the powers of ten are equal.

For example, let's solve the problem $3.7 \times 10^6 + 4.3 \times 10^5$ where the powers of ten are different. First we need to make the powers of ten be the same. Since 3.7×10^6 is the larger number, we'll leave that alone and convert 4.3×10^5 into an equal number that has $\times 10^6$ at the end. Since we want the power of ten to be bigger by a single factor of ten, we'll need to make the 4.3 smaller by a factor of ten as follows:

$$4.3 \times 10^5 = 0.43 \times 10^6$$

Now that we have the same power of ten (which means the same place value), we can solve as follows:

$$3.7 \times 10^6 + 0.43 \times 10^6 = 4.13 \times 10^6$$

Notice that we simply added the 3.7 and the 0.43 together. Notice that we could have turned the scientific notation numbers into standard form numbers and then added. However, this is only convenient with small powers of ten.

$$3.7 \times 10^6 = 3{,}700{,}000$$

$$4.3 \times 10^5 = 430{,}000$$

$$3.7 \times 10^6 + 4.3 \times 10^5 = 3{,}700{,}000 + 430{,}000 = 4{,}130{,}000 = 4.13 \times 10^6$$

Estimating Very Large Numbers with Powers of 10

Scientists have measured the temperature at the edge of the sun to be around 5,400° C. Let's round that number to a single digit times a power of ten. Looking at the leftmost digit, which is 5, should that stay a 5 or round up to 6? Since the next number is only a four, it will stay a 5. That means $5{,}4000 \approx 5{,}000$. Now let's look at our place value.

This shows us that we can write $5{,}000$ as 5×10^3. We may be tempted to simply count the number of zeros and use that as the power, but that will only work for this specific type of problem. Instead we should continue to think about place value.

Let's look at one final example of rounding a large number to a single digit times a power of ten. In a penny there may be approximately $19{,}370{,}000{,}000{,}000{,}000{,}000{,}000$ atoms.

$$19{,}370{,}000{,}000{,}000{,}000{,}000{,}000 \approx 20{,}000{,}000{,}000{,}000{,}000{,}000{,}000$$

$$20{,}000{,}000{,}000{,}000{,}000{,}000{,}000 = 2 \times 10^{22}$$

$$19{,}370{,}000{,}000{,}000{,}000{,}000{,}000 \approx 2 \times 10^{22}$$

So there are approximately 2×10^{22} atoms in a penny. Notice that we could simply count the place values from right to left starting from the where the decimal would be to find our exponent. In other words, we could count how many places the decimal "moved" to get the new number.

Estimating Very Small Numbers with Powers of 10

This will work the same way except that now we will have negative powers of ten since we will be dealing with very small decimals. For example, a single atom in that penny is approximately 0.0000000312 cm across. We can round that using powers of ten in nearly the same way.

First, look at the leftmost non-zero digit, which is a 3. Should that stay a 3 or should it round up to a 4? It should stay a 3 because the next digit is a one and we only round up when the number is five or greater. That means that $0.0000000312 \approx 0.00000003$. Now let's examine that in our place value chart. Notice that we have negative exponents because the tenths place is really the fraction $\frac{1}{10}$ which is 10^{-1} and so forth.

Now we see that $0.0000000312 \approx 3 \times 10^{-8}$.

Let's do one final example of rounding a small number using a single digit times a power of ten. Round the number $0.000\ 000\ 000\ 000\ 000\ 871$ to single digit times a power of ten. (Notice that sometimes we put spaces between every three zeros to make it easier to count how many zeros are there.)

$$0.000\ 000\ 000\ 000\ 000\ 871 \approx 0.000\ 000\ 000\ 000\ 000\ 9$$

$$0.000\ 000\ 000\ 000\ 000\ 9 = 9 \times 10^{-16}$$

$$0.000\ 000\ 000\ 000\ 000\ 871 \approx 9 \times 10^{-16}$$

Notice again that we could simply count the number of places the decimal "moved" to make it 9. That took 16 places for the decimal to move; therefore we will use the negative 16th power as our exponent.

Estimating Operations

When faced with an operation involving scientific notation, we can estimate the final solution by rounding the numbers first then performing the operation. This will make your final solution an estimate as well.

Lesson 1.5

Estimate each of the following as a single digit times a power of ten. Then compute each of the following giving your answer in scientific notation.

1. $(3 \times 10^{-6})(3 \times 10^{9})$

2. $\frac{6.8\times10^{9}}{2\times10^{5}}$

3. $4.5 \times 10^{7} + 4{,}000{,}000$

4. $8.4 \times 10^{7} - 3.1 \times 10^{7}$

5. $(2.4 \times 10^{4})(7{,}000)$

6. $\frac{5.4\times10^{8}}{9{,}000}$

7. $3.9 \times 10^{13} + 4.2 \times 10^{13}$

8. $8.2 \times 10^{-5} - 0.000\,059$

9. $(2.5 \times 10^{-4})(7 \times 10^{11})$

10. $\frac{4.5\times10^{9}}{1.5\times10^{13}}$

11. $1.3 \times 10^{7} + 4 \times 10^{6}$

12. $5.2 \times 10^{7} - 120{,}000$

Answer the following questions giving both an estimated answer (single digit times a power of ten) and a precise answer (scientific notation).

13. How many times bigger is the distance from Earth to the sun of 9.3×10^6 miles than the furthest distance from Earth to the moon of 3×10^5 miles? (Hint: round the moon distance up.)

14. The temperature halfway to the Sun from Mercury is approximately $1{,}800°\ C$ and scientists theorize that it may be up to $26{,}000$ times hotter at the center of the Sun. Approximately how hot is it at the center of the Sun?

15. Each shrimp weighs approximately $0.000\ 27\ g$ and a shrimp company can bring in over $3{,}100{,}000{,}000$ shrimp per year. Approximately how much would that many shrimp weigh?

16. The Earth has a mass of about 6×10^{24} kg. Neptune has a mass of 1.8×10^{27} kg. How many times bigger is Neptune than Earth?

17. A country has an area of approximately $8{,}400{,}000{,}000$ square miles and has approximately $210{,}000$ people. How much area is this per person?

18. A blue whale can eat $300{,}000{,}000$ krill in a day. All of that krill weighs approximately $6{,}300{,}000{,}000\ mg$. About how much does each krill weigh?

19. The US spends on average $9{,}000$ dollars on each student per year. There are about $77{,}000{,}000$ students in the United States. How about much money total is spent on students each year?

20. McDonald's has about $210{,}000$ managers and each makes on average $40{,}000$ dollars per year. How much money does McDonald's spend on managers each year?

Review Unit 1: Exponents

No calculator necessary. Please do not use a calculator.

Unit 1 Goals

- *Know and apply the properties of integer exponents to generate equivalent numerical expressions. (8.EE.1)*
- *Use numbers expressed in the form of a single digit times an integer power of* 10 *to estimate very large or very small quantities, and to express how many times as much one is than the other. (8.EE.3)*
- *Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities. Interpret scientific notation that has been generated by technology. (8.EE.4)*

Evaluate, meaning multiply out the exponent, giving your answer as a fraction when necessary.

1. $4^4 \times 4^{-2}$

2. $(3^2)(3^{-5})$

3. $(6^2)^0$

4. $\frac{3^4}{3^9}$

5. $(v^6)^{-3}$

6. $\frac{b^8}{b^{-2}}$

7. $(k^{-10})(k^{-2})$

8. $(5^{-3})^{-1}$

9. $\frac{2^{-6} \times 2^3}{2^{-5}}$

10. $(j^3)^{-2} \times j^6$

11. $\frac{(m^{-3})^5}{m^5}$

12. $\frac{4^6}{4^{-2}} \times 4^{-6}$

Determine if the following equations are true. Justify your answer.

13. $8^{-5} \times 8^6 = 8^0 \times 8$

14. $(j^2)^{-5} = \frac{j^{10}}{j^2}$

15. $\frac{m^3 \times m^{-5}}{m^2} = \frac{m^4}{m^0}$

16. $(4^{-5})^4 = (4^{10})^{-2}$

Determine the appropriate exponent to make the equation true.

17. $\frac{p^5}{p^{-5}} = (p^2)^{\square}$

18. $2^{-10} \times 2^2 = 2^{-4} \times 2^{\square}$

19. $\frac{f^0}{f^{-7}} = \frac{f^{14}}{f^{\square}}$

20. $\frac{(9^2)^4}{9^{-2}} = \left(9^{\square}\right)^2$

Write the following numbers in scientific notation.

21. 9,089,000,000 22. 810,000,000,000 23. 0.000 000 27 24. 0.001 06

Write the following numbers in standard form.

25. 5.14×10^{-6} 26. 4.07×10^{9} 27. 3.1×10^{-7} 28. 7.109×10^{4}

Choose the best unit of measurement for the following problems.

29. Would the weight of a strand of hair, about 6.9×10^{-5} grams, be best expressed using tons, pounds, or milligrams?

30. Would the area of the City of Charleston, about 24.9×10^{11} square centimeters, be best expressed using square kilometers, square meters, or square millimeters?

Estimate each of the following as a single digit times a power of ten. Then compute each of the following giving your answer in scientific notation.

31. $(2.1 \times 10^{6})(500)$ 32. $\frac{6.3\times10^{7}}{3\times10^{3}}$ 33. $4.2 \times 10^{7} - 3{,}000{,}000$

34. $6.2 \times 10^{-6} + 0.000\,005$ 35. $\frac{4.8\times10^{3}}{4\times10^{6}}$ 36. $(3 \times 10^{-9})(2{,}300)$

Answer the following questions giving <u>both</u> an estimated answer (single digit times a power of ten) and a precise answer (scientific notation).

37. In the United States, there are approximately 300,000,000 people that use a total of approximately 360,000,000,000 gallons of water in a day. How much water does each person use?

38. A football field has an area of 56,560 ft^2 and a tennis court 2,800 ft^2. How many times bigger in area is a football field than a tennis court?

39. Three thousand people sued McDonald's for hot coffee and were compensated at least $280,000 for each person. How much money was distributed in total?

40. There were 9,700,000 Macintosh computers sold in 2008. If Macintosh earns an average of $200 profit per computer sold, how much profit did the company make selling computers in 2008?

Unit 2: Similar and Congruent

Pre-Test Unit 2: Similar and Congruent

No calculator necessary. Please do not use a calculator.

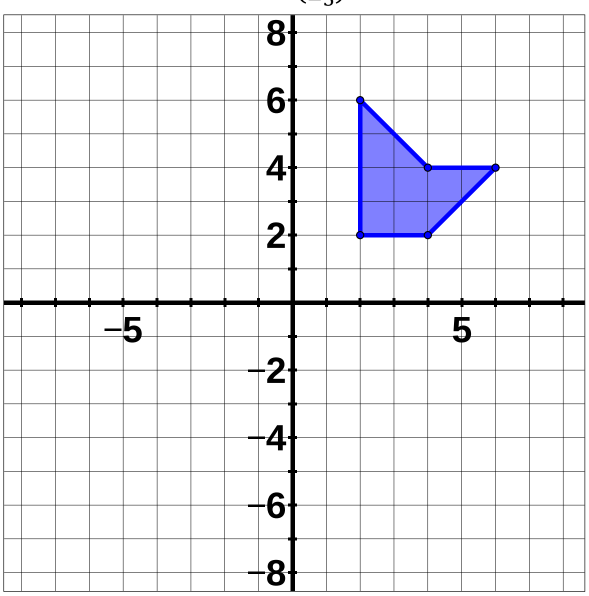
Answers the following questions. *(5 pts; no partial credit)*

1. List all the transformations that lead to congruent images.

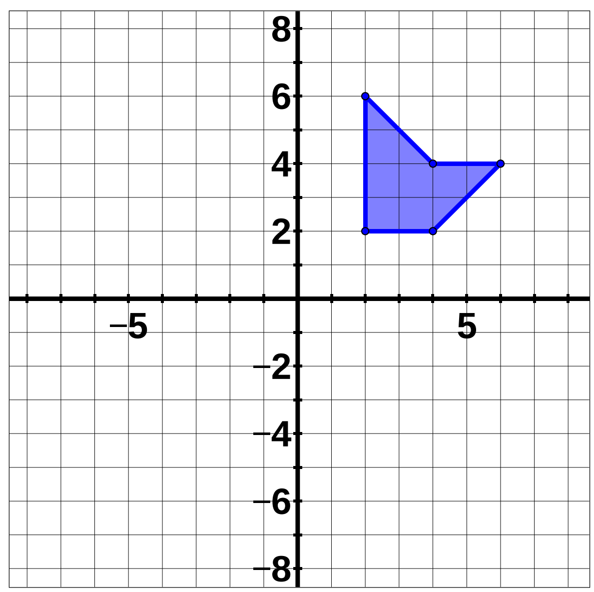
2. List all the transformations that lead to similar, but not congruent, images.

Apply the given transformation to the given pre-image. *(5 pts; no partial credit)*

3. Translation by vector $\binom{2}{-5}$

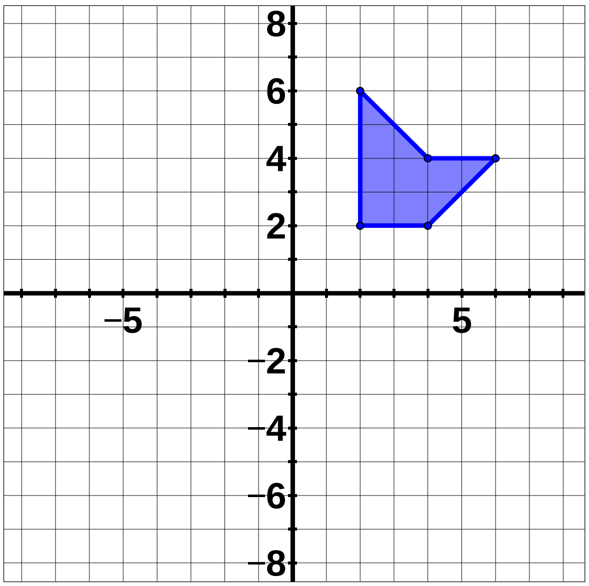


4. Rotation by $90°$

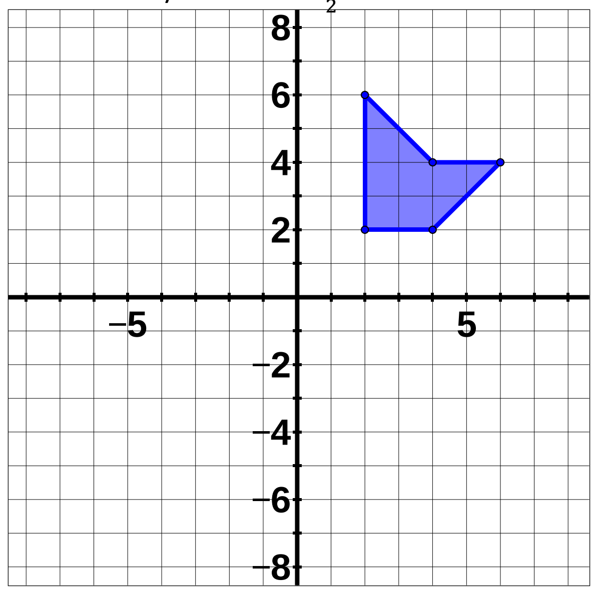


Apply the given transformation to the given pre-image. *(5 pts; no partial credit)*

5. Reflection across the x-axis

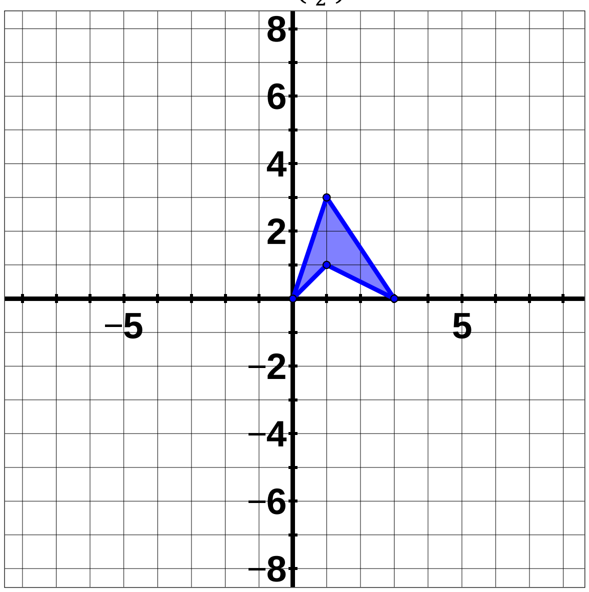


6. Dilation by scale factor $\frac{1}{2}$

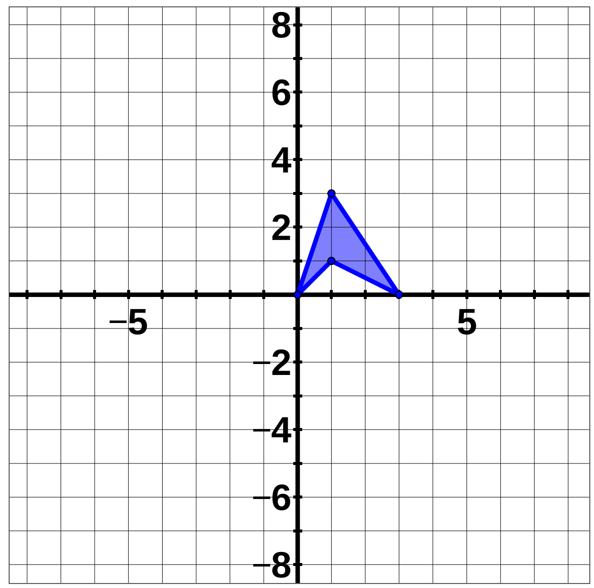


Apply the given series of transformations to the given pre-image. *(5 pts; 2 pts for each transformation, 1 pt for correct order)*

7. Translation by vector $\binom{-6}{2}$ and rotation by $180°$

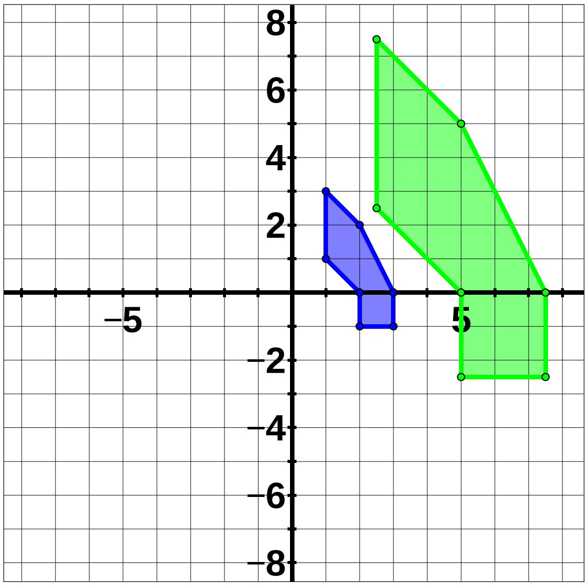


8. Dilation by scale factor 2 and reflect across y-axis

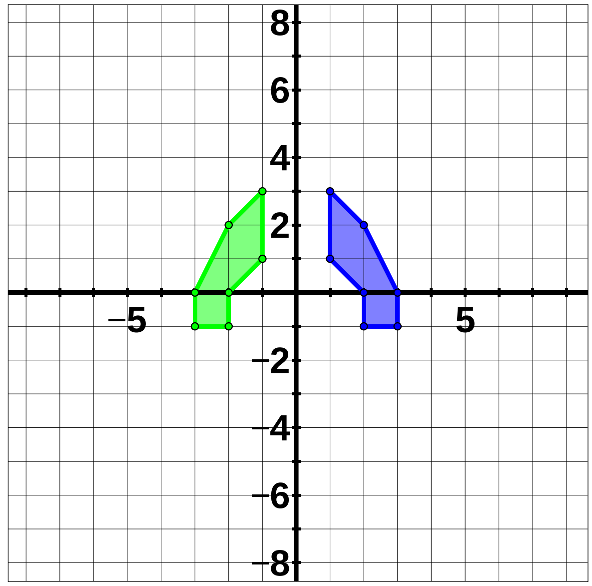


Identify a specific series of transformations that would take the pre-image (darker in blue) to the image (lighter in green). Then tell whether the pre-image and image are congruent or similar.
(5 pts; 2 pts for transformation(s), 2 pts for specific vector(s), rotation angle(s), reflection line(s), or scale factor(s), 1 pt for cong/sim)

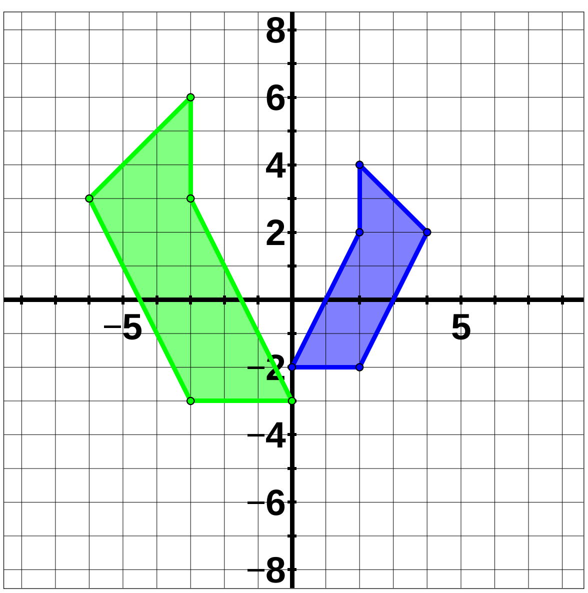
9.



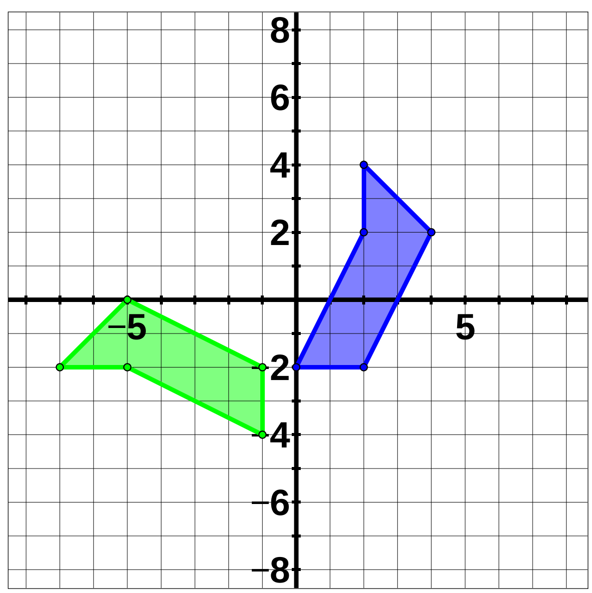
10.



11.

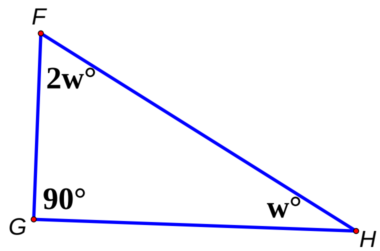


12.

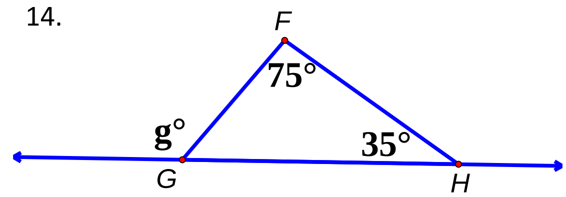


Find the angle measure of each missing angle. *(5 pts; 3 pts for computation error or only solving for the variable on problem 13)*

13.

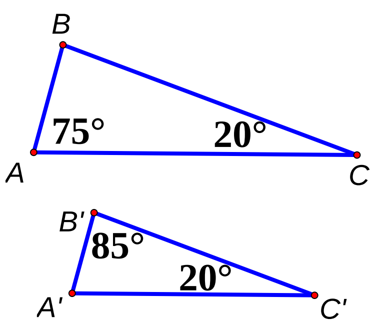


14.

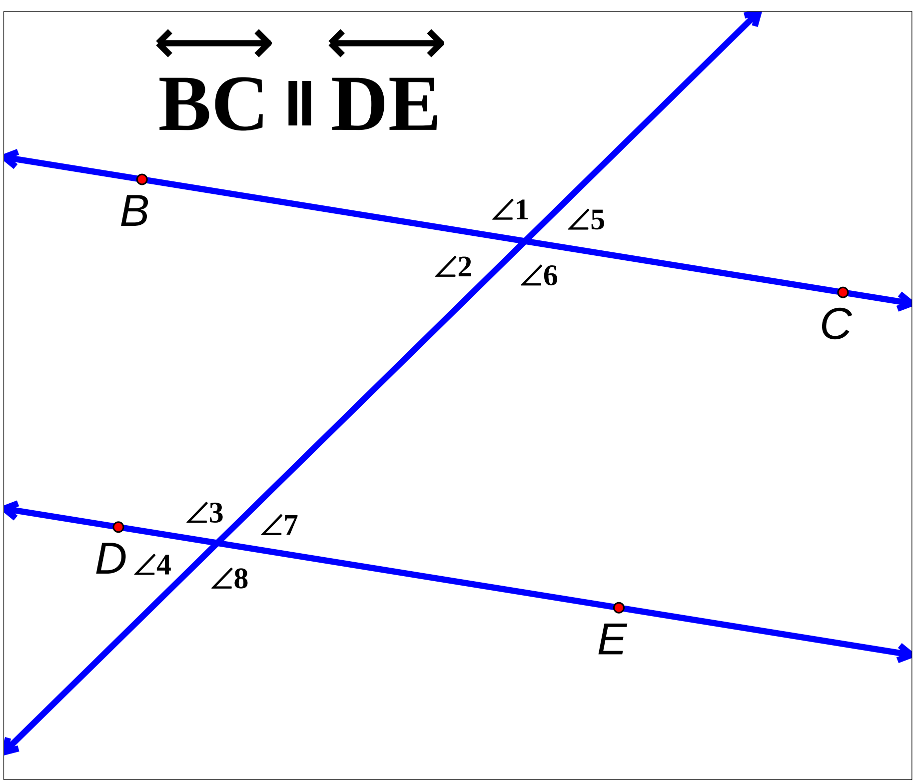


Determine if the following triangles are similar or not and explain why or why not. *(5 pts; 2 pts for answer, 3 pts for explanation)*

15.



Use the picture to answer the following questions. *(5 pts; no partial credit)*



16. Name a pair of corresponding angles.

17. Name a pair of alternate interior angles.

18. Name a pair of alternate exterior angles.

19. Name a pair of vertical angles.

20. If $m\angle 1 = 135°$, what is $m\angle 7$?

2.0 Transformation Basics

Transformations are when we take a picture or shape and change it using four different methods of sliding, rotating, reflecting, or shrinking/enlarging. We refer to the picture before the change as the **pre-image** and may give it a name such as A. We might also name the points that make up the picture the same way.

The picture after we apply the transformation is called the **image**. The name of the image is based on the name of pre-image. For example, if the pre-image were named A, then the image would be name A'. That would be pronounced, "A prime." We add the prime mark, really just a tick mark like an apostrophe, to show that it is the image, or picture after the transformation happened.

Some transformations keep the pre-image and image congruent. **Congruent** means that they are the same size and shape or that they have the same measurements. They make not have the same orientation, meaning they could be flipped or turned, but they do have the same measurements. Most transformations lead to congruent shapes.

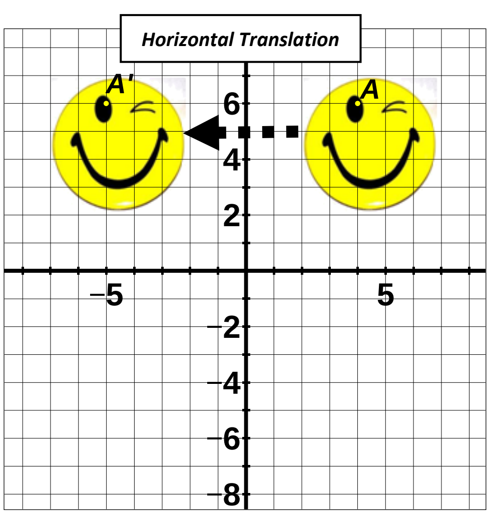
One transformation makes the image similar to the pre-image. **Similar** means that they are the same shape and proportion, but not the same size. In other words, the image was either shrunk or enlarged since it's not the same size. One of our main jobs throughout this unit is to make sure we know the difference between congruent and similar transformations.

Transformations Leading to Congruent Shapes

Three of the four transformations preserve the size and shape of the pre-image: translations, rotations, and reflections. In other words, if you translate a pre-image square that has side lengths of $4\ cm$, it will still have side lengths of $4\ cm$ after it is slid to its new position.

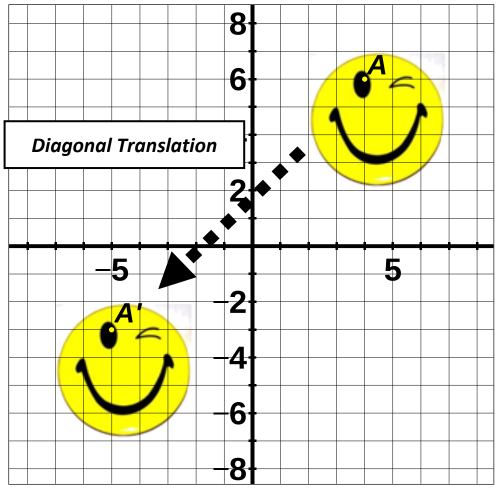
Translations

When we slide a picture or shape, it is called a **translation**. There are two directions we can slide when our pre-image is in a plane: vertically or horizontally. Even diagonal translations are combinations of vertical and horizontal movement. The following transformation is a horizontal translation to the left.



Note that the pre-image, the original picture, is labeled as A and is the smiley face in quadrant one. To create this transformation we just slid the smiley face to the left nine places. We'll talk about how to write that out mathematically a little bit later. For right, we just need to be able to recognize that this transformation is a translation.

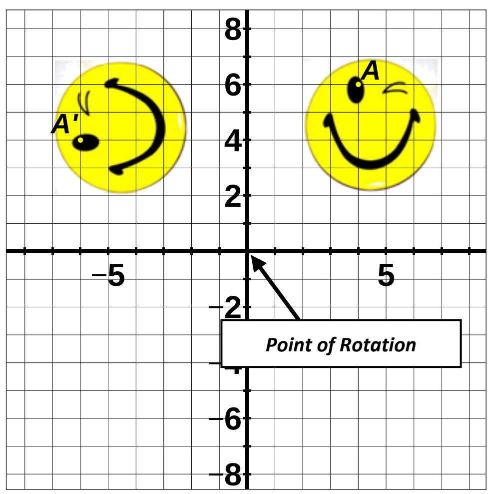
Next we'll look at an example of a translation diagonally.



Note that this translation is really two parts: a horizontal translation left and a vertical translation down. What this shows us is that we can translate in any direction we want. If we need a diagonal translation we simply put together a horizontal and vertical part to get the diagonal we want.

Rotations

When we rotate a picture or shape by a specific angle measurement about a specific point, it is called a **rotation**. Rotations always occur in a counter-clockwise fashion. For our purposes, we will only rotate in 90° increments and we will always rotate about the origin. Here's an example of rotating our smiley about the origin:



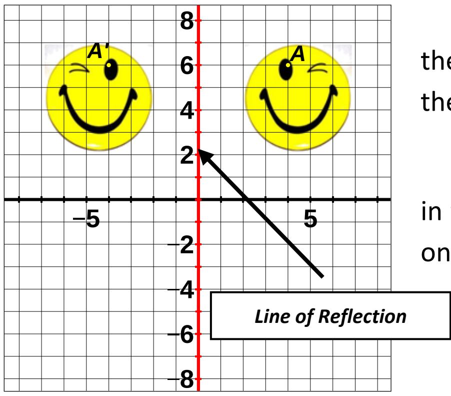
We can see that by rotating a figure, we did not change its size (so the smiley faces are still congruent), but the orientation did change. The rotated smiley face is on its side at this point.

What angle measurement would keep the orientation the same? If we did a 360° rotation, the image would be exactly the pre-image. They would be sitting right on top of each other. Because of this, we typically don't talk about a rotation of 360°.

The only rotations we will use will be 90° (a quarter turn), 180° (a half turn), and 270° (a three-quarters turn). Again, at this point we just need to be able to recognize a rotation.

Reflections

When we reflect a picture or shape across a line, it is called a **reflection**. Mathematically what is happening is every point on the pre-image is sent to a corresponding point on the other side of the line so that the distance from each point to the line is the same. Also, if you were to draw line between the point in the pre-image and its image, that line would be perpendicular to the reflection line. Here's an example of a reflection across the y-axis:



Notice that with the reflection the orientation changed (the wink is on the left in the image instead of the right like the pre-image), but the size stayed the same again maintaining congruence.

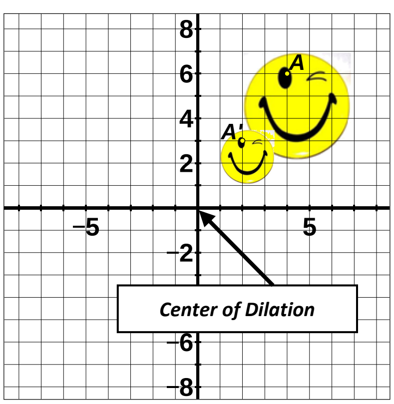
We could also reflect across the x-axis which would have put the image in the quadrant four (below the original smiley face). The two axes will be the only lines we will reflect across at the 8^{th} grade level.

Transformations Leading to Similar Shapes

The fourth transformation does not preserve the size of the pre-image, only the shape. This transformation is called a dilation.

Dilations

When we shrink or enlarge a picture or shape, it is called a **dilation**. We need a scale factor to dilate by so we know how much to shrink or enlarge. The scale factor is the ratio of the size of the image to the size of the pre-image. So a scale factor of $\frac{1}{2}$ represents $\frac{1\ unit\ on\ image}{2\ units\ on\ pre-image}$, which is a shrinking scale since the pre-image is bigger. An enlarging scale would be a number greater than one, and a shrinking scale would be numbers between zero and one.

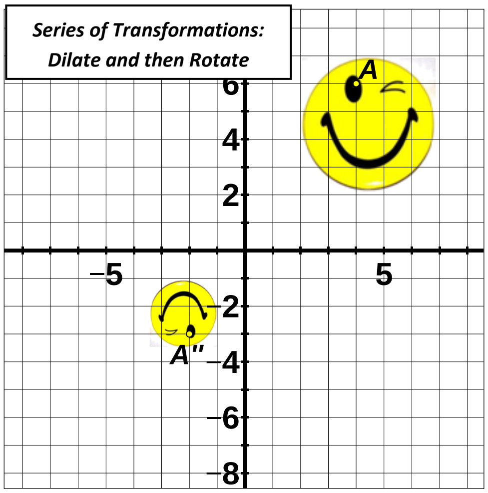


Besides a scale factor, we also need a center of dilation. The center of dilation is where things get compressed towards if we are shrinking or where things get stretched away from if we are enlarging. Again, for the ease of the math involved, we will only use the origin as our center of dilation. Here is an example of a shrinking dilation:

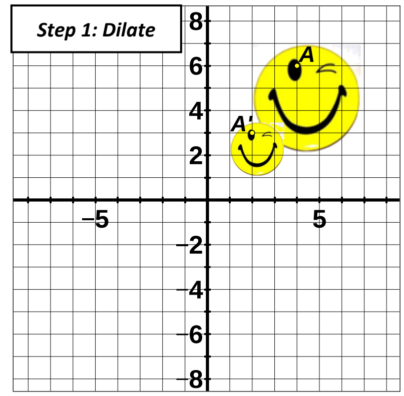
Notice that the smiley face was also pulled toward the origin during the shrinking process. Also notice that in this case the pre-image and image are the same shape but not the same size. This makes them similar instead of congruent.

Combining Transformations

Finally, we can combine multiple transformations into a series. For example, we could dilate by half (shrink it) and then rotate by $180°$ as follows:

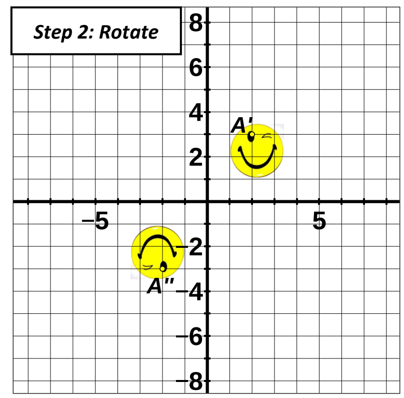


One thing to note is that this image is labeled as A'' instead of A'. This is because there were two transformations applied to the pre-image A to get the image.



Let's walk through the process. First, the image is dilated by half. That gives us the picture on the right.

The second step in the process is to rotate A' by $180°$. In other words, we're taking the new picture we just created and rotating it. We're not rotating the original picture. You can see this step to the right again.



Another interesting thing to note about our series is that we could get the same image in another way. We could dilate by half, reflect across the x-axis, and then reflect across the y-axis. This highlights the fact that there may be more than one solution when deciding what transformations occurred in a series.

2.1 Constructing Dilations

It is useful to be able to perform transformations using the coordinate plane. This allows us to specify the exact coordinates of the pre-image, image, point of rotation, and line of reflection.

To help us do so, let's review our notation. Remember that a capital letter, such as A, represents the pre-image which is usually a point or a whole picture. In the following section, it will most often refer to a point. The prime following the capital letter, such as A', means the image. So we might say that A' is the image of A. Another way to think about it is with the words "new" and "old" so that A is the old point and A' is the new point.

We'll also apply this same notation to x- and y-coordinates so that x means the x-coordinate of the old point (pre-image) and x' means the new x-coordinate of the new point (image). Similarly y is the y-coordinate of the old point (pre-image) and y' means the new y-coordinate of the new point (image).

Dilations

We focus on dilations first because it is usually best to perform this operation first in a series since dilating also pulls towards or away from the origin. Also remember that dilation is the only transformation that creates similar but not congruent figures because of the shrinking or enlarging. This means that any time we dilate we are dealing with similarity.

For now, we will only be dilating with the origin as the center of dilation. That means every shrinking scale will pull every point closer to the origin and every enlarging scale will push points further away from the origin. Because of this, dilations will often produce images that overlap with the pre-image.

In order to dilate we need a scale factor. We'll call this scale factor c for now. In order to dilate a shape, we can dilate each point using the following formulas for each coordinate:

$$x' = cx$$

$$y' = cy$$

This means that the new x coordinate is the scale factor times the old x coordinate and the same for the y coordinates.

A Shrinking Triangle

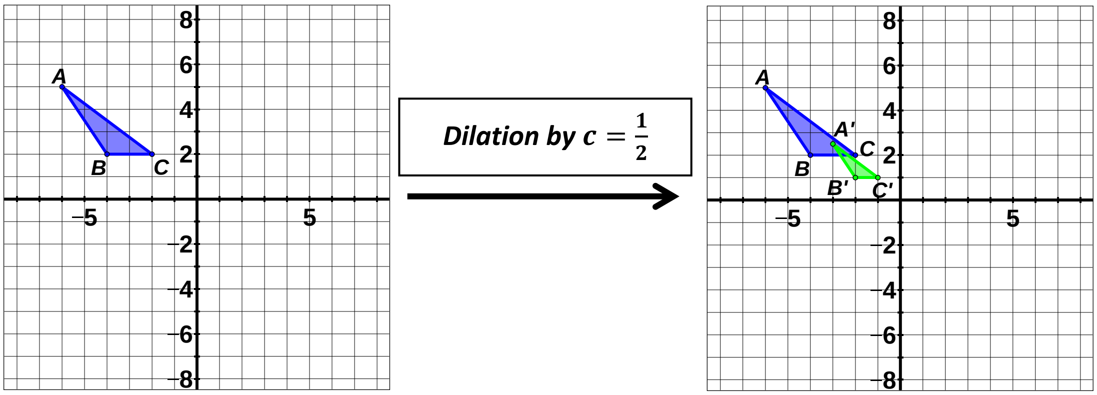
For example, consider the pre-image triangle with points $A: (-6,5), B: (-4,2), C: (-2,2)$ being dilated by a scale factor of $c = \frac{1}{2}$. To get each new point in the image, we simply multiply each coordinate by half.

$$A': \left(\frac{1}{2} * -6, \frac{1}{2} * 5\right) \; or \; A': (-3, 2.5)$$

$$B': \left(\frac{1}{2} * -4, \frac{1}{2} * 2\right) \; or \; B': (-2, 1)$$

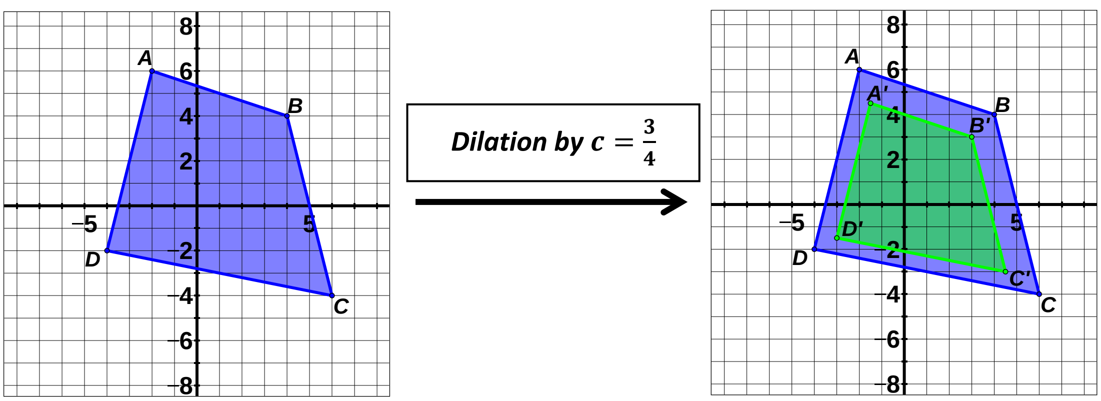
$$C': \left(\frac{1}{2} * -2, \frac{1}{2} * 2\right) \; or \; C': (-1,1)$$

It's okay that we got a decimal value for the y-coordinate of A'. Just move up two and half when plotting that point. Here is the picture of the dilation process where the pre-image is the dark triangle (in blue) and the image is the lighter triangle (in green):



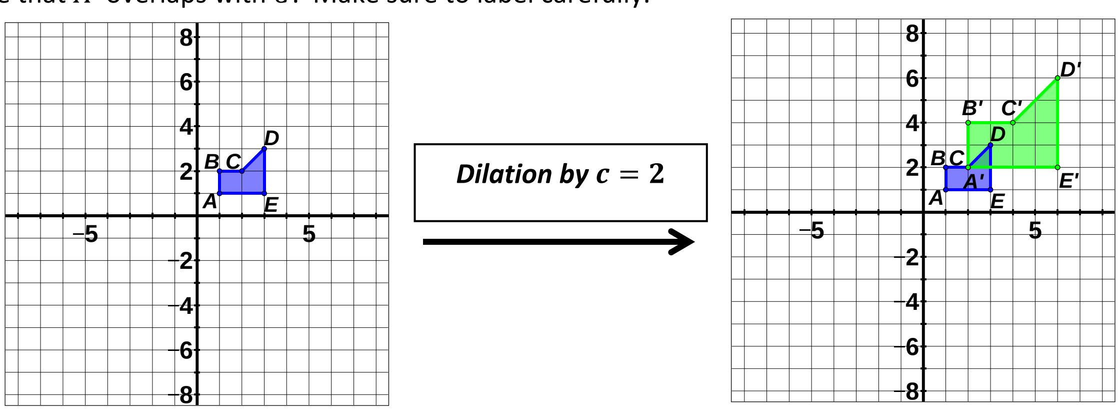
A Shrinking Quadrilateral

Now let's dilate a shape where the points are not all in the same quadrant. Consider the following quadrilateral with points $A: (-2, 6), B: (4, 4), C: (6, -4), D: (-4, -2)$. Let's dilate by a scale factor of $c = \frac{3}{4}$. That means we'll take each coordinate in the pre-image multiplied by $\frac{3}{4}$ to get the image. Verify that gives us the points $A': (-1.5, 4.5), B': (3, 3), C': (4.5, -3), D': (-3, -1.5)$.



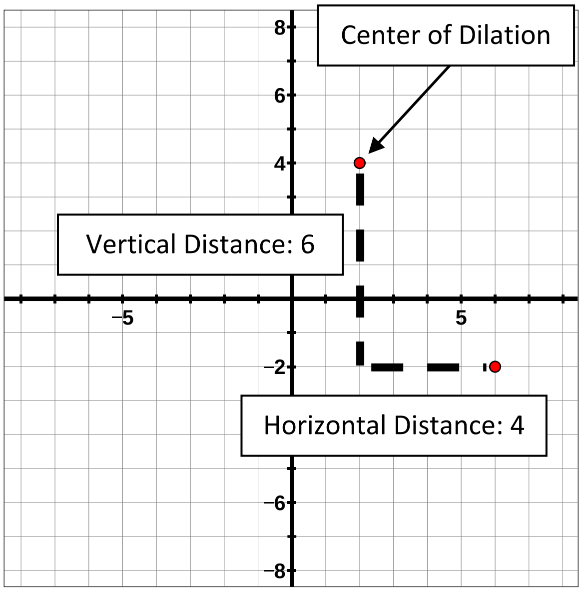
An Enlarging Polygon

Finally we should look at an enlarging scale factor. Consider using a scale factor of $c = 2$ to enlarge this polygon with points $A: (1, 1), B: (1, 2), C: (2, 2), D: (3, 3), E: (3, 1)$. That means just multiply all coordinates by two. You might note that A' overlaps with C. Make sure to label carefully.



Non-Origin Centers of Dilation

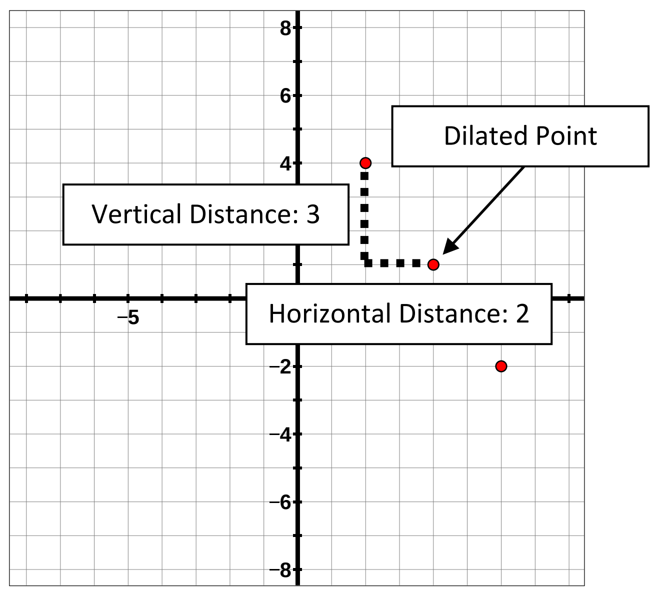
If the center of dilation is not the origin, then we multiply the horizontal and vertical distance from the center of dilation by the scale factor. For example, consider dilating the point $(6,-2)$ by a scale factor of $k=\frac{1}{2}$ using a center of dilation of $(2,4)$.



Note that the vertical distance between the point and the center of dilation is six units. If we take that distance times the scale factor of half, we get that the new point, or image, should be three units down from the center of dilation.

The horizontal distance between the point and the center of dilation is four units. Following the same idea, half of that distance is two units to the right of the center of dilation.

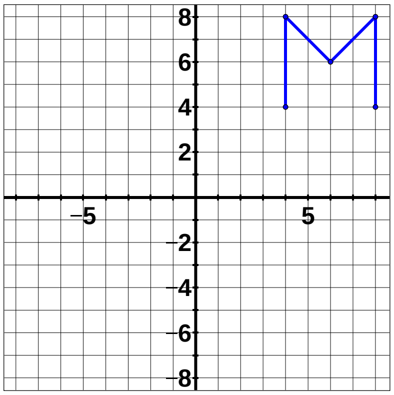
Using this information we can plot the new point at $(4,1)$.



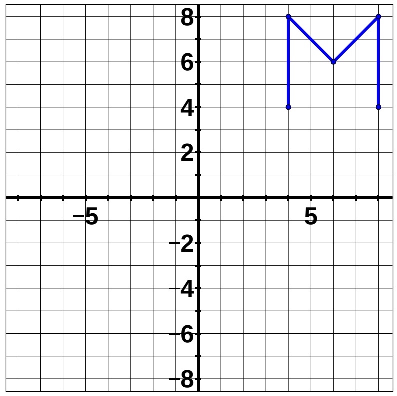
Lesson 2.1

Perform the given dilation on each given pre-image with the given center of dilation.

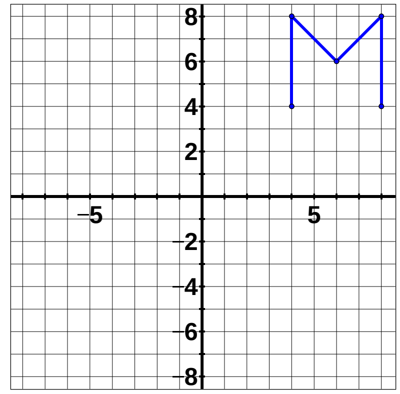
1. Dilate by $c = \frac{1}{4}$, center $(0,0)$



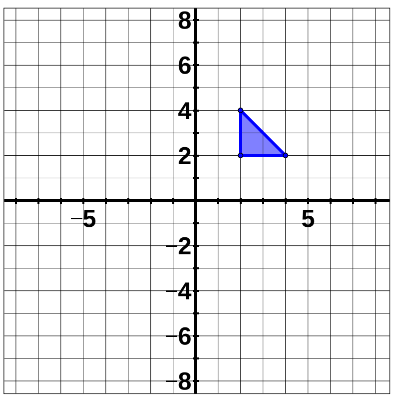
2. Dilate by $c = \frac{1}{2}$, center $(2,2)$



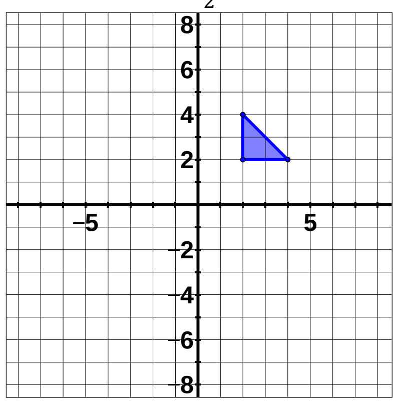
3. Dilate by $c = \frac{3}{4}$, center $(0,0)$



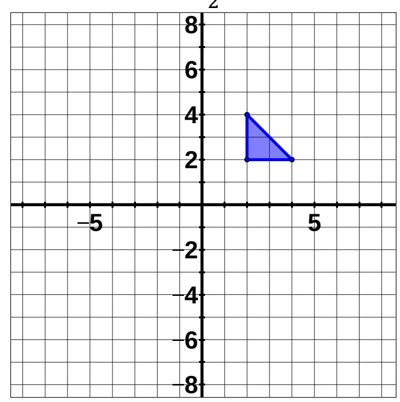
4. Dilate by $c = 2$, center $(6,4)$



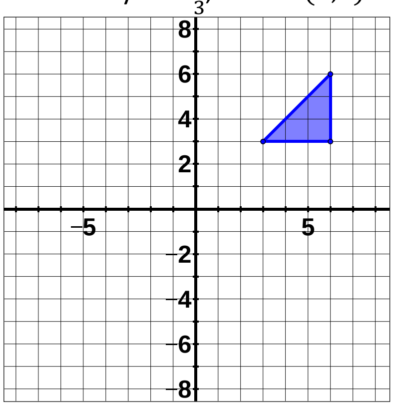
5. Dilate by $c = \frac{3}{2}$, center $(0,0)$



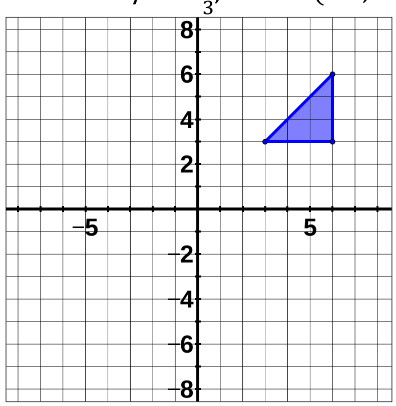
6. Dilate by $c = \frac{1}{2}$, center $(-6,2)$



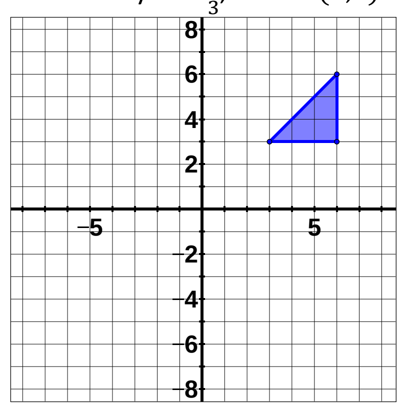
7. Dilate by $c = \frac{1}{3}$, center $(0,0)$



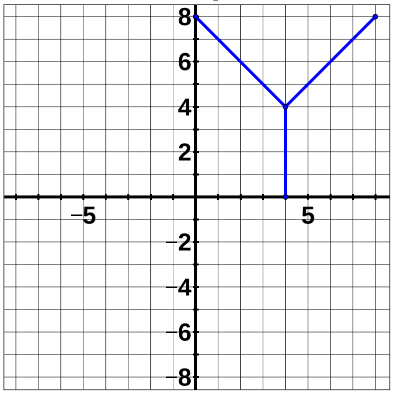
8. Dilate by $c = \frac{2}{3}$, center $(-3,-6)$



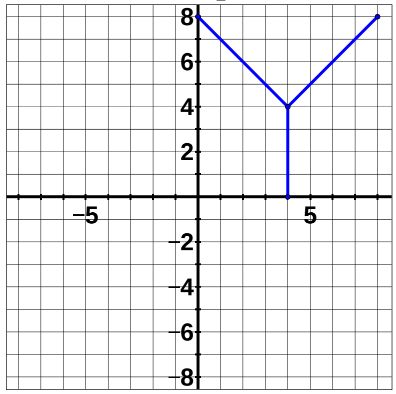
9. Dilate by $c = \frac{4}{3}$, center $(0,0)$



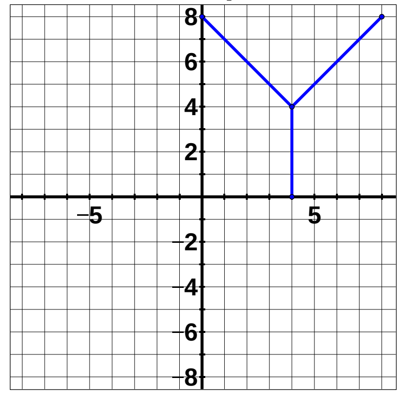
10. Dilate by $c = \frac{1}{4}$, center $(4,4)$



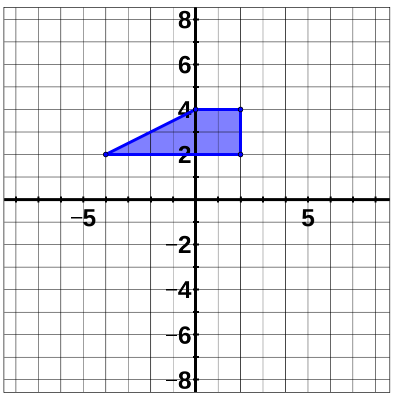
11. Dilate by $c = \frac{1}{2}$, center $(0,0)$



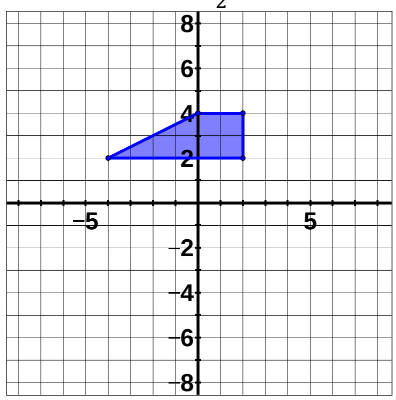
12. Dilate by $c = \frac{3}{4}$, center $(-4,8)$



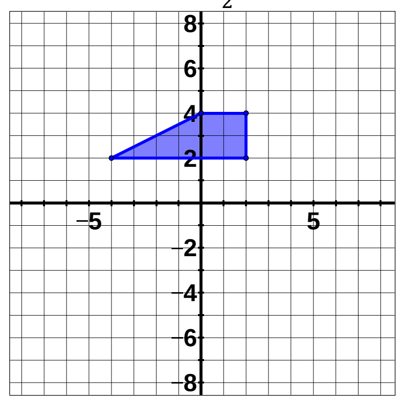
13. Dilate by $c = 2$, center $(0,0)$



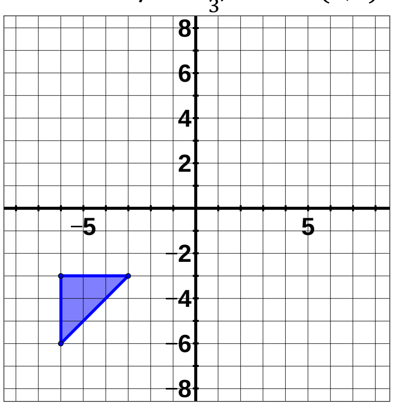
14. Dilate by $c = \frac{3}{2}$, center $(-4,-2)$



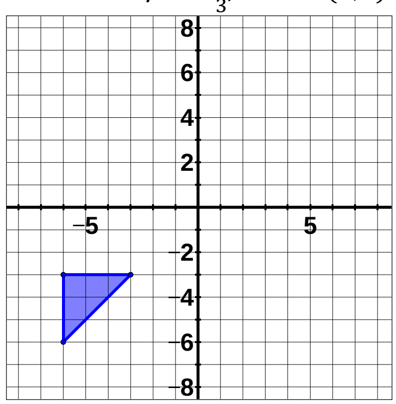
15. Dilate by $c = \frac{1}{2}$, center $(0,0)$



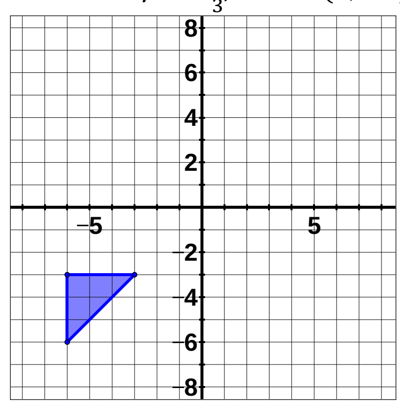
16. Dilate by $c = \frac{1}{3}$, center $(3,0)$



17. Dilate by $c = \frac{2}{3}$, center $(0,0)$



18. Dilate by $c = \frac{4}{3}$, center $(0,-6)$



2.2 Constructing Reflections

Now we begin to look at transformations that yield congruent images. We'll begin with reflections and then move into a **series** of transformations. A series of transformations applies more than one transformation one at a time to a pre-image.

Reflections

Reflections are like a mirror image, or flip, of the pre-image. This results in a congruent image, meaning it is not only the same shape but also the same size. At the 8^{th} grade, we will only be reflecting across the x-axis or the y-axis. Both cases need different formulas for the coordinates, so we'll break it down one at a time.

Reflecting Across the x-Axis

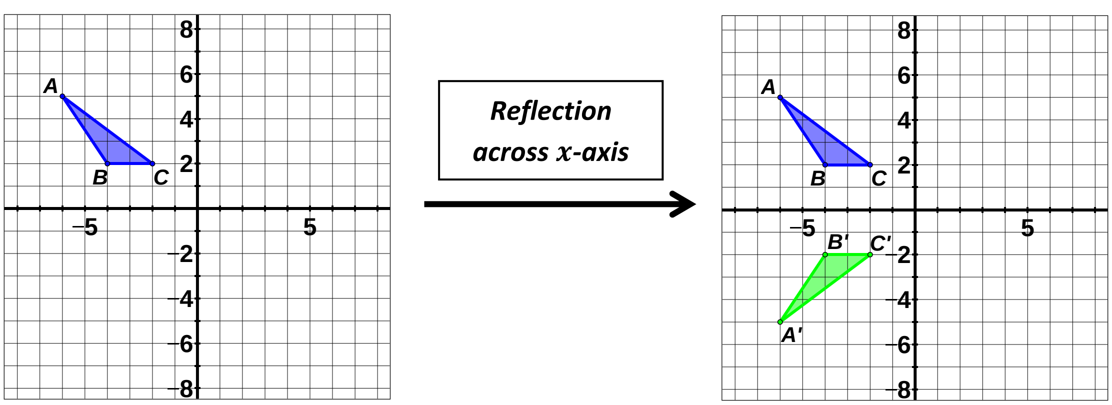
A reflection across the x-axis takes points above the x-axis to below the x-axis and vice versa. It basically flips the shape up and down. To do so, it utilizes these formulas:

$$x' = x$$

$$y' = -y$$

Notice that the x-coordinate stays the same in the image as it was in the pre-image. The only real change is that you take the opposite y value. So if the y-coordinate was positive in the pre-image, it would be negative in the image. If it was negative in the pre-image, it would be positive in the image. Basically, if you want to apply a reflection across the x-axis, just change the sign of the y-coordinates.

Let's take the triangle from section 11.2 and reflect it across the x-axis. Recall that those coordinates were $A: (-6,5), B: (-4,2), C: (-2,2)$. Since we only need to change the sign of the y-coordinates we end up with the image of a triangle with the points $A': (-6, -5), B': (-4, -2), C': (-2, -2)$.



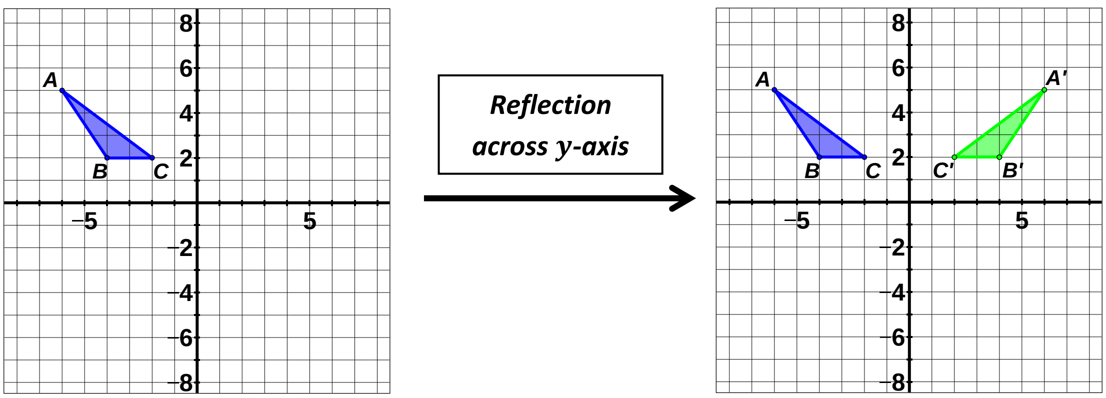
Reflecting Across the y-Axis

Seeing how to reflect across the x-axis, what do you think will happen if you reflect across the y-axis? As expected, it is now the x-coordinates that change, and the y-coordinates that stay the same. We use these formulas to reflect across the y-axis:

$$x' = -x$$

$$y' = y$$

Just change the sign of the x-coordinate. Let's look at the triangle from our previous example and reflect it across the y-axis. You should verify that it leads to the points A': $(6,5)$, B': $(4,2)$, C: $(2,2)$.

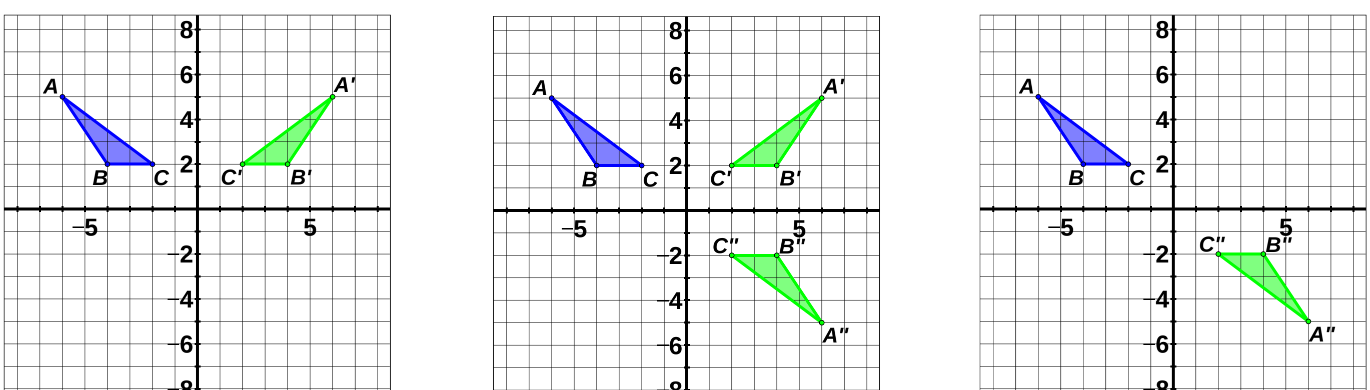


A Series of Transformations

We can also apply multiple transformations to a shape. When we do so, we apply only one transformation at a time. So if we wanted to dilate, translate, and then reflect we would first dilate the pre-image, then translate that new picture, then reflect that translation. For now we'll stick with just the two transformations we have covered so far.

Reflect-Reflect

Taking our faithful triangle, we can apply a reflection across the y-axis and then reflect that across the x-axis as follows:



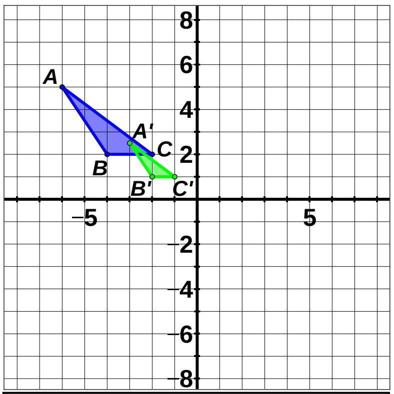
Reflection across y-axis | ***Reflect that across x-axis*** | ***Final Picture of the Series***

Notice that the in-between step is not shown on the final graph. You are welcome to leave the in-between step as you are graphing. Just be sure to label appropriately so we know which is which.

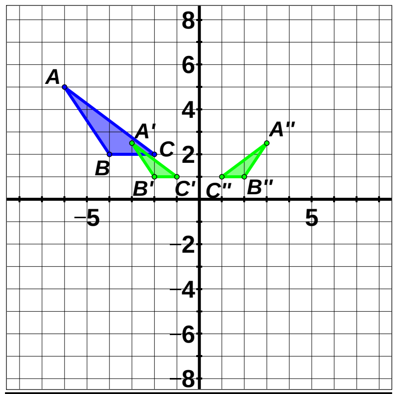
Also notice that we now have double prime to represent the second reflection and we end up with the final image points of $A'': (6, -5), B'': (4, -2), C'': (2, -2)$. You might notice that this also looks like a rotation by $180°$ which is true. It turns out that reflecting across both axes gives you a full half turn.

Dilate-Reflect

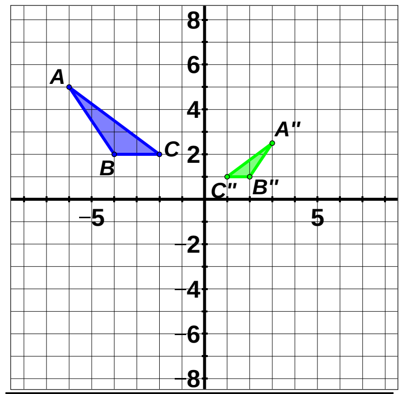
Typically if we are going to dilate in a series, we perform that operation first. Otherwise it can throw off your other transformations because it moved the points closer or farther away from the origin. Let's again take that same triangle and dilate by $c = \frac{1}{2}$ and then reflect across the y-axis. Take a look.



Dilation by $c = \frac{1}{2}$



Reflect that across y-axis



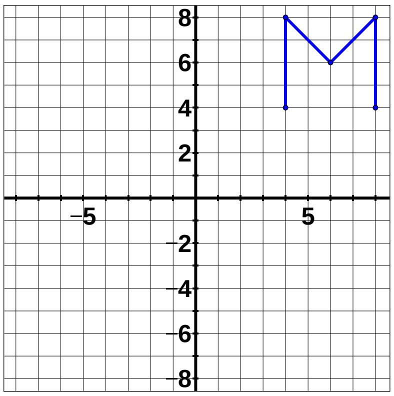
Final Picture of the Series

The dilation gave us the points $A': (-3, 2.5), B': (-2, 1), C': (-1, 1)$ and then reflecting those points gave us the final points of $A'': (3, 2.5), B'': (2, 1), C'': (1, 1)$.

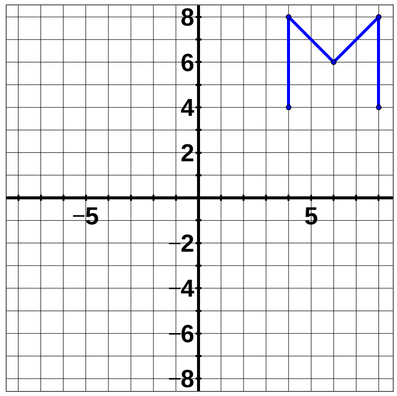
Lesson 2.2

Perform the given reflection or series of transformations on each given pre-image. When performing a dilation, use the origin as the center of dilation.

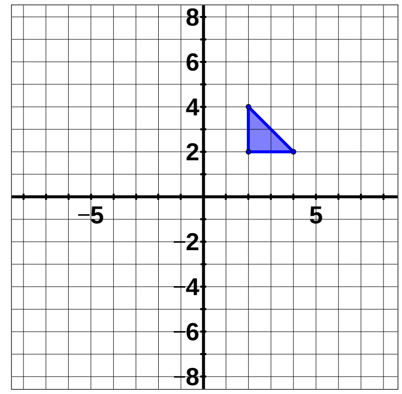
1. Reflect across x-axis



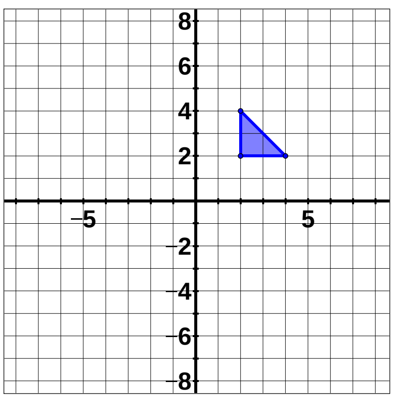
2. Reflect across y-axis



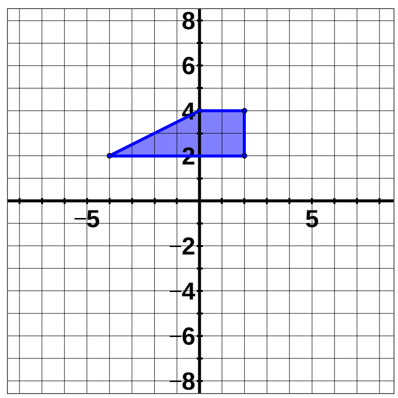
3. Reflect across y-axis



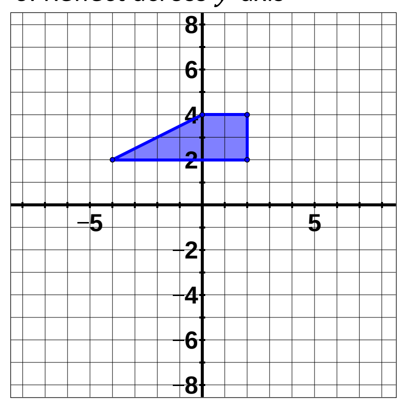
4. Reflect across x-axis



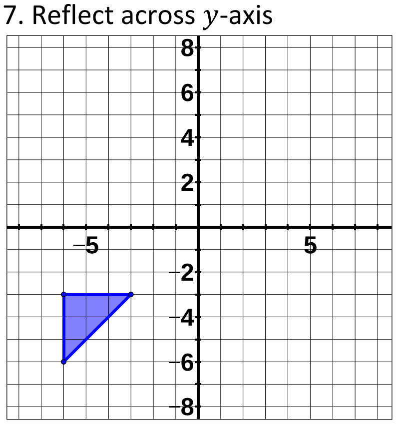
5. Reflect across x-axis



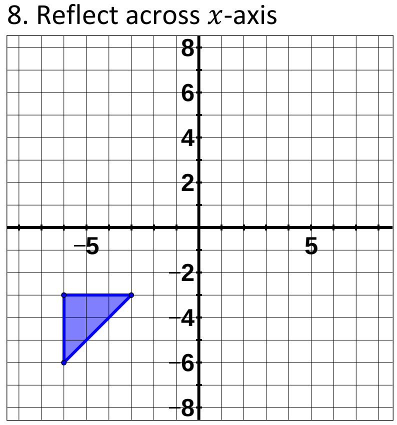
6. Reflect across y-axis



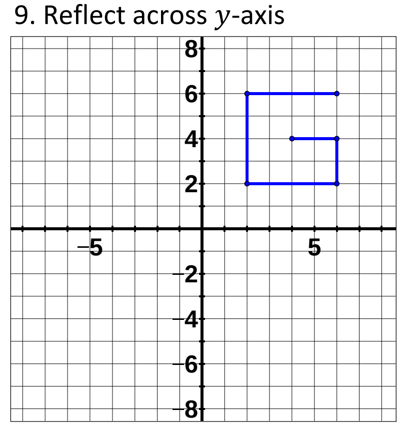
7. Reflect across y-axis



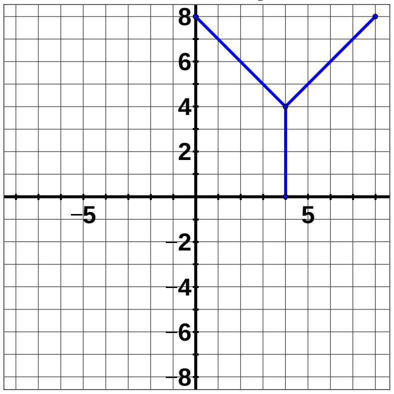
8. Reflect across x-axis



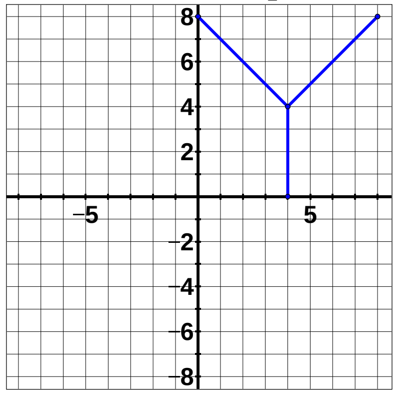
9. Reflect across y-axis



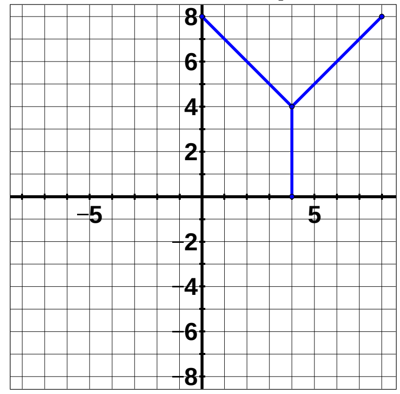
10. Reflect across x-axis
and dilate by $c = \frac{1}{4}$



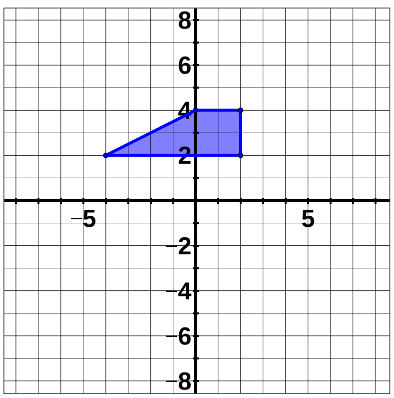
11. Reflect across y-axis
and dilate by $c = \frac{1}{2}$



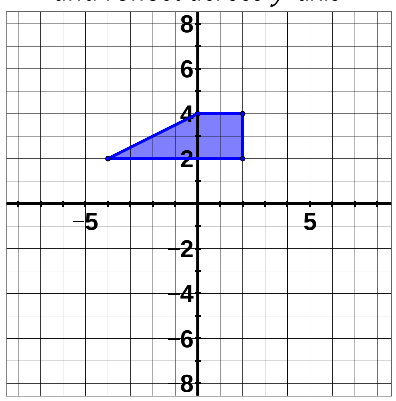
12. Reflect across y-axis
and dilate by $c = \frac{3}{4}$



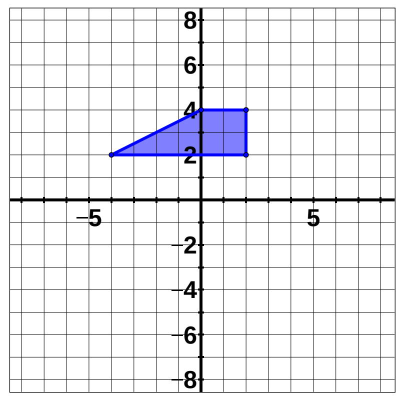
13. Dilate by $c = 2$
and reflect across x-axis



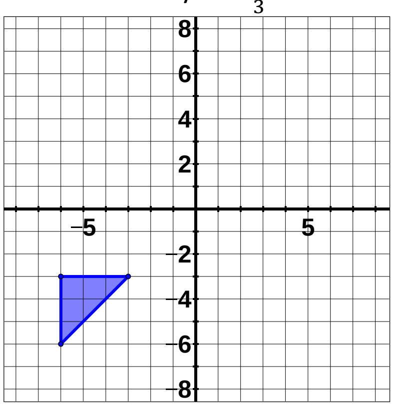
14. Dilate by $c = \frac{3}{2}$
and reflect across y-axis



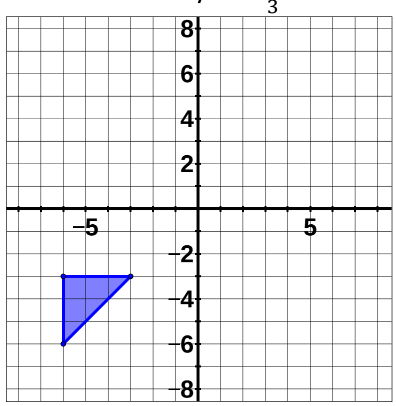
15. Dilate by $c = \frac{1}{2}$
and reflect across x-axis



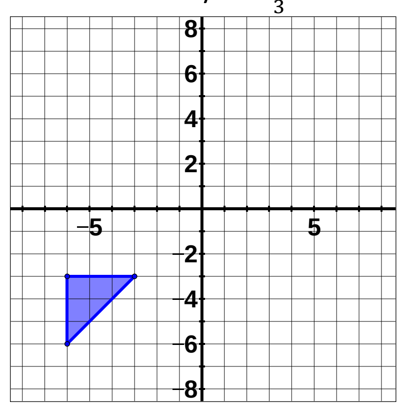
16. Reflect across x-axis
and dilate by $c = \frac{1}{3}$



17. Reflect across y-axis
and dilate by $c = \frac{2}{3}$



18. Reflect across x-axis
and dilate by $c = \frac{4}{3}$



2.3 Constructing Rotations

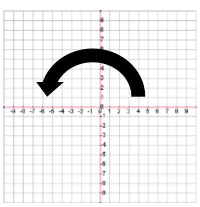
We're halfway through the transformations and our next one, the rotation, gives a congruent image just like the reflection did. Just remember that a series of transformations with a rotation or reflection doesn't guarantee that it will be congruent because it could have a dilation in there. That would make the image similar to the pre-image instead of congruent.

Rotations

At the 8th grade, we will only be rotating about the origin and only in increments of 90°. That means that we could rotate 90°, 180° or 270°. We won't be doing a 360° because that would turn the shape in a complete circle giving us an image right on top of the pre-image. That's rather boring, so we'll stick with the other three.

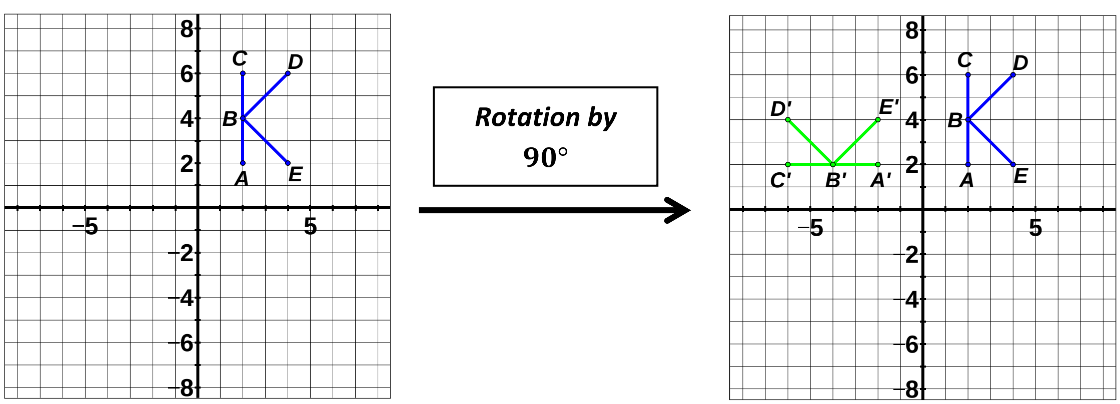
Rotations are also known as turns. So we may say a ninety degree turn, or a quarter turn, at times. Another name for a rotation would be a spin because we are spinning the picture about the origin.

90° Rotation about the Origin



For a 90° about the origin first remember that we rotate counterclockwise. That means we rotate like the arrow to the left.

Looking at a capital letter may be an easy example to think about to make sure we get the concept of rotations down. Take the following capital "K" and rotate it counterclockwise 90° in you mind. You can also do so physically by putting your finger on the origin and turning the book a quarter turn counterclockwise. You should be able to visualize in your mind the "back" of the K turning to lie flat parallel to the x-axis. Here's what it would look like:



How did that happen? We can use the following formulas to execute a 90° turn:

$$x' = -y$$

$$y' = x$$

At first glance those formulas may be confusing. Written out in words it says, "The new x is equal to the opposite of the old y," and, "The new y is equal to the old x." Let's look at where our "K" picture started in quadrant one. It had the points $A: (2,2), B: (2,4), C: (2,6), D: (4,6), E: (4,2)$. Let's just rotate the point E.

First find the new x value, or x'. The new x is the opposite of the old y. (Remember opposite means change the sign.) The old y was 2, so that means the new x must be -2. The new y is the old x. The old x was 4, so that means the new y is also 4. That gives us the point $E': (-2,4)$. So we went from $E: (4,2)$ to $E': (-2,4)$.

Switch the coordinates and flip the sign on the new first coordinate. Switch and flip. Start with $(4,2)$, switch to $(2,4)$ and then flip the sign to $(-2,4)$. Switch and flip for a $90°$ rotation.

One other strategy that you can apply is only using the formulas to rotate one point. Since we already rotated point E, we could simply build the rest of the picture from that point if we want. We know that the back of the "K" was four units long. Since rotations give congruent images, the laying down "K" will have a four unit long back as long. So just draw it in. In the same way you can find the other two points and finish the "K."

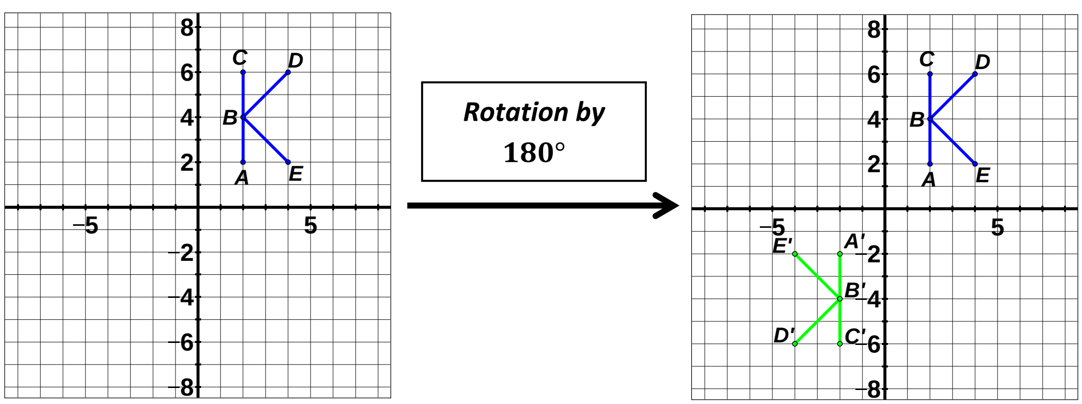
180° Rotation about the Origin

This rotation, or the half turn, is much easier. It follows the formulas:

$$x' = -x$$

$$y' = -y$$

This means we just change the sign of both coordinates. To rotate our "K" by $180°$, we simply change the sign of all the coordinates giving us the points $A: (-2,-2), B: (-2,-4), C: (-2,-6), D: (-4,-6), E: (-4,-2)$. Take a look:



Alternatively, we could just do a $90°$ rotation to get the half turn. However, it's probably easier to just change the sign of all the coordinates.

270° Rotation about the Origin

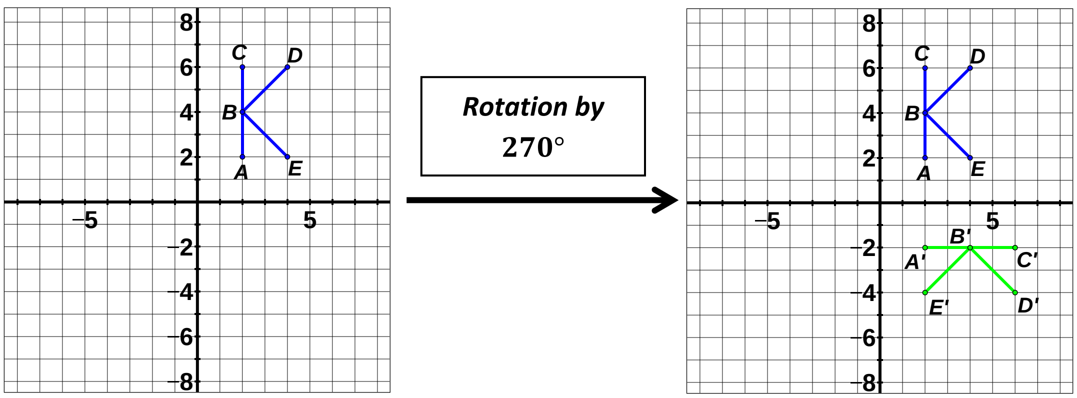
This rotation, or the three-quarters turn, may be more difficult to see. You could do a $90°$ three times or a $180°$ rotation followed by a $90°$ rotation or you could use these formulas:

$$x' = y$$

$$y' = -x$$

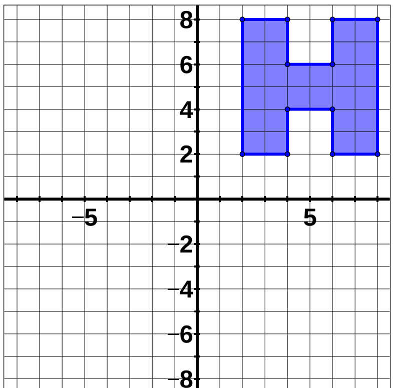
This means that the new x is equal to the old y and the new y is equal to the opposite of the old x.

Let's again take the point $E{:}\,(4,2)$ on our "K" to rotate $270°$. The new x is equal to the old y so the new x is equal to 2. The new y is equal to the opposite of the old x so the new y is equal to -4. That means we get the point $E'{:}\,(2,-4)$. This is also a switch and flip, but we flip the sign of the second coordinate instead of the first. Take a look at the transformation:

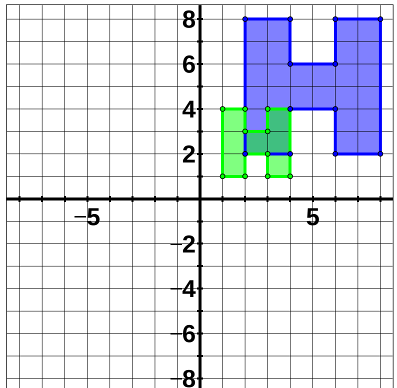


A Series Transformations

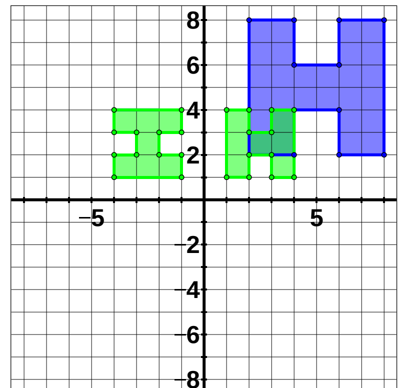
Let's take the following block letter "H" and dilate it by $c = \frac{1}{2}$, rotate it 90°, and then reflect it across the x-axis. That's three transformations in a row, which is perfectly acceptable. Here's what it would look like a step at a time:



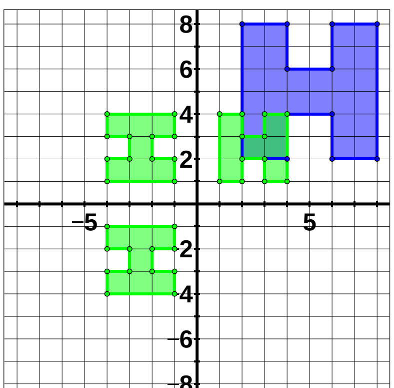
Start with the "H"



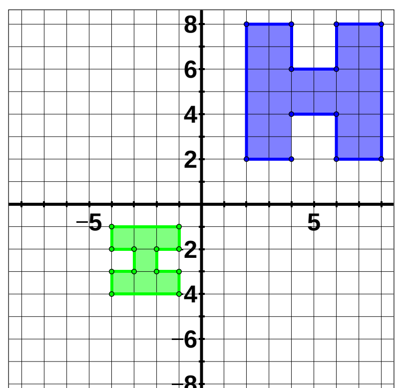
Dilate by $c = \frac{1}{2}$



Rotate by $90°$



Reflect across x-axis



Final Picture of Series

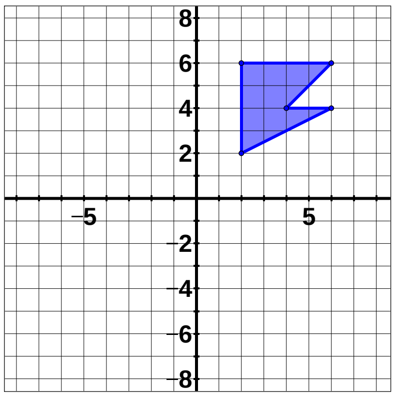
Again in this case it would be more beneficial to not use the formulas for every single point. It would be better to use the bottom left corner of the original "H" as an anchor point. Always use the formula for that point and then build the rest from what we know.

For example, you might dilate the bottom left point $(2,2)$ by the factor $c = \frac{1}{2}$ giving a new point of $(1,1)$. From there you know that the thickness of the "H" will be one unit since it was originally two units. Just build a half size "H" from there. Follow the same process for the other two transformations.

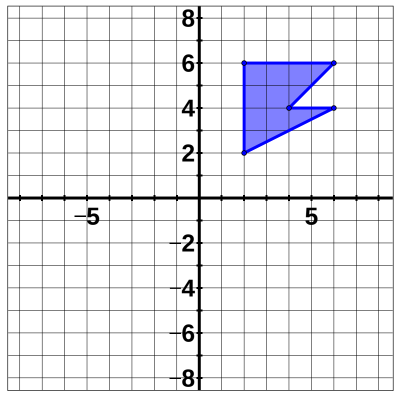
Lesson 2.3

Perform the given rotation or series of transformations on each given pre-image. When performing a dilation, use the origin as the center of dilation.

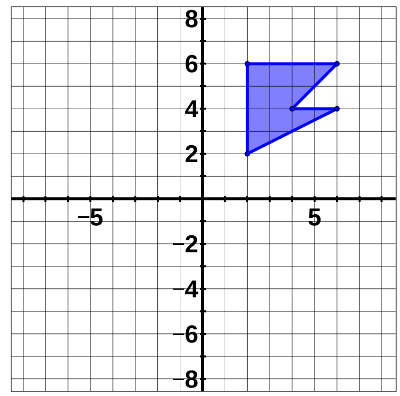
1. Rotate 90°



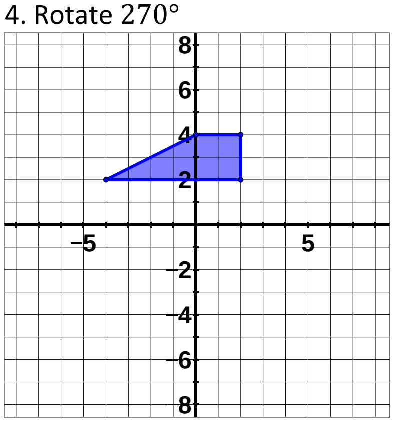
2. Rotate 180°



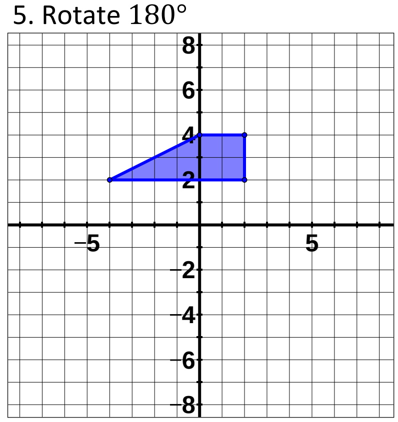
3. Rotate 270°



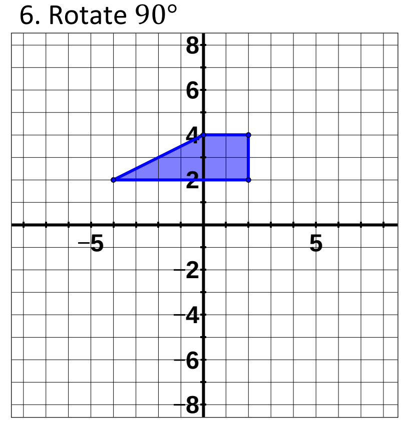
4. Rotate 270°



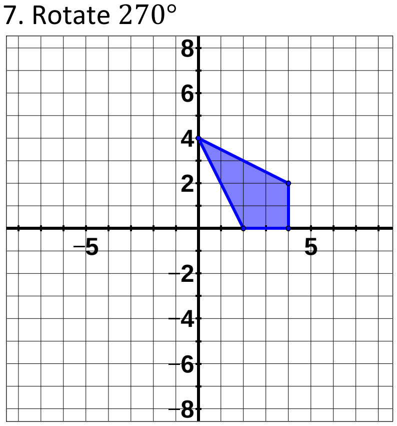
5. Rotate 180°



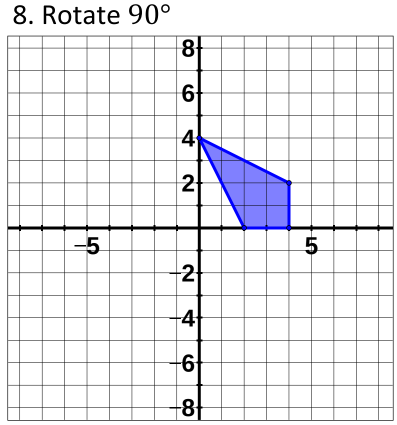
6. Rotate 90°



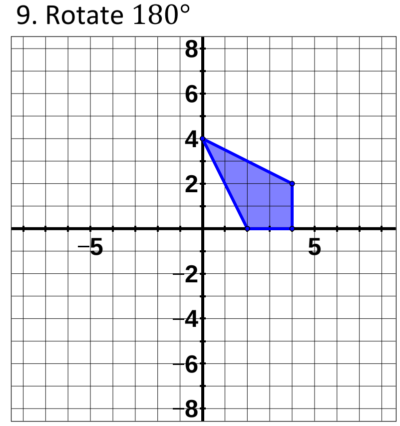
7. Rotate 270°



8. Rotate 90°

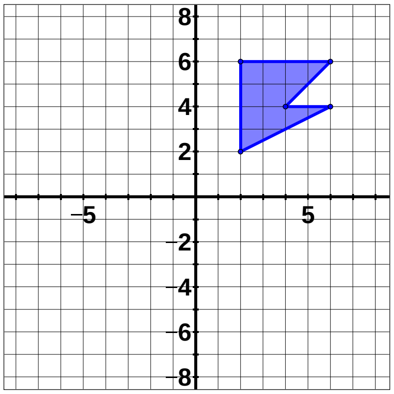


9. Rotate 180°



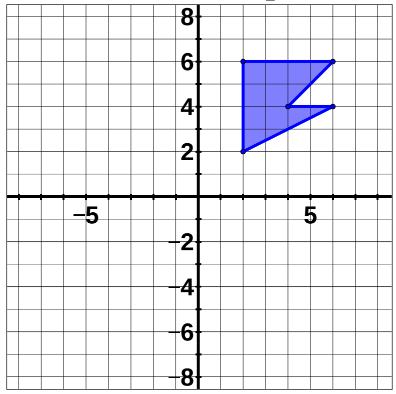
10. Rotate $90°$

and reflect across x-axis



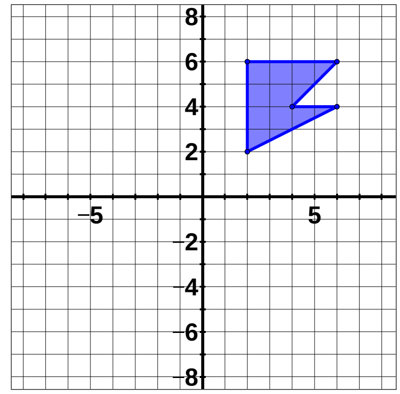
11. Rotate $180°$

and dilate by $c = \frac{1}{2}$



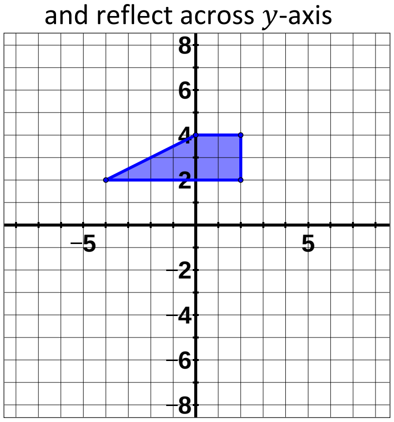
12. Rotate $270°$

and reflect across y-axis



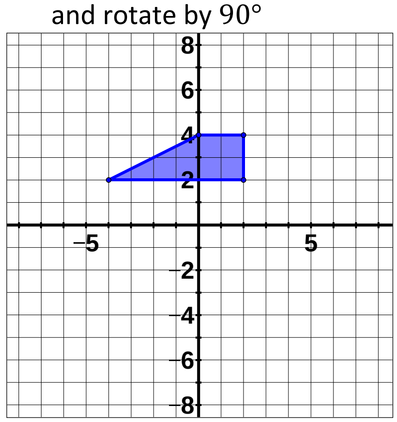
13. Dilate by $c = 2$

and reflect across y-axis



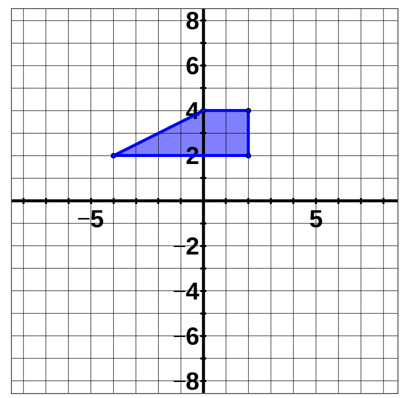
14. Dilate by $c = \frac{1}{2}$

and rotate by $90°$



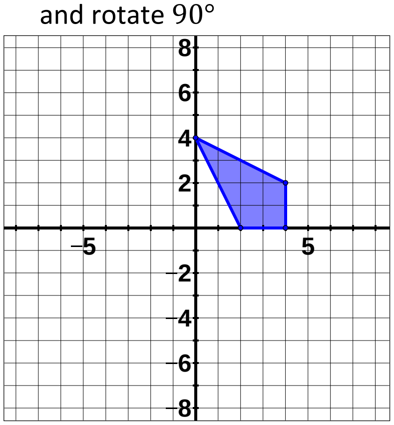
15. Rotate $180°$

and reflect across x-axis



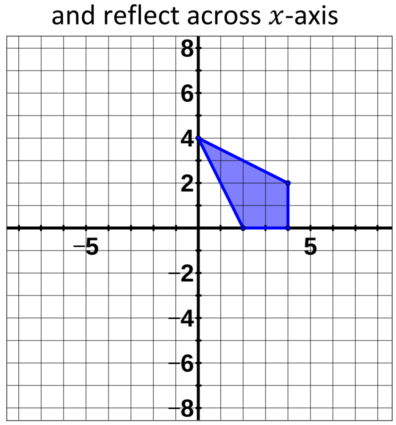
16. Reflect across y-axis

and rotate $90°$



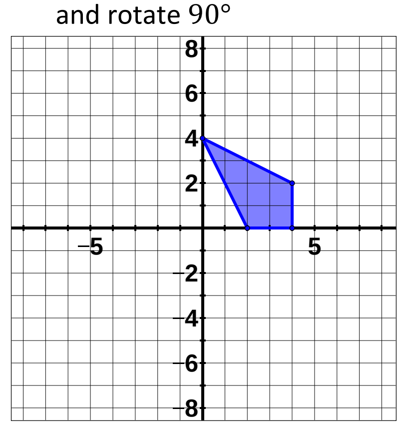
17. Dilate by $c = \frac{1}{2}$

and reflect across x-axis



18. Dilate by $c = 2$

and rotate $90°$



2.4 Constructing Translations

Our final transformation also creates images that are congruent to their pre-images. Translations are also sometimes called a “slide” because it appears that they slide the pre-image to a new position on the coordinate plane.

Translation Vectors

To translate a shape, we use a **translation vector**. The translation vector looks like this: $\binom{a}{b}$ where a represents how much to add to the x-value of each point in the pre-image and b represents how much to add to the y-value of each point in the pre-image. This leads to the formulas:

$$x' = x + a$$

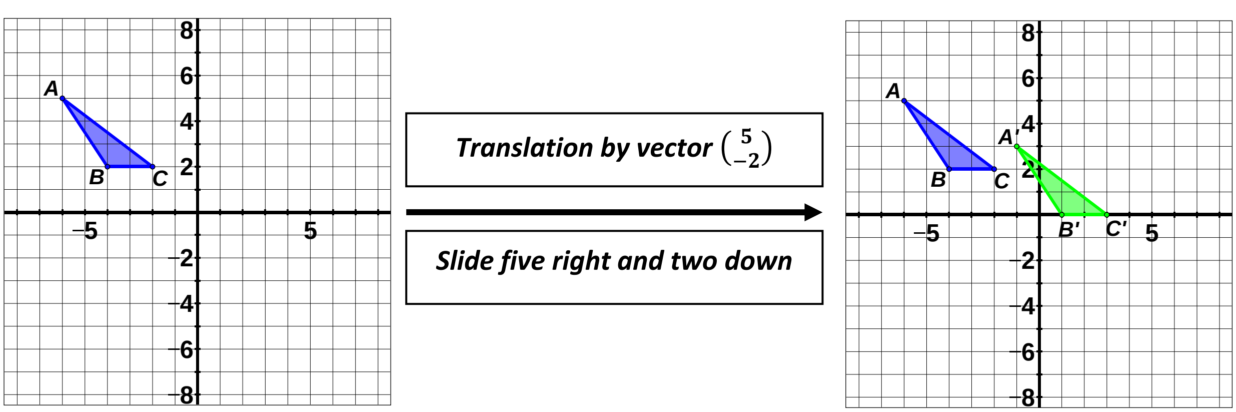
$$y' = y + b$$

Let’s look at a couple of specific vectors to get the main idea down. The vector $\binom{2}{-3}$ would slide a shape two to the right and three down. Remember that the top number in the vector is added to the x-coordinate. That means left or right movement. Since it’s positive two, the shape will move two to the right. The bottom number of the vector is added to the y-coordinate and therefore represents up and down movement. Since it’s a negative three, the shape will move three down.

In the same manner, the vector $\binom{-1}{0}$ would slide a shape one left and not at all up or down. This is just a horizontal slide. A vertical slide would have a zero in the top of the vector. Let’s look at some examples with specific pre-images.

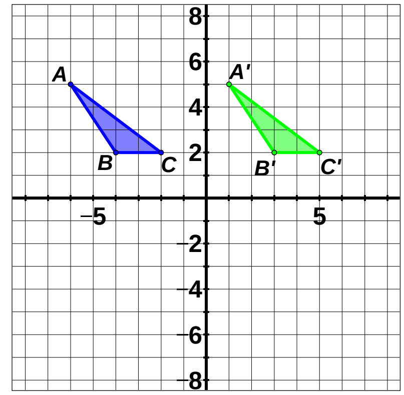
Let the pre-image be the points $A: (-6,5), B: (-4,2), C: (-2,2)$ giving us a triangle and let’s translate that using the translation vector $\binom{5}{-2}$. This means add five to each x-coordinate and negative two to each y-coordinate or move the triangle five right and two down.

To get our new points, we do the addition for each point in the pre-image. $A': (-6+5, 5+(-2))$ or $A': (-1,3)$. Similarly we find that $B': (1,0)$ and $C': (3,0)$. Instead of using the formulas, we can always just count five right and two down for each pre-image point and plot the new image point. The final result should be as follows:



A Horizontal Slide

Sometimes throwing in a zero can confuse people, so let's work through a horizontal translation with the vector $\binom{7}{0}$. Notice that this will slide the pre-image shape seven to the right. Let's use the same triangle as before. You should see that the new points will be A': $(1,5)$, B': $(3,2)$, C': $(5,2)$ as seen to the right:



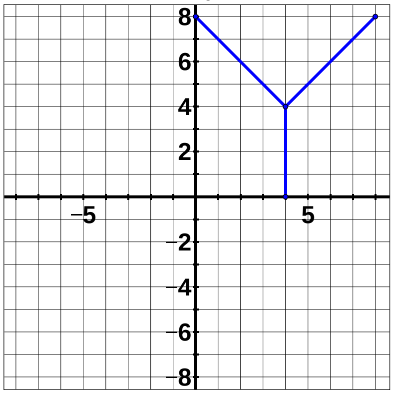
A Series of Transformations

Just as with all previous examples, we can add a translation to a series of transformations to make more interesting pictures. Keep in mind that if the series contains only reflections, rotations and/or translations, the image will be congruent to the pre-image. If we use a dilation at any point, we will end up with a similar, not congruent, image.

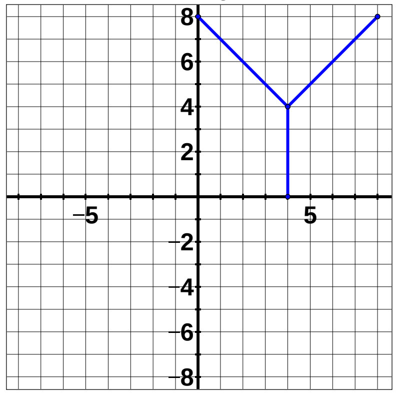
Lesson 2.4

Perform the given translation or series of transformations on each given pre-image. When performing a dilation, use the origin as the center of dilation.

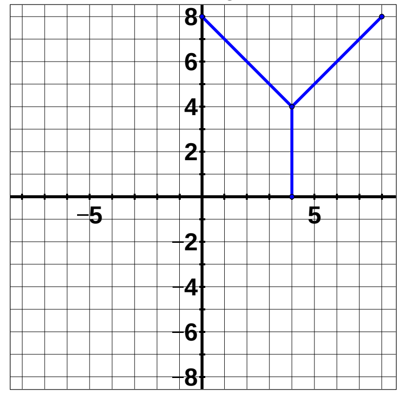
1. Translate by $\binom{-8}{0}$



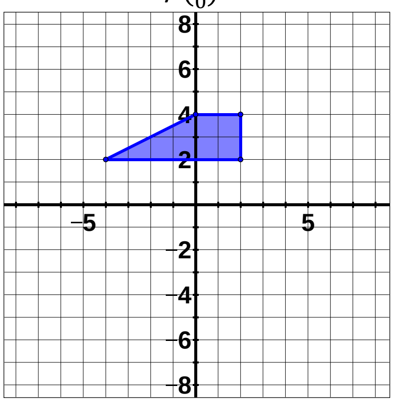
2. Translate by $\binom{0}{-8}$



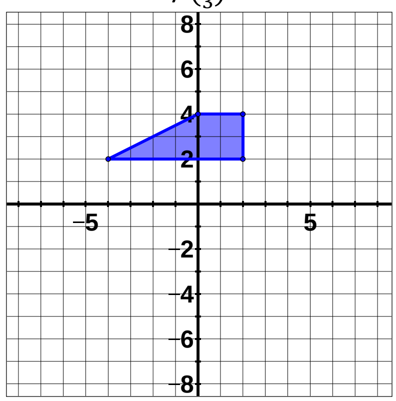
3. Translate by $\binom{-3}{-6}$



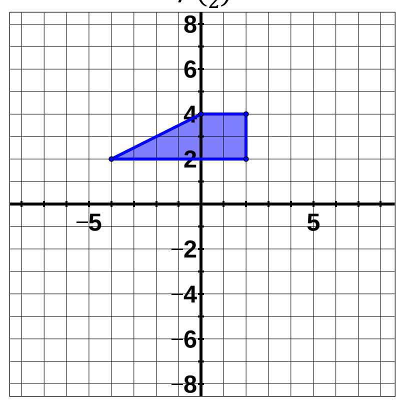
4. Translate by $\binom{5}{0}$



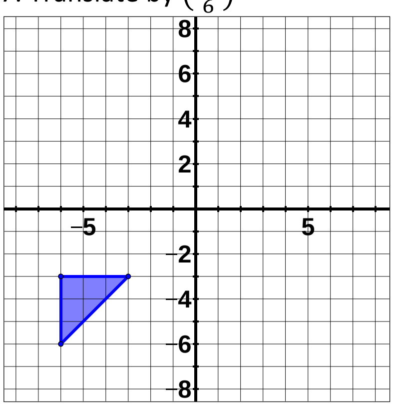
5. Translate by $\binom{0}{3}$



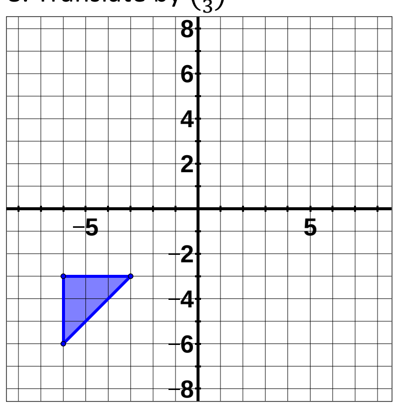
6. Translate by $\binom{4}{2}$



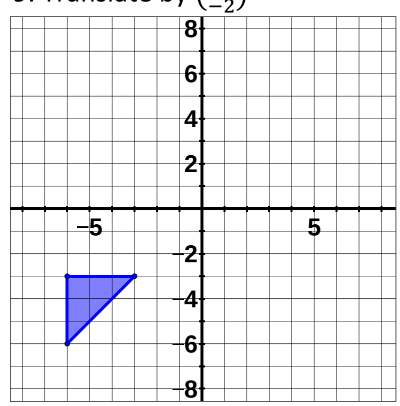
7. Translate by $\binom{-2}{6}$



8. Translate by $\binom{3}{3}$

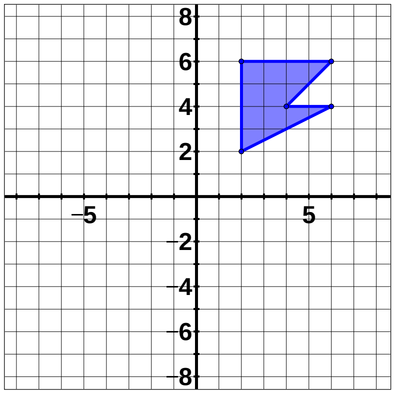


9. Translate by $\binom{8}{-2}$



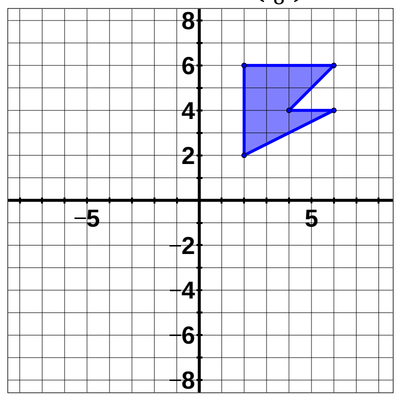
10. Rotate 90°

and translate by $\binom{3}{-4}$



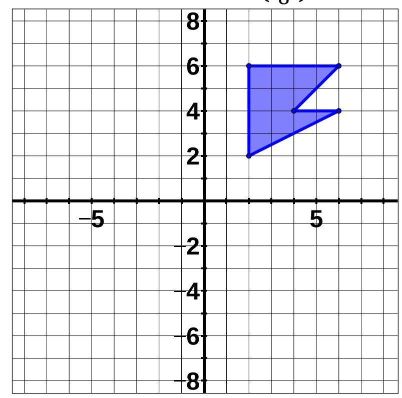
11. Rotate 180°

and translate by $\binom{-1}{8}$



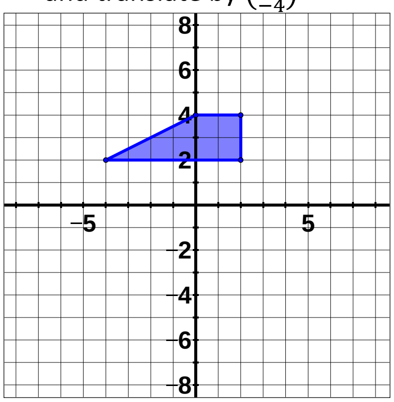
12. Reflect across x-axis

and translate by $\binom{-8}{8}$



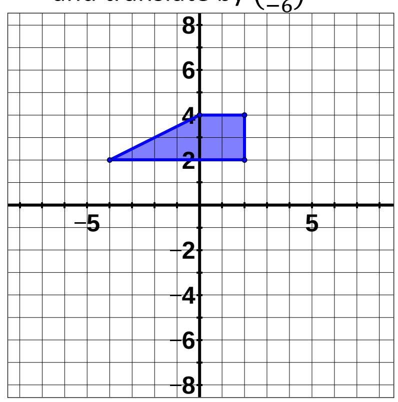
13. Reflect across y-axis

and translate by $\binom{0}{-4}$



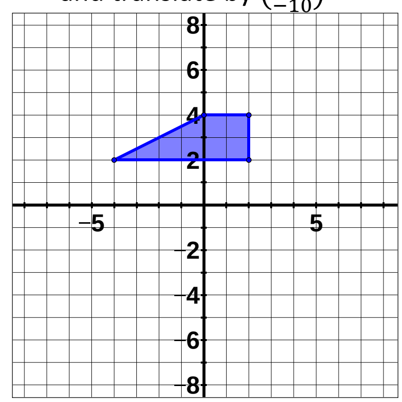
14. Dilate by $c = \frac{1}{2}$

and translate by $\binom{4}{-6}$



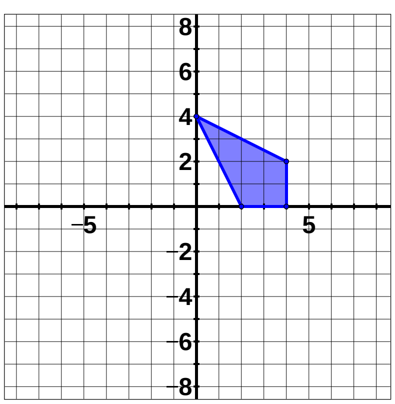
15. Dilate by $c = 2$

and translate by $\binom{2}{-10}$



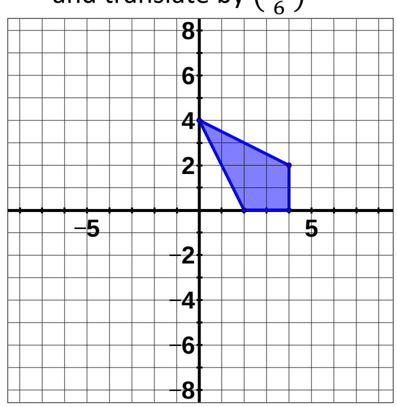
16. Dilate by $c = \frac{1}{2}$, rotate 90°,

and reflect across x-axis



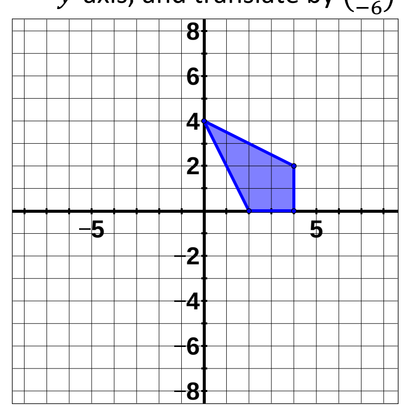
17. Dilate by $c = \frac{1}{2}$, rotate 180°,

and translate by $\binom{-4}{6}$



18. Dilate by $c = 2$, reflect across

y-axis, and translate by $\binom{4}{-6}$



2.5 Identifying a Series and Determining Congruence or Similarity

Now that we understand how to create each type of transformation and how to combine with a series, we should be able to identify the specific transformations we see happening on the coordinate plane. There is no exact method for doing this other than intuition based on what we see.

A Series of Transformations

When identifying a series of transformation (or a single transformation), there is a three step process:

1.) Look for dilations
2.) Look for reflections and/or rotations
3.) Look for translations

Looking for Dilations

Check if there is a dilation first. To do so ask the question, **"Has the picture been shrunk or enlarged?"** If so, try to decide by what scale factor. (For a series, we will assume the origin is the center of dilation.) One way to do this is find a side length that is easily measurable in the pre-image and check the corresponding side length on the image. If the pre-image has a side length of 2 units and the image has a side length of 1 unit, then the dilation had a scale factor of $c = \frac{1}{2}$. If the pre-image has a side length of 3 units and the image has a side length of 5 units, then the dilation had a scale factor of $c = \frac{5}{3}$.

Looking for Reflections/Rotations

The next question to ask yourself is, **"Has the picture orientation changed?"** If so, then a reflection or rotation was involved. Experiment to see which or if both have been used perhaps by using a piece of tissue paper to trace the pre-image. Next try turning, rotating, that tissue paper in $90°$ increments with it anchored at the origin. Remember that we're trying to find a way to get the orientations to match.

If turning the pre-image doesn't get the orientation of the image, try flipping the tissue paper over the axes. Try the x- and y-axis one at a time to get the orientation to match.

Looking for Translations

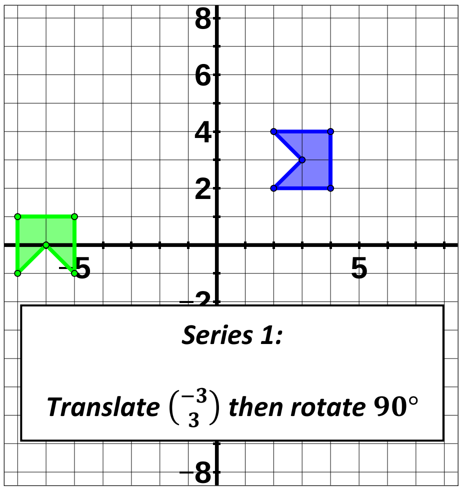
Finally check for a translation. Once you have the size and orientation matching, find the translation that you need to move the shape to the image. Remember that a translation takes into account both horizontal and vertical distance, so be sure to note both using the vector form.

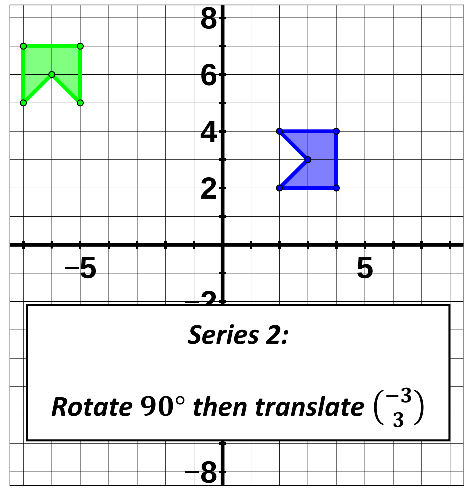
Is the Series Unique?

That means, is the series that we find following this method the only series that will take the pre-image to the image? No. There may be other series that would work as well. Therefore if you are working and find a series that doesn't match the series of someone else or the answer document, that's OK. Just make sure that you can prove that your series works.

Does Order Matter?

For example, remember that a translation could occur before or after a reflection or rotation in a series. Each would produce a different result. Consider the following pictures (remember that the pre-image is darker in blue and the image is lighter in green). The first is a translation by the vector $\binom{-3}{3}$ followed by a $90°$ rotation. The second is a rotation by $90°$ followed by a translation by the vector $\binom{-3}{3}$. Notice that the two different series produce different results even though they use the same transformations. **The order matters!**

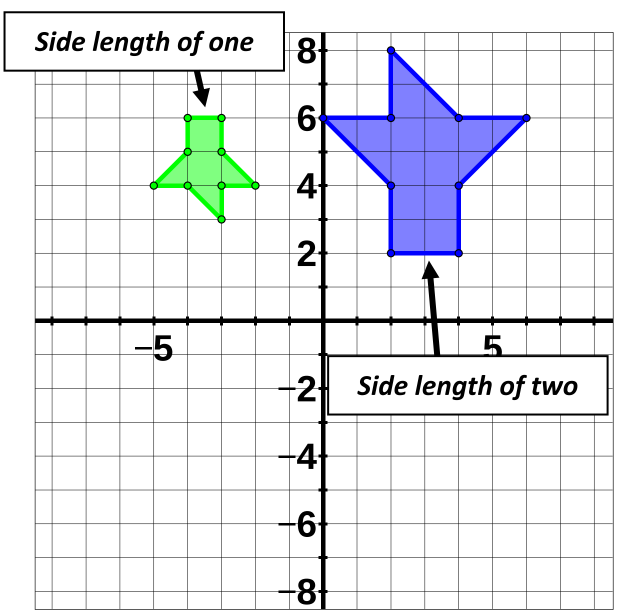




Consider again that there may be more than one series of transformations to take the pre-image to the image. For example, Series 1 could be a $90°$ rotation followed by a translation by vector $\binom{-3}{-3}$. That's just one example that there are often multiple ways to get a series.

All Together Now!

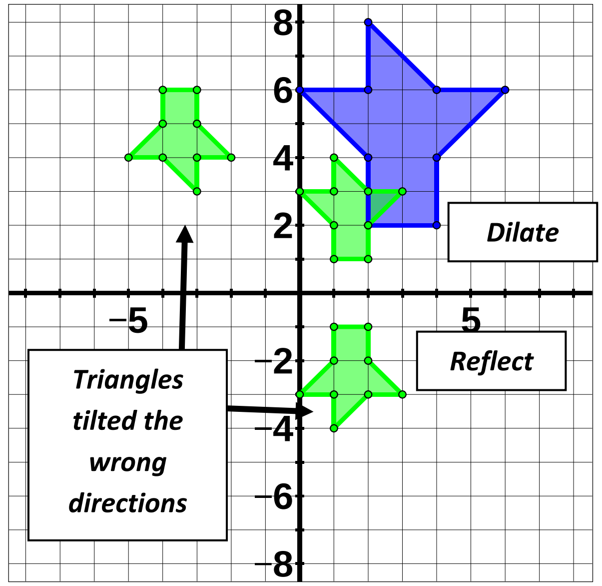
Now that we know the three step process (dilations, then reflections/rotations, then translations), let's work through an example problem.

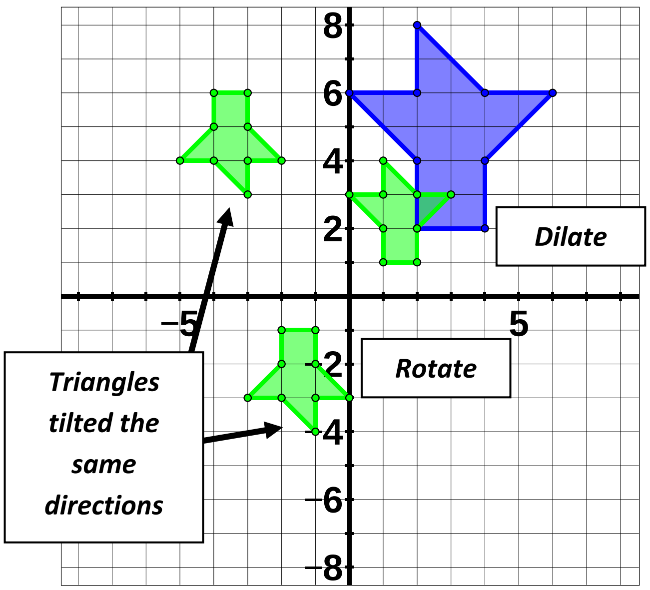


Step 1: Dilation? Yes, this has clearly been dilated at some point because the image has been shrunk. Notice that the bottom of the pre-image has a length of two units and the corresponding side on the image has a length of one unit. So it has been cut in half. That means there was a dilation by a scale factor of $c = \frac{1}{2}$ with the origin as the center of dilation.

Step 2: Reflection/Rotation? Yes, there had to be one of these transformations because the orientation has changed. Now we need to experiment to decide if there a flip (reflection) or a turn (rotation) involved. Remember that it's possible that both were used as well.

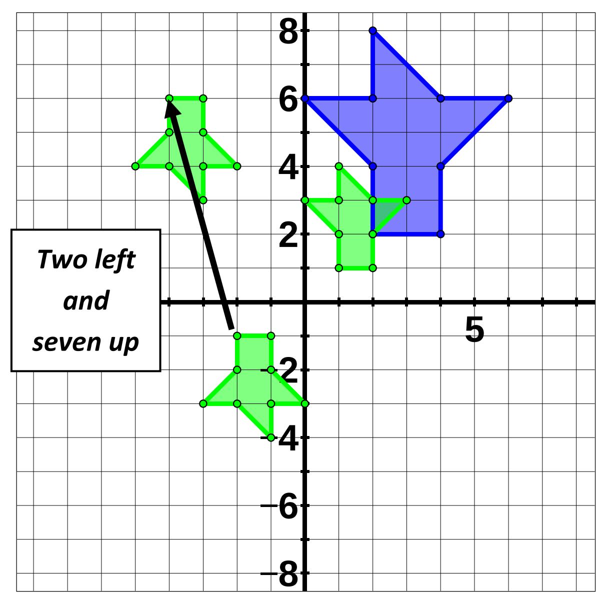
Just thinking and visualizing in our head a bit, a reflection across the y-axis won't work. The pointy ends would still be sticking up. So it might be a reflection across the x-axis. The picture to the right shows a dilation by $c = \frac{1}{2}$ and then a reflection across the x-axis. Does the orientation match that of the image? No. It's very close, but that doesn't do it because the middle triangle part is going the wrong direction.





Since reflections didn't work, we should try some rotations to match up the orientation. Picturing some rotations in our head, we should see that a quarter turn (90° rotation) won't work. That would lay the "tree" on its side instead of upside down. In the same way a three-quarters turn (270° rotation) won't work. So take a look at a dilation by $c = \frac{1}{2}$ followed by a $180°$ rotation at the left. Does the orientation now match that of the image? Yes. So the rotation by $180°$ was what we needed.

Step 3: Translation? Now we know the first two transformations in the series, but it still doesn't line up exactly with the image. That means we need to slide (translate) it to where the image is. Pick one set of corresponding points (one point from the picture we currently have after dilation and rotation and the other from the image) and check how we need to move that single point. That translation vector will work for the whole picture now that we have the size and orientation correct. Note that this translation vector is two left and seven up or $\binom{-2}{7}$.

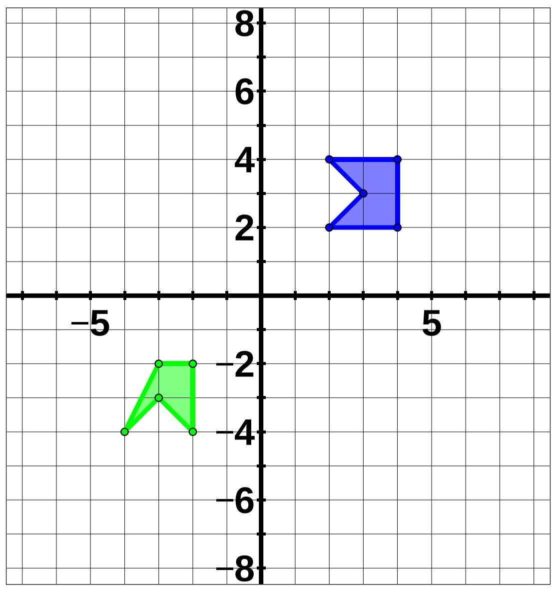


Complete Series: So the complete series of transformations is a dilation by $c = \frac{1}{2}$, rotation of $180°$, and translation by vector $\binom{-2}{7}$.

Congruent or Similar?

Just as with all previous examples, we can add a translation to a series of transformations to make more interesting pictures. Keep in mind that if the series contains only reflections, rotations and/or translations, the image will be congruent to the pre-image. If we use a dilation at any point, we will end up with a similar, not congruent, image. You will be expected to accurately identify whether the image is congruent or similar to the pre-image.

A third option does exist: the image and pre-image could be *neither* congruent nor similar. Consider a picture where the pre-image is Kermit the Frog and the image is Miss Piggy. Clearly those are neither congruent nor similar. No geometric transformation could turn Kermit into Miss Piggy. Consider the following example on the coordinate plane.

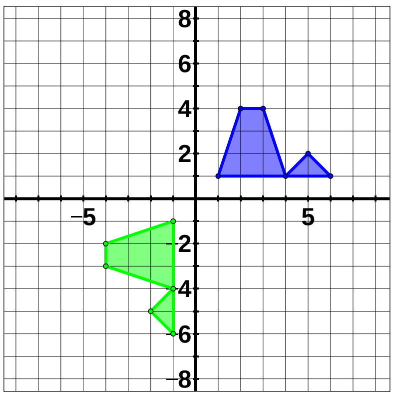


Can you tell that the image is neither congruent nor similar to the pre-image? Yes, the two are not the same shape. One point seems to be out of place. The point $(-3, -2)$ on the image should be at $(-4, -2)$. That would make them congruent. In reality the image and pre-image are not really a true pre-image and image because no transformation will take one to the other. These shapes are neither congruent nor similar.

Lesson 2.5

Determine the specific series of transformations that took the pre-image (darker in blue) to the image (lighter in green). Be sure to give the specific vector, rotation angle, line of reflection and/or scale factor. Then determine if the pre-image and image are similar or congruent.

1.



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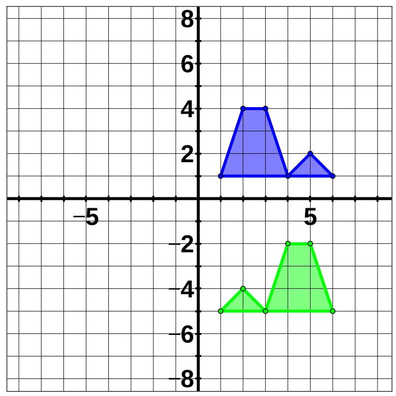

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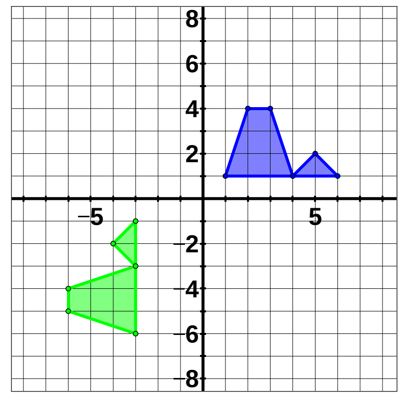

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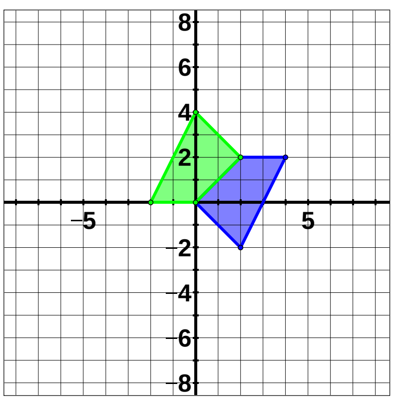

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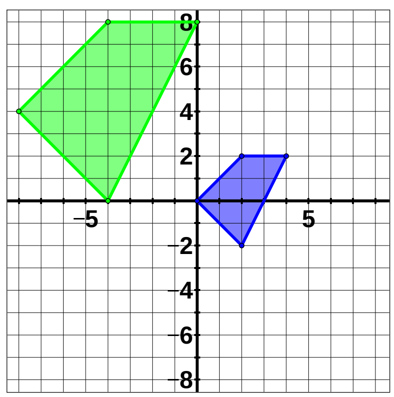

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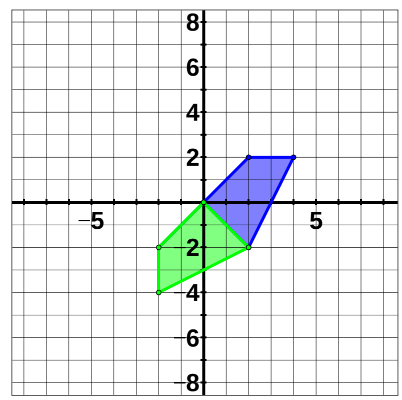

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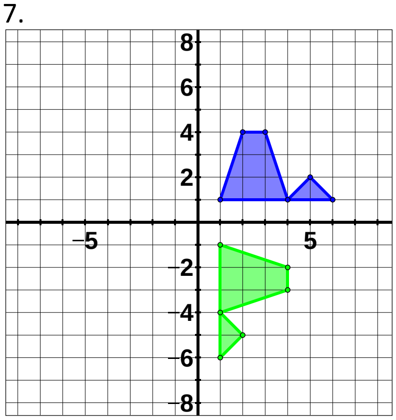

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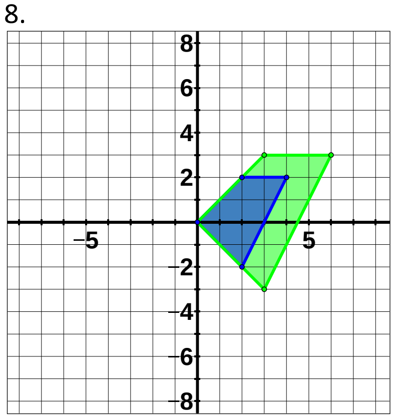

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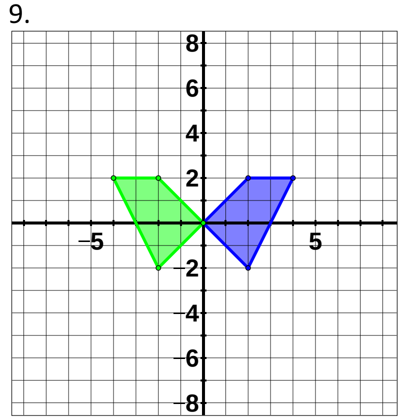

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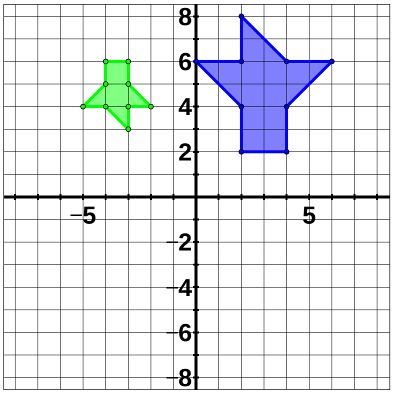

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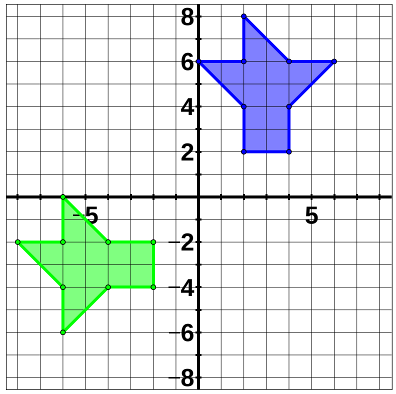

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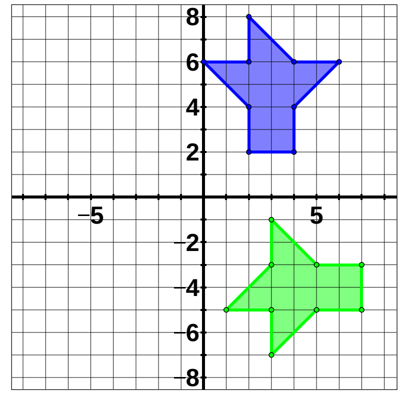

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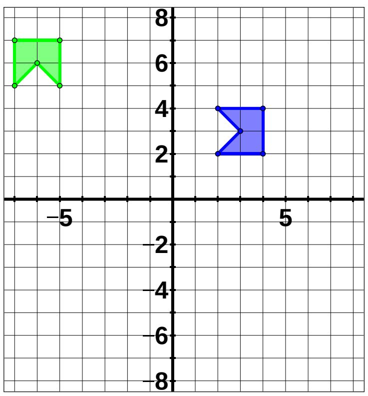

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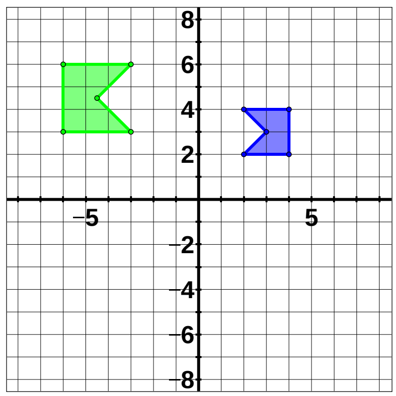

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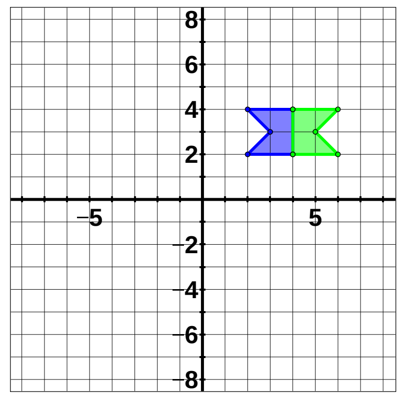

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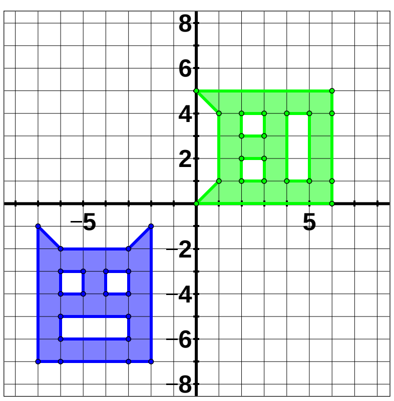

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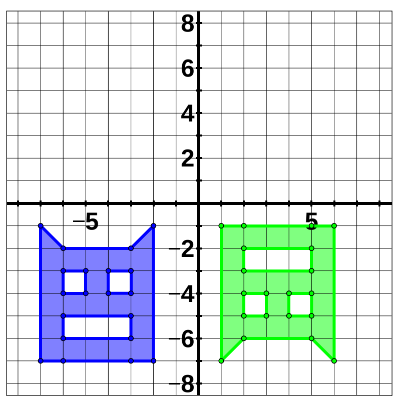

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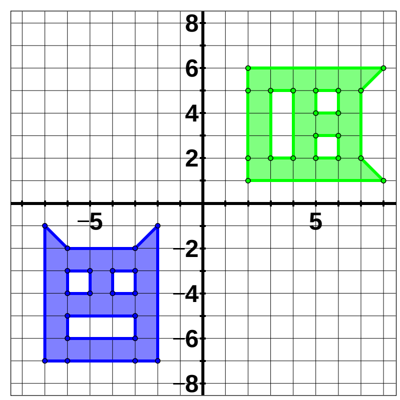


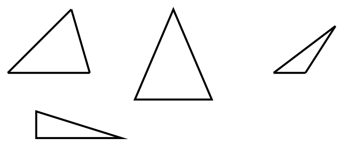
2.6 Sum of the Angles of a Triangle

There are other ways to determine if shapes are similar besides using transformations. We will explore one of these ways using triangles. To do so, we'll need to learn a few things about triangles in general before moving into the similar triangles.

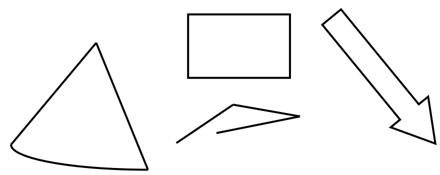
Reviewing the Triangle Definition

Let's look at some examples of triangles and non-triangles to find a definition of a triangle.

Triangles



Non-Triangles

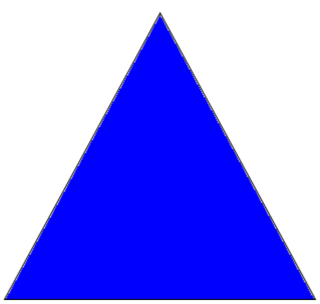


Take a moment to write down a definition of a triangle based on what you see above.

A **triangle** is a closed, three-sided figure where each side is a line segment. Closed means that the figure has no gaps in its perimeter, and requiring line segments to be the sides means that there won't be a curve of any kind.

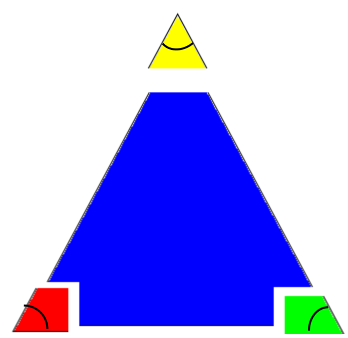
The Sum of the Angles of a Triangle

Now we can look at the angles of a triangle. Take a look at the following images:

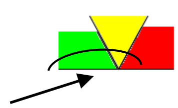


The shape to the left is a basic triangle. Now examine the shape to the right and see that we have cut off each of the angles. (As a side note, we should recall that each of these angles are called acute angles because they are less than 90°.)

Now look what happens if we stick all those angles together. What total angle measurement do we get?



What is the overall angle measurement?



The three angles put together made a straight angle, which measure 180°. As an experiment on your own, cut out a triangle from a sheet of paper. It can be any kind of triangle such as right scalene, obtuse isosceles, acute scalene, etc. Now rip off each angle (corner) of the triangle and put the vertices of the angles together to form one big angle. Does it make a straight angle as well?

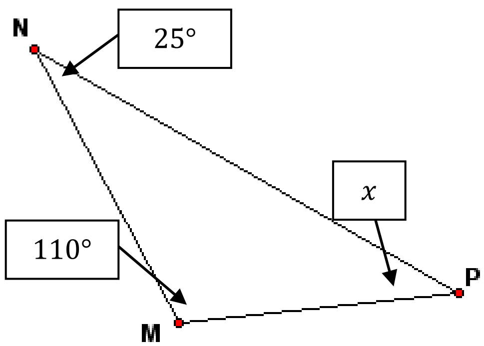
In fact, the angles of a triangle will always add up to 180°. This is sometimes called the Triangle Sum Theorem and allows us to find the missing angle measurements in a triangle.

Finding Missing Angles

Since the angles of a triangle (let's call them $\angle 1$, $\angle 2$, and $\angle 3$) add up to $180°$, we can write an equation to represent this.

$$m\angle 1 + m\angle 2 + m\angle 3 = 180°$$

Remember that the little m in front of the angle symbol means "the measure of". This means if we know the measurement of two of the angles of a triangle, we can solve for the third. Let's look at an example.



In order to solve for x, we substitute what we know into the equation. In this case we know that $m\angle M = 110°$, $m\angle N = 25°$, and we're missing the measure of $\angle P$. Since they all add up to $180°$, we get the following equation:

$$x + 110° + 25° = 180°$$

$$x + 135° = 180°$$

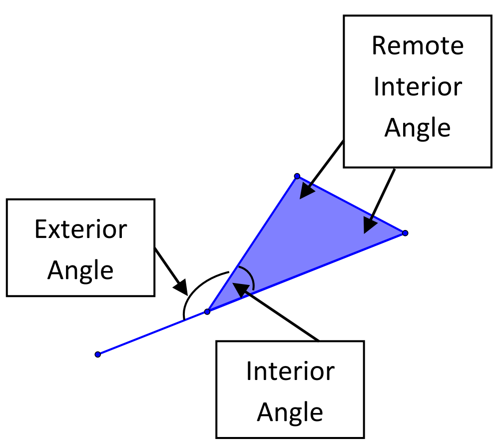
$$x + 135° - 135° = 180° - 135°$$

$$x = 45°$$

So the missing angle measurement must be $45°$ in order for all the angles to add up to $180°$.

Exterior Angles

If we look at any angle in a triangle (we call it the **interior angle**) and extend one of the line segments beyond the angle, we get the **exterior angle**. The **remote interior** angles are the other two angles in the triangle. Looking at the picture below, what do we know about the exterior angle?



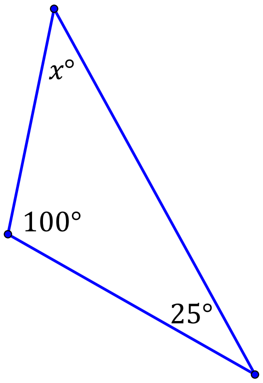
The exterior angle measurement plus the interior angle measurement together make a straight angle. That means that they should add up to $180°$. So if the interior angle measure is $35°$, then the exterior angle must measure $145°$.

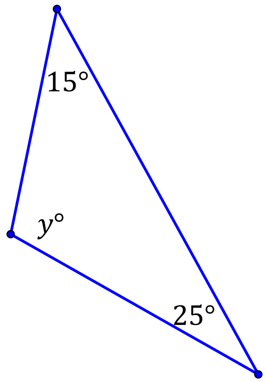
Also note that the exterior angles must be equal to the sum of the remote interior angles. Why? Since all the interior angle and remote interior angles add up to $180°$ and the interior angle and the exterior angle add up to $180°$.

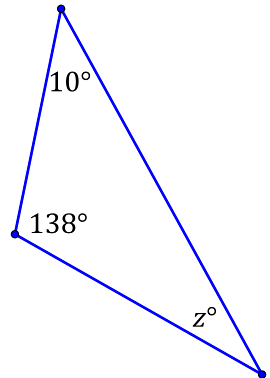
This means we can find the angle measurement of an exterior angle given the interior angle and vice versa.

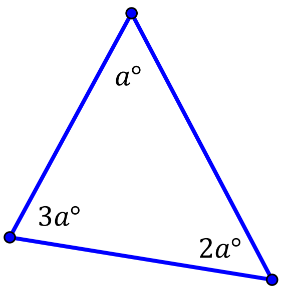
Lesson 2.6

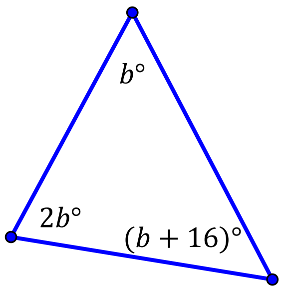
Solve for the variable.

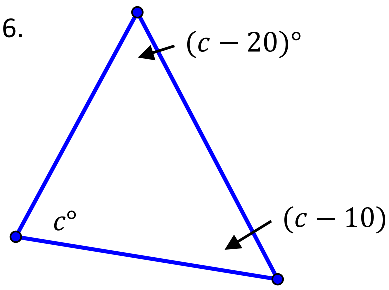
1. 

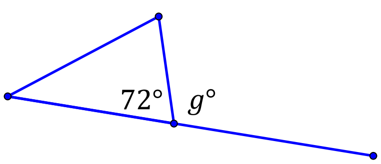
2. 

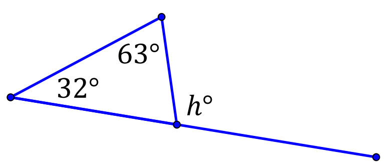
3. 

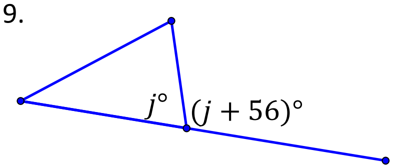
4. 

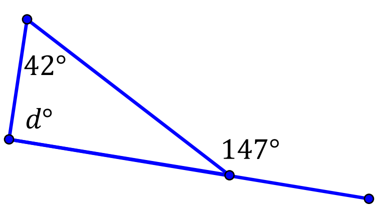
5. 

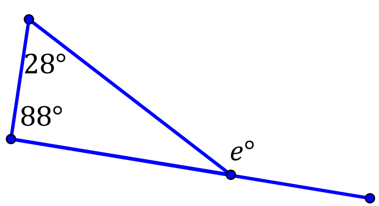
6. 

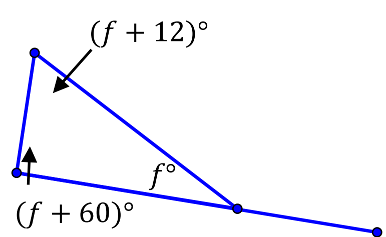
7. 

8. 

9. 

10. 

11. 

12. 

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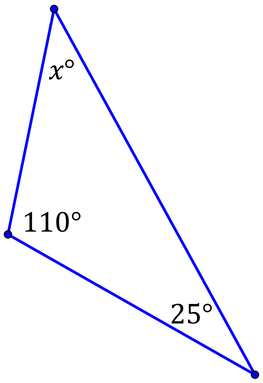

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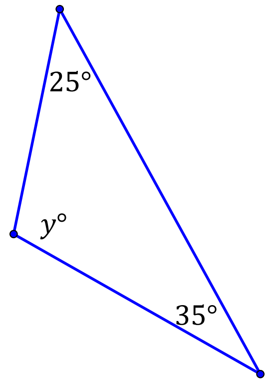

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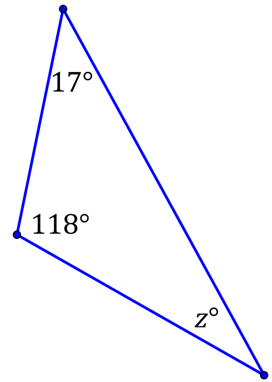

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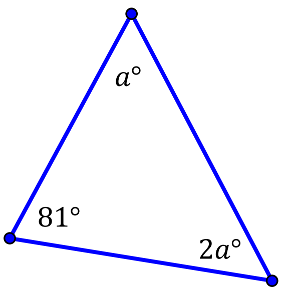

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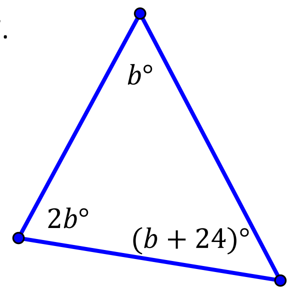

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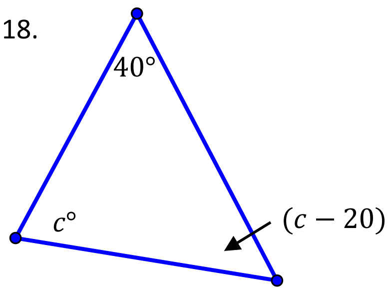

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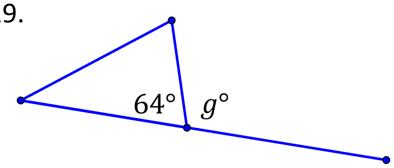

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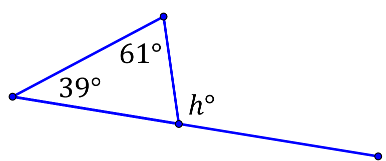

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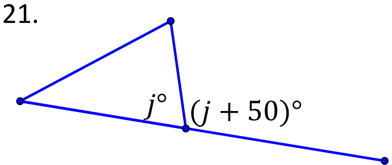

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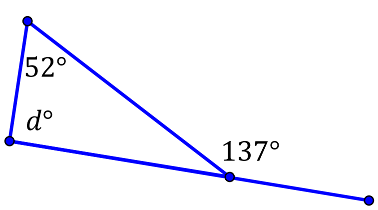

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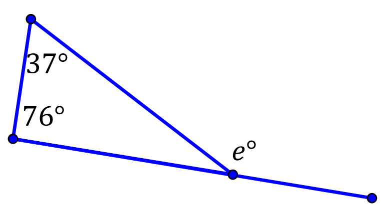

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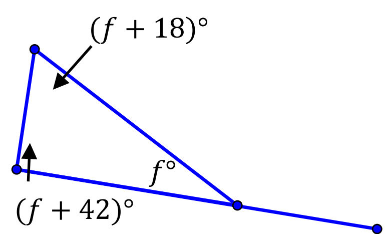


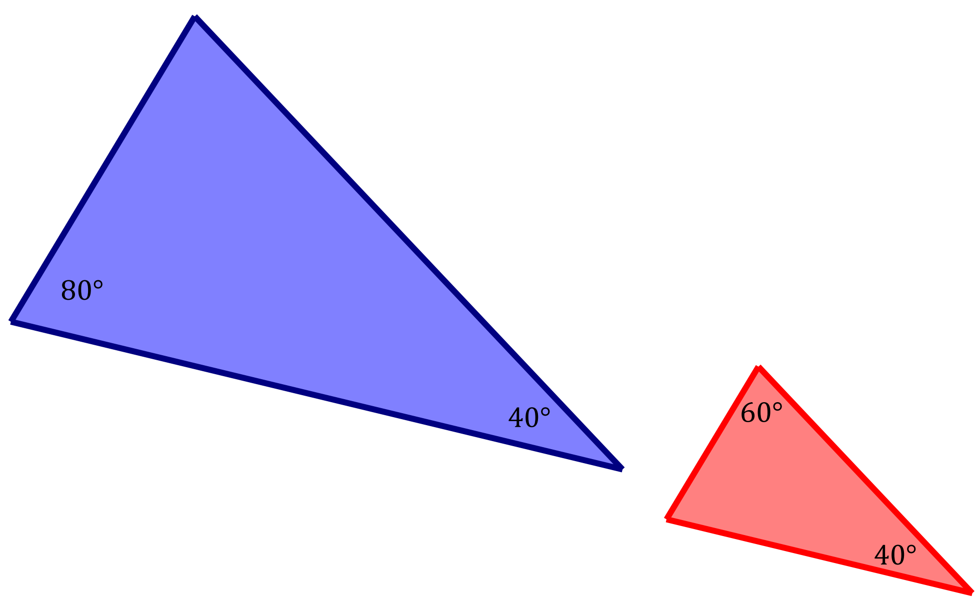
2.7 Similar Triangles

We have already talked about similar triangles when discussing proportions noting that similar triangles are triangles where the side lengths are proportional. Another way to think about similar triangles is that they are triangles of the same shape, but not necessarily the same size. Congruent triangles would have both the same size and shape.

We have also talked about similar triangles in the context of transformations in that triangles would be similar if there exists a set of translations, rotations, reflections, and/or dilations that take one shape to the other. Now we will examine similarity through examining their angles.

Angle-Angle Criterion for Similarity

We know that the angles of a triangle must add up to $180°$. This means that if a triangle has two angle measurements of $40°$ and $80°$, then the third angle must be $60°$. Now if a second triangle has two angle measurements of $40°$ and $60°$, we know the third angle must be $80°$. This means the two triangles are the same shape, but not necessarily the same size. Alternately we may think of one as a dilation of the other. Either way we know that the triangles are similar. We call this the angle-angle criterion for similarity.



Lesson 2.7

Decide if the following triangles are similar and explain why using the angle-angle criterion.

1. Triangle 1 – $m\angle 1 = 45°, m\angle 2 = 45°$
   Triangle 2 – $m\angle 1 = 45°, m\angle 2 = 90°$

2. Triangle 1 – $m\angle 1 = 75°, m\angle 2 = 65°$
   Triangle 2 – $m\angle 1 = 65°, m\angle 2 = 140°$

3. Triangle 1 – $m\angle 1 = 50°, m\angle 2 = 30°$
   Triangle 2 – $m\angle 1 = 30°, m\angle 2 = 100°$

4. Triangle 1 – $m\angle 1 = 80°, m\angle 2 = 20°$
   Triangle 2 – $m\angle 1 = 80°, m\angle 2 = 80°$

5. Triangle 1 – $m\angle 1 = 60°, m\angle 2 = 20°$
   Triangle 2 – $m\angle 1 = 40°, m\angle 2 = 100°$

6. Triangle 1 – $m\angle 1 = 45°, m\angle 2 = 30°$
   Triangle 2 – $m\angle 1 = 30°, m\angle 2 = 100°$

7. Triangle 1 – $m\angle 1 = 40°, m\angle 2 = 30°$
   Triangle 2 – $m\angle 1 = 90°, m\angle 2 = 30°$

8. Triangle 1 – $m\angle 1 = 80°, m\angle 2 = 40°$
   Triangle 2 – $m\angle 1 = 40°, m\angle 2 = 60°$

9. Triangle 1 – $m\angle 1 = 35°, m\angle 2 = 95°$
   Triangle 2 – $m\angle 1 = 35°, m\angle 2 = 40°$

10. Triangle 1 – $m\angle 1 = 105°, m\angle 2 = 35°$
    Triangle 2 – $m\angle 1 = 40°, m\angle 2 = 105°$

11. Triangle 1 – $m\angle 1 = 35°, m\angle 2 = 95°$
    Triangle 2 – $m\angle 1 = 35°, m\angle 2 = 50°$

12. Triangle 1 – $m\angle 1 = 50°, m\angle 2 = 50°$
    Triangle 2 – $m\angle 1 = 50°, m\angle 2 = 90°$

13. Triangle 1 – $m\angle 1 = 25°, m\angle 2 = 115°$
    Triangle 2 – $m\angle 1 = 25°, m\angle 2 = 40°$

14. Triangle 1 – $m\angle 1 = 70°, m\angle 2 = 45°$
    Triangle 2 – $m\angle 1 = 45°, m\angle 2 = 65°$

15. Triangle 1 – $m\angle 1 = 5°, m\angle 2 = 15°$
    Triangle 2 – $m\angle 1 = 120°, m\angle 2 = 15°$

16. Triangle 1 – $m\angle 1 = 90°, m\angle 2 = 20°$
    Triangle 2 – $m\angle 1 = 90°, m\angle 2 = 80°$

17. Triangle 1 – $m\angle 1 = 5°, m\angle 2 = 15°$
    Triangle 2 – $m\angle 1 = 160°, m\angle 2 = 15°$

18. Triangle 1 – $m\angle 1 = 80°, m\angle 2 = 30°$
    Triangle 2 – $m\angle 1 = 70°, m\angle 2 = 30°$

19. Triangle 1 – $m\angle 1 = 45°, m\angle 2 = 55°$
    Triangle 2 – $m\angle 1 = 55°, m\angle 2 = 90°$

20. Triangle 1 – $m\angle 1 = 72°, m\angle 2 = 23°$
    Triangle 2 – $m\angle 1 = 85°, m\angle 2 = 23°$

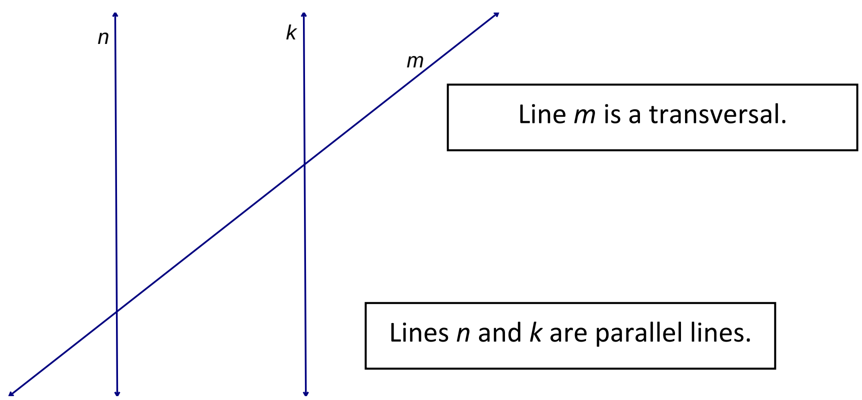
2.8 Parallel Lines Cut By A Transversal

When we extended the sides of triangles to find interior and exterior angle measurements, we created a situation that was very similar to parallel lines cut by a transversal. In this section we will explore what types of angles are created when we have a transversal cutting parallel lines.

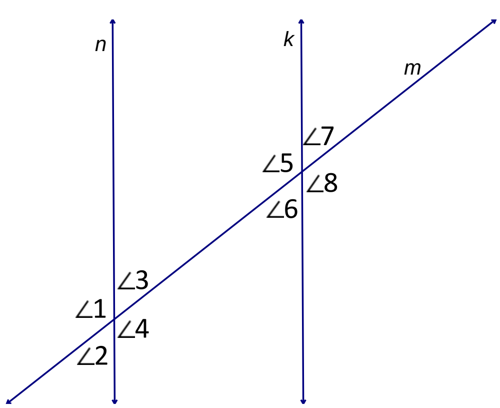
Reviewing Vocabulary

Parallel lines in two dimensions are lines that never cross or intersect. This means that the lines have the same orientation. If one line is going straight up and down, the parallel line will also be going straight up and down. For our purposes we will only look at parallel lines that are not overlapping. In other words, one line will not be sitting right on top of the other. Instead, our parallel lines will be more like railroad tracks.

A **transversal** is a line that intersects one or more parallel lines. This means that the transversal will have a different orientation from the parallel lines. So if the parallel lines are going straight up and down, then the transversal might be going left or right. The transversal could also be at some other angle (think of a positive 2 slope for example).



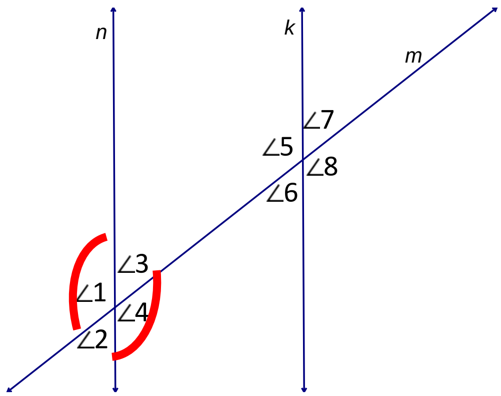
Notice that in this picture there are eight angles that are created. We typically name those angles using the numbers 1 through 8. It could look something like this.



You should notice right away that several of these angles look like they have the same angle measurement. In fact it looks like the four acute angles have equal measurement and the four obtuse angles have equal measurement. In fact this is the case, but let's examine why this is true and classify the different types of angles we find here.

Vertical Angles

Note that $\angle 1$ and $\angle 3$ must add up to $180°$ because they sit on a line. They are like the exterior and interior angles of a triangle adding up to $180°$. However, the same argument can be made for $\angle 4$ and $\angle 3$. They must also add up to $180°$. Therefore we know that $\angle 4$ and $\angle 1$ must have the same measurement. In other words we know that $\angle 4 \cong \angle 1$ (pronounced "angle 4 is congruent to angle 1") or $m\angle 4 = m\angle 1$ (pronounced "the measure of angle 4 is equal to the measure of angle 1").



We call this type of congruent angle **vertical angles**. One way to remember vertical angles is to remember that they sit in a "V". Where are the other vertical angles in our picture?

The vertical angles come in pairs. A second pair of vertical angles is $\angle 2$ and $\angle 3$. A third pair of vertical angles is $\angle 5$ and $\angle 8$. The fourth pair of vertical angles is $\angle 6$ and $\angle 7$.

We can now ask questions such as: what is $m\angle 3$ if $m\angle 2 = 40°$? Since we know that they are vertical angles, they must be congruent. Therefore the answer is $m\angle 2 = 40°$.

Corresponding Angles

Now imagine taking the angles formed by line *n* and line *m* and sliding them up so that they overlap the angles formed by line *k* and line *m*. Now which angles do we know are congruent? In other words, angles 1 through 4 will be sitting right on top of angles 5 through 8. Which ones line up? These angles are called **corresponding angles** and are congruent.

For starters, you should notice that $\angle 1$ sits on top of $\angle 5$. So $\angle 1$ and $\angle 5$ are a pair of corresponding angles. A second pair would be $\angle 2$ and $\angle 6$. A third pair would be $\angle 3$ and $\angle 7$. The final pair would be $\angle 4$ and $\angle 8$. This means that if $m\angle 2 = 40°$ then $m\angle 6 = 40°$ must be true since they are corresponding angles. One way to remember corresponding angles is to think of the angles that are on the same corner. Cor-ner and Cor-responding angles.

Alternate Interior Angles

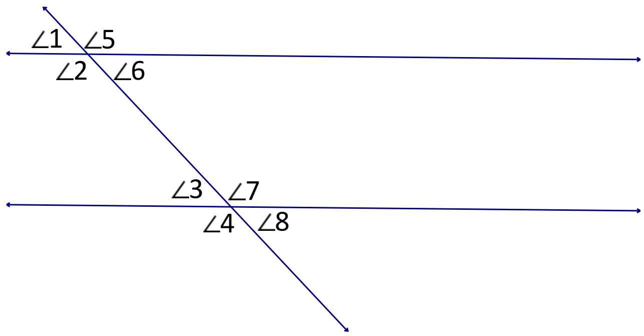
Using the same picture above, look at $\angle 3$ and $\angle 6$. These are called **alternate interior angles** and are also congruent. They are called alternate interior angles because they alternate which side of the transversal they are on ($\angle 3$ is on top of the transversal in this case and $\angle 6$ is on bottom) and because they are inside the parallel lines (hence the word "interior"). Alternate interior angles are also congruent. There are two pairs of this type of angle in our picture: $\angle 3$ and $\angle 6$ and then $\angle 4$ and $\angle 5$.

Alternate Exterior Angles

In a similar fashion, if the angles lie on alternate sides of the transversal and are outside the parallel lines, the angles are called **alternate exterior angles**. Alternate exterior angles are congruent as well. In our picture there are two pairs of alternate exterior angles which are $\angle 1$ and $\angle 8$ and then $\angle 2$ and $\angle 7$.

Lesson 2.8

Use the following picture to answer the questions.



1. Name a pair of vertical angles.

2. Name a pair of corresponding angles.

3. Name a pair of alternate interior angles.

4. Name a pair of alternate exterior angles.

5. If $m\angle 2 = 110°$, what is $m\angle 5$?

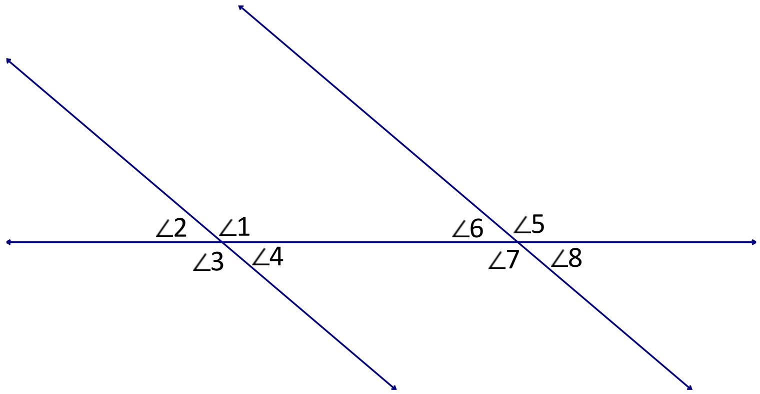
6. If $m\angle 2 = 110°$, what is $m\angle 4$?

7. If $m\angle 2 = 110°$, what is $m\angle 7$?

8. If $m\angle 1 = 70°$, what is $m\angle 8$?

9. If $m\angle 1 = 70°$, what is $m\angle 7$?

10. If $m\angle 2 = 140°$, what are the measures of all the other angles?



11. Name all the pairs of vertical angles.

12. Name all the pairs of corresponding angles.

13. Name all the pairs of alternate interior angles.

14. Name all the pairs of alternate exterior angles.

15. If $m\angle 2 = 40°$, what is $m\angle 8$?

16. If $m\angle 2 = 40°$, what is $m\angle 4$?

17. If $m\angle 1 = 140°$, what is $m\angle 7$?

18. If $m\angle 1 = 140°$, what is $m\angle 5$?

19. If $m\angle 1 = 140°$, what is $m\angle 6$?

20. If $m\angle 2 = 35°$, what are the measures of all the other angles?

Review Unit 2: Congruence and Similarity

You may not use a calculator.

Unit 2 Goals

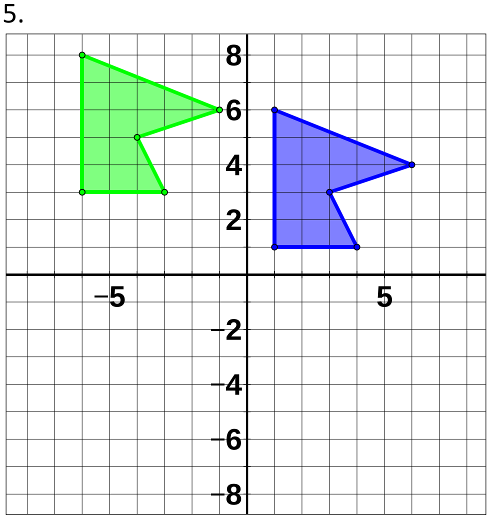
- *Verify experimentally the properties of rotations, reflections, and translations: a) Lines are taken to lines, and line segments to line segments of the same length; b) angles are taken to angles of the same measure; c) parallel lines are taken to parallel lines. (8.G.1)*
- *Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them. (8.G.2)*
- *Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates. (8.G.3)*
- *Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them. (8.G.4)*
- *Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. (8.G.5)*

Answers the following questions.

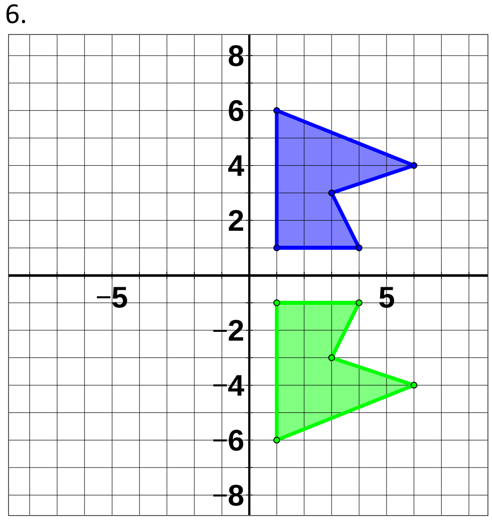
1. If you translate the pre-image A, will the image A' be congruent or only similar?
2. If you dilate the pre-image B, will the image B' be congruent or only similar?
3. If you rotate the pre-image C, will the image C' be congruent or only similar?
4. If you reflect the pre-image D, will the image D' be congruent or only similar?

Name the specific transformation shown in each picture as a translation, rotation, reflection, or dilation. Then determine if the pre-image (darker in blue) and the image (lighter in green) are similar or congruent.

5.

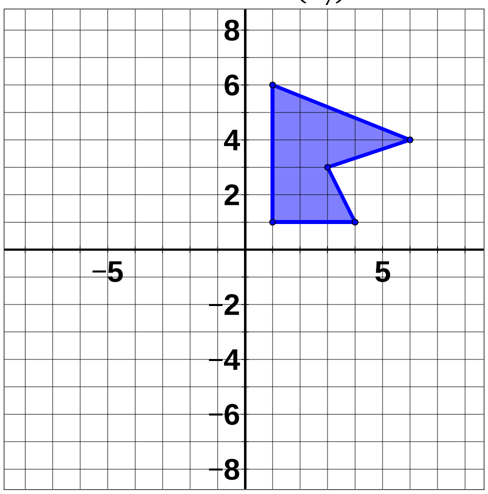


6.

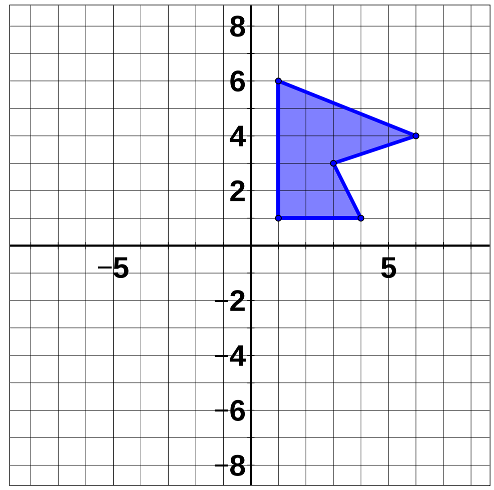


Apply the given transformation or series of transformations to the given pre-image.

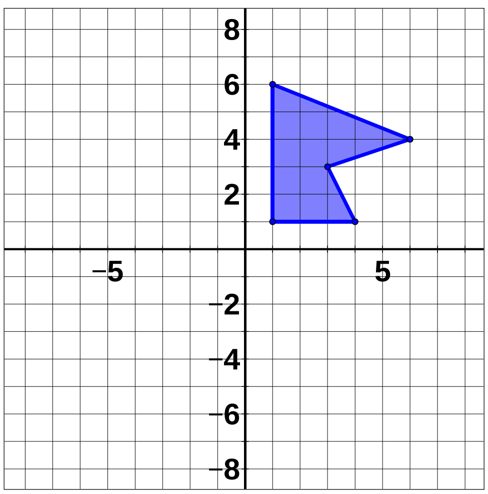
7. Translation by vector $\binom{-6}{-7}$



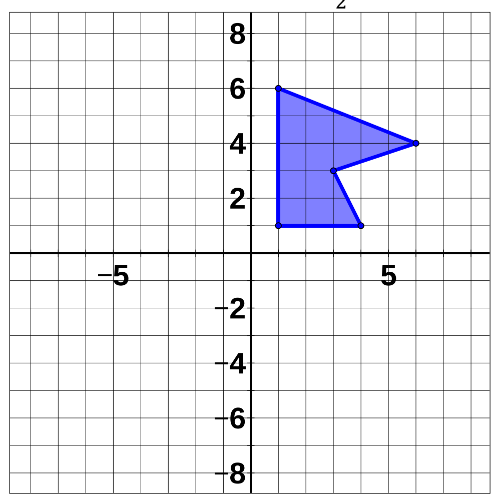
8. Rotation by $180°$



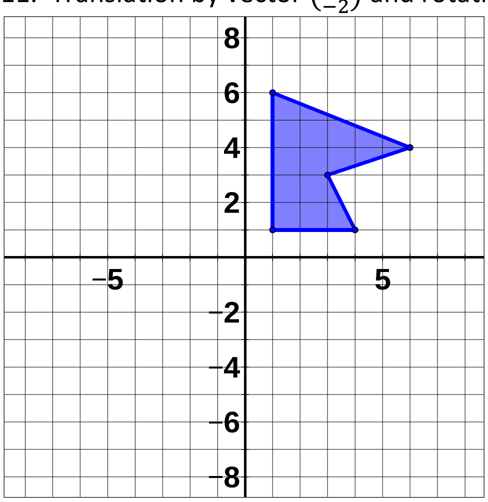
9. Reflection across the x-axis



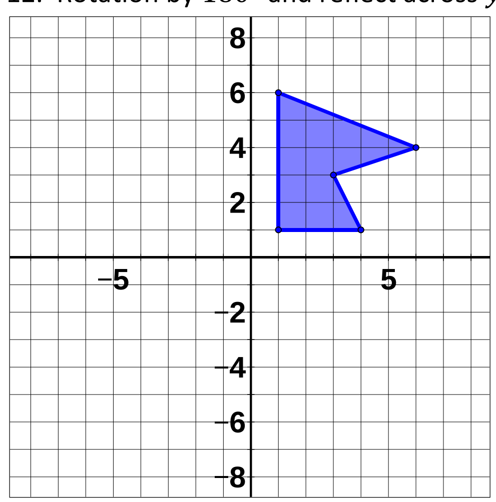
10. Dilation by scale factor $\frac{1}{2}$



11. Translation by vector $\binom{2}{-2}$ and rotation by $90°$

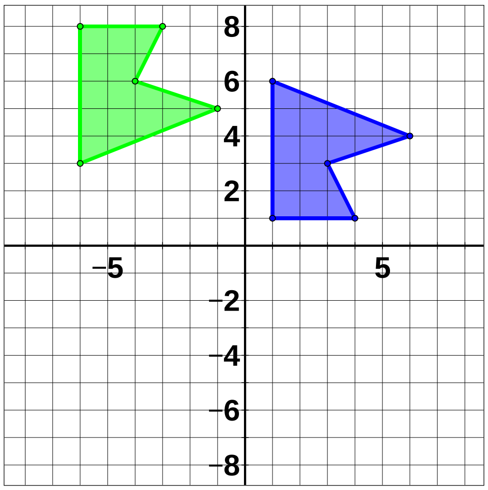


12. Rotation by $180°$ and reflect across y-axis

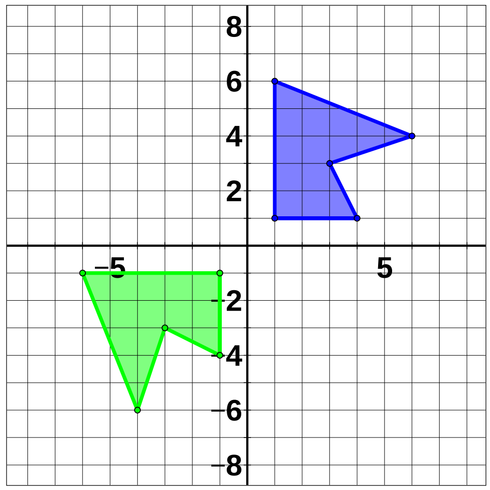


Identify a specific series of transformations that would take the pre-image (darker in blue) to the image (lighter in green). Then tell whether the pre-image and image are congruent or similar.

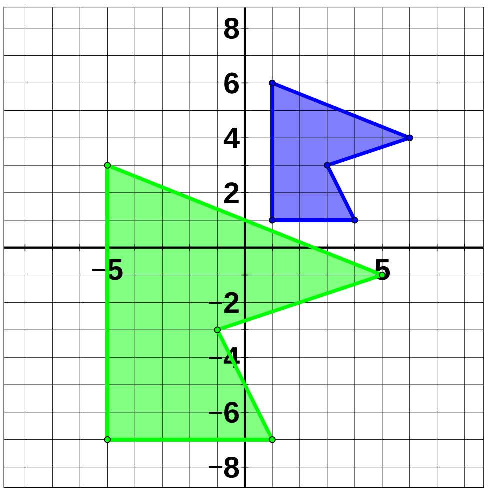
13.



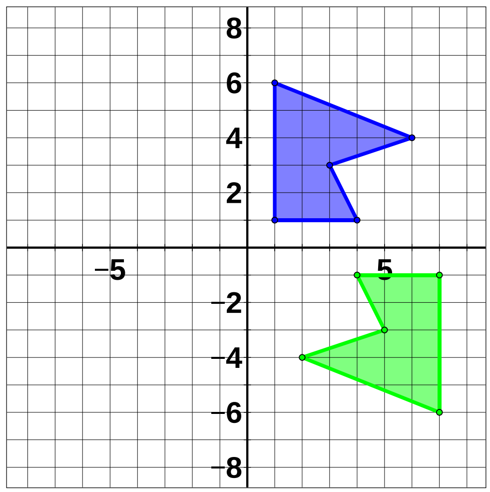
14.



15.

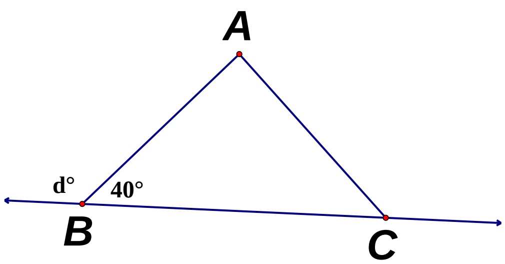


16.

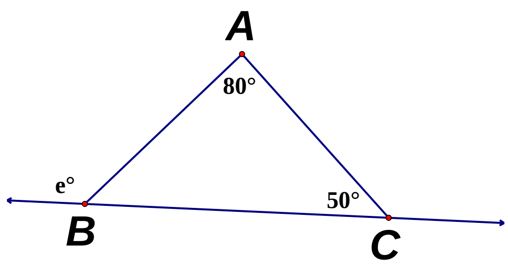


Find the angle measure of each missing angle.

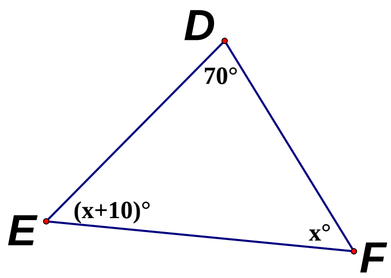
17.



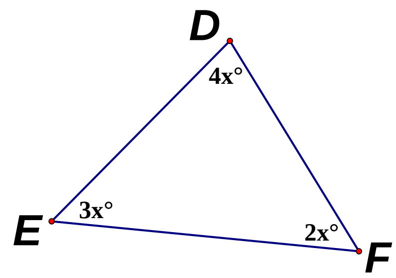
18.



19.

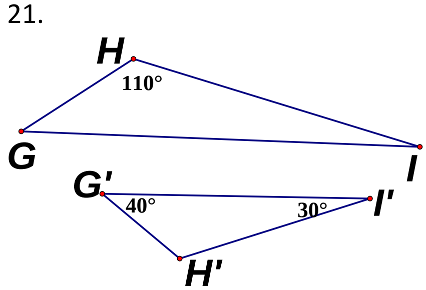


20.

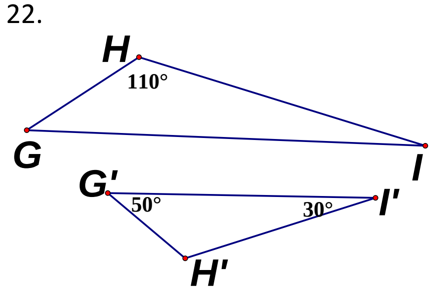


Determine if the following triangles are similar or not and explain why or why not.

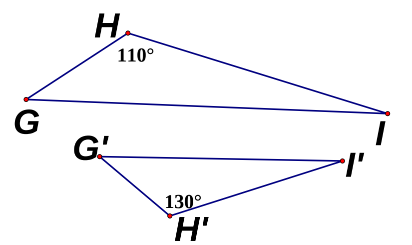
21.



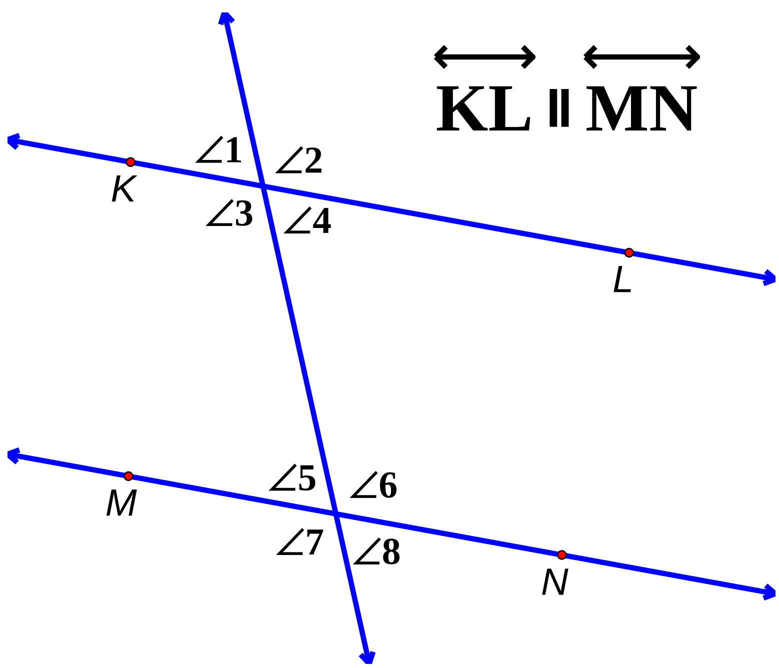
22.



23.



Use the picture to answer the following questions.



24. What type of angles are $\angle 1$ and $\angle 4$?

25. What type of angles are $\angle 1$ and $\angle 5$?

26. What type of angles are $\angle 1$ and $\angle 8$?

27. What type of angles are $\angle 3$ and $\angle 6$?

28. List all the angles congruent to $\angle 4$.

29. If $m\angle 1 = 75°$, what is $m\angle 8$?

30. If $m\angle 2 = 135°$, what is $m\angle 4$?

1^{st} Quarter Review

No calculator necessary. Please do not use a calculator.

1. What is $(5^7)(5^9)$ as a number to a single power?

2. Evaluate $\frac{6^5}{6^7}$.

3. Evaluate 3^{-3}.

4. Evaluate $(2^{-2})^2$.

5. Evaluate $\frac{4^6}{4^{-4}} \times 4^{-8}$

6. Determine appropriate exponent to make the equation true: $\left(8^{\square}\right)^6 = (8^3)^{-4}$.

7. Determine appropriate exponent to make the equation true: $\frac{5^{\square}}{5^2} = (5^2)(5^{-9})$.

8. The population of New Zealand is approximately 4.4×10^6. If the population of China is approximately 2.2×10^9, about how many times bigger is the population of China than New Zealand?

9. Write 5,300,000,000,000 in scientific notation.

10. Write 0.000 000 083 in scientific notation.

11. Write 4.6×10^{-3} in standard form.

12. Write 4.35×10^7 in standard form.

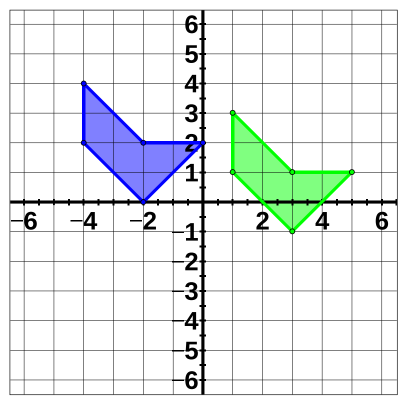
13. What is the best unit of measurement for the amount an ice shelf lowered in a year that lost approximately 3.9×10^1 meters of height, millimeters, meters, or kilometers?

14. What is $(2.1 \times 10^{-7})(4 \times 10^{15})$ in scientific notation?

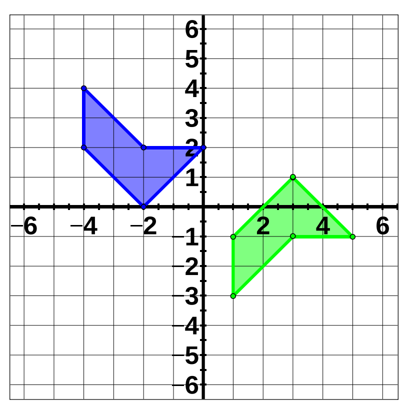
15. What is $\frac{6.8 \times 10^9}{2{,}000{,}000}$ in scientific notation?

16. What is $2.12 \times 10^6 + 9 \times 10^5$ in sci. not.?

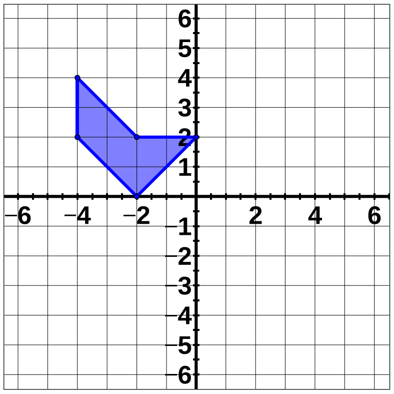
17. In the picture below the pre-image is darker (in blue) and the image is lighter (in green). What ***single*** transformation does this show? Give the translation vector, rotation angle, line of reflection, or dilation factor.



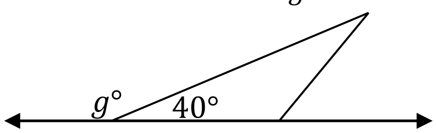
18. In the picture above the pre-image is darker (in blue) and the image is lighter (in green). What ***two*** transformations does this show? Give the translation vector, rotation angle, line of reflection, or dilation factor.



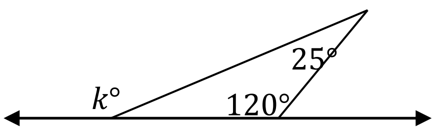
19. Rotate the given pre-image by $90°$ and then dilate by a scale factor of $\frac{1}{2}$.



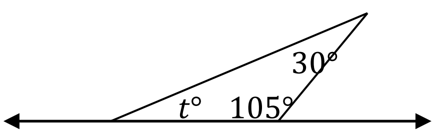
20. What is the value of g?

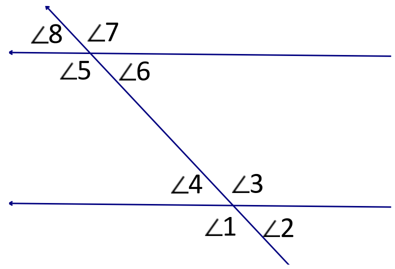


21. What is the value of k?



22. What is the value of t?





23. Name a pair of vertical angles and a pair of corresponding angles.

24. Name a pair of alternate interior angles and a pair of alternate exterior angles.

25. If $m\angle 1 = 150°$, find the measures of all other angles.

Unit 3: Functions

Pre-Test Unit 3: Functions

No calculator necessary. Please do not use a calculator.

Determine if each of the following is a true function based on the equation or table. Explain how you know. *(5 pts; 2 pts for answer only)*

1. $y = x^2$

x	-2	-1	0	1	2
y	4	1	0	1	4

2. $y^2 = x$

x	0	1	4	9	16
y	0	±1	±2	±3	±4

Evaluate the given function using the given value as inputs. *(5 pts; 3 pts for computation error only)*

3. $a = 3b - 2$
   $b = -2$

4. $g = h^2 - 3$
   $h = 3$

Answer the following question in complete sentences. *(5 pts; partial credit at teacher discretion)*

5. Determine if the following describes a true function or not. Explain why or why not.

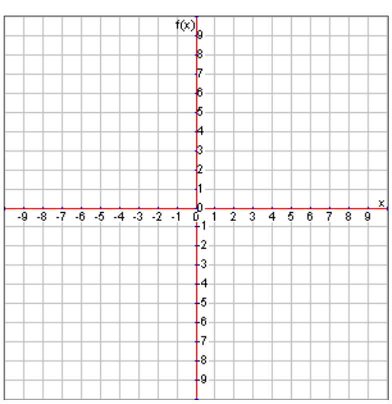
   Input: Age of an author, Output: Amount of money earned

6. Give an example of a function in words and explain what the input and output are.

Graph the following functions by filling out the x/y ***chart using the inputs (*** x ***values) that you think are appropriate.*** *(5 pts; 1 pt for appropriate* x *values, 2 pts for correct table, 2 pts for graph following table)*

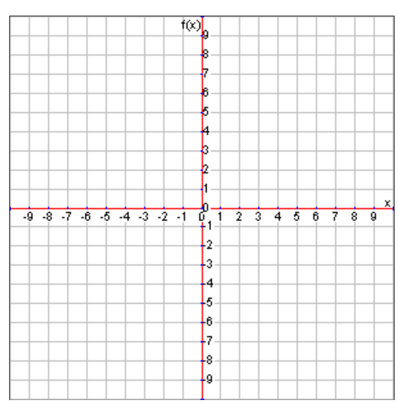
7. $y = x^2 + 2$

x					
y					



8. $y = \frac{1}{4}x + 1$

x					
y					



Determine whether the following functions are linear or non-linear and explain how you know. *(5 pts; 2 pts for correct answer only)*

9. $y = x^3$

10. $y = \frac{3}{4}x + 1$

Answer the following question about different types of functions. *(5 pts; 3 pts for correct example with incorrect or missing explanation)*

11. Give an example of a linear function in equation form and explain how you know it is linear.

Identify the rate of change, initial value, independent variable, and dependent variable. Then describe what the rate of change and initial value mean in the context of each situation. Finally, give the equation of the line if necessary. *(6 pts; 1 pt for each)*

12. The amount of money in dollars a farmer gets paid (p) to leave land fallow for a season based on the acres of land he or she owns (a) is modeled by the following function: $p = 300a - 50$.

Rate of Change:___________

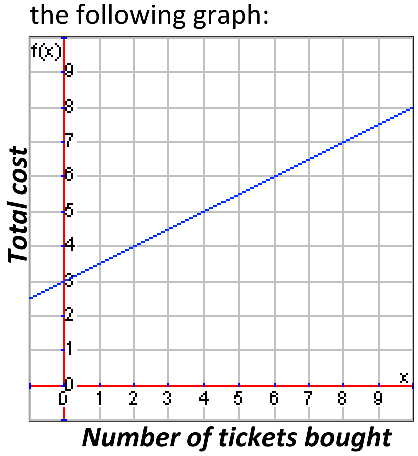
Initial Value:______________

Independent Variable:_____

Dependent Variable:______

Contextual Description of Rate of Change	Contextual Description of Initial Value

13. The function relating the cost in dollars of entering a carnival (c) to how many tickets you buy (t) is shown by the following graph:



Rate of Change:___________

Initial Value:______________

Independent Variable:_____

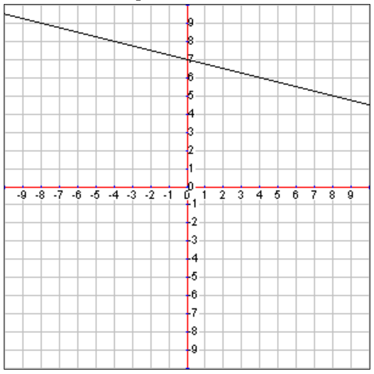
Dependent Variable:_______

EQ of Line:_________________

Contextual Description of Rate of Change	Contextual Description of Initial Value

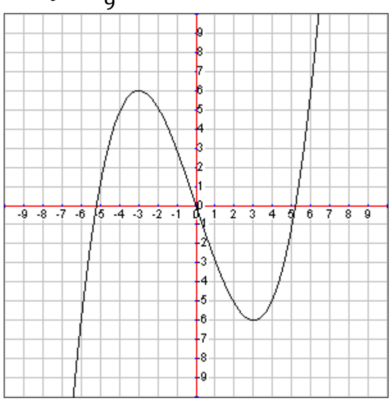
Tell whether the following linear function is increasing, decreasing, or constant. *(3 pts; no partial credit)*

14. $y = -\frac{1}{4}x + 7$

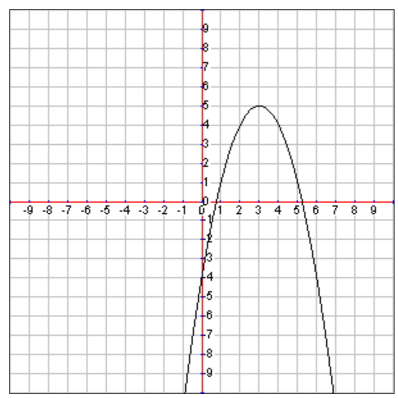


For the following functions tell where they are increasing and where they are decreasing. Then give the max or min of the function. *(5 pts; 3 pts for increasing/decreasing, 2 pts for max/min)*

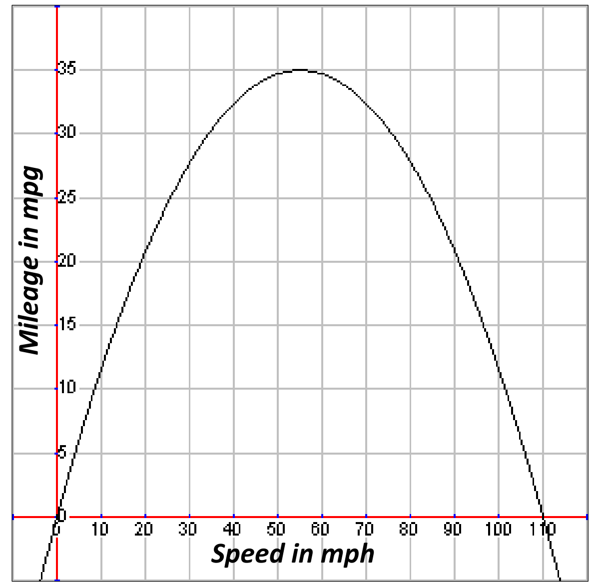
15. $y = \frac{1}{9}x^3 - 3x$



16. $y = -(x-3)^2 + 5$



Use the following graph showing a function modeling the miles per gallon (m) a car gets in terms of its speed (s) to answer the questions. *(5 pts; partial credit at teacher discretion)*

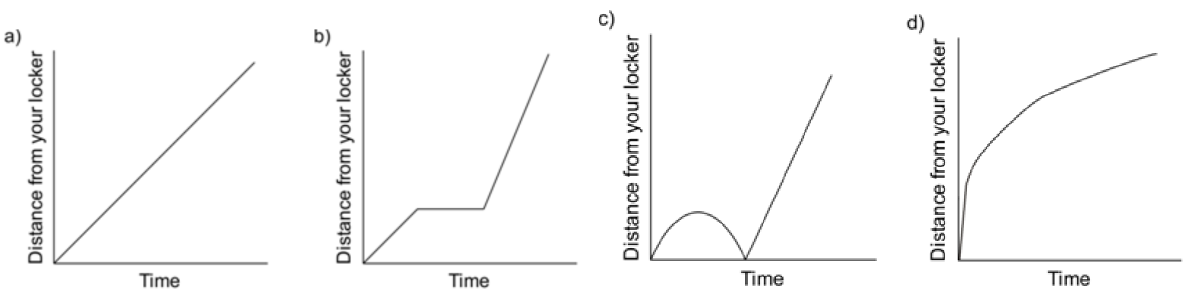


17. What appears to be the best mileage this car will get and at what speed does it occur?

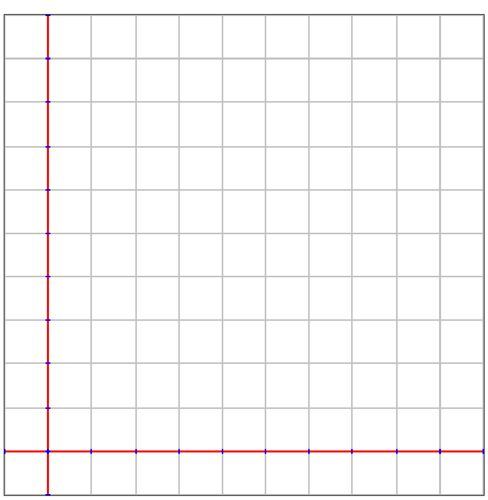
18. What are all the possible speeds this car can drive at?

Determine which graph matches the story and explain why. *(5 pts; 2 pts for correct answer with no explanation)*

19. I started to walk to class, but I realized I had forgotten my notebook, so I went back to my locker and then I went quickly at a constant rate to class.



Sketch a graph modeling a function for the following situations. *(5 pts; partial credit at teacher discretion)*



20. A dog is sleeping when he hears the cat "meow" in the next room. He quickly runs to the next room where he slowly walks around looking for the cat. When he doesn't find the cat, he sits down and goes back to sleep. Sketch a graph of a function of the dog's speed in terms of time.

3.1 Intro to Functions

Functions govern many interactions in our society today. Whether buying a cup of coffee at the local coffee shop or playing a video game, we are using a function in some fashion.

Definition of a Function

A **function** is a rule or relationship between two quantities, often referred to as the input and output, such that for every input there is exactly one output. If we input a specific value into the function, we get a specific output as an answer. We won't get the possibility of two answers or else it wouldn't be a function. The most common example of a function is an equation such as:

$$y = 2x + 3$$

In this case, x is the input and y is the output. If we substitute a value for x, say $x = 3$, then we will get an answer for y, namely $y = 9$, as the output. Notice that every time we input $x = 3$ you will get the output of $y = 9$. Since we always get only a single output for any value we input, this is a true function.

An example of an equation that is not a function would be $y^2 = x$. Notice that if we input $x = 4$, then $y = 2$ could be the output or $y = -2$ could be the output. Therefore this is not a true function unless we make the function $y = \sqrt{x}$ where we take only the principal (or positive) square root.

Parts of a Function

While input and output make sense for a function, we sometimes refer to them as the independent and dependent variables. The input is the **independent variable** because we can plug in anything we want for the input as long as it is legal. The output is called the **dependent variable** because it depends on what you input into the function. So in the previous example of $y = 2x + 3$, the x is the independent variable and the y is the dependent variable. In other words, the value of y depends on what value we substitute in for x.

Let's put this in context and say that a carnival charges \$3 to get in plus \$2 per ride ticket that you have to buy at the entrance. Your total cost would be modeled by the function $y = 2x + 3$ where x is the number of tickets that you buy and y is the total cost. In this context, the input (or independent variable) is how many tickets you buy and the output (or dependent variable) is how much money you pay.

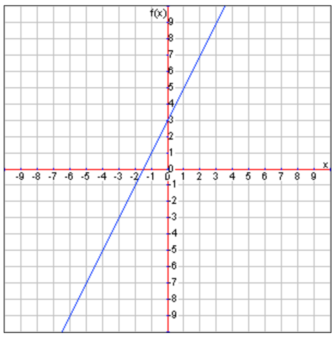
The set of all possible inputs is called the **domain** of a function. For $y = 2x + 3$ the domain is all real numbers. Any number we want could be input into the function as x. In the context of the carnival as described above, it would only make sense to think about the domain $x \geq 0$ since we wouldn't buy negative tickets.

Some functions have limited domains or ranges. For example, $y = \sqrt{x + 5}$ has a domain of $x \geq -5$ because we can't input a negative value since we can't take the square root of a negative. Another example of a limited domain is the function $y = \frac{100}{x}$ which has a domain of any number except 0 (since we can't divide by 0). We might write this out by saying the domain is $x \neq 0$.

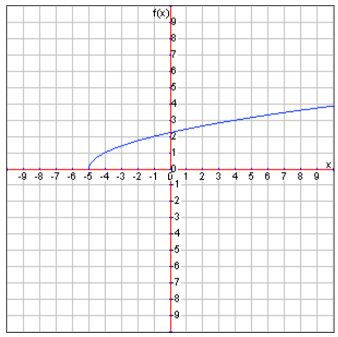
Similar to the domain, all of the outputs we could possibly get are called the **range**. More precisely, the range is the set of all possible outputs for a function. The range for the function $y = 2x + 3$ is all real numbers, or any y. This means the output, or y value, could be any number. The function $y = x^2 - 2$ has a range of $y \geq -2$ because no matter value we plug in for x, we will always get a number greater than or equal to -2 for y.

It is often easier to see the domain and range from the graph of a function. Let's consider a few examples.

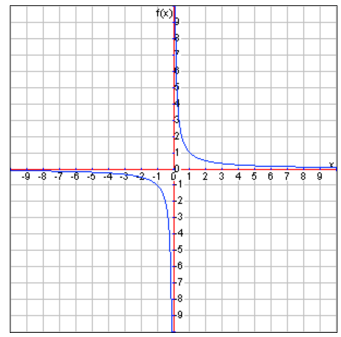
$y = 2x + 3$



$y = \sqrt{x + 5}$



$y = \frac{1}{x}$



$y = 2x + 3$	$y = \sqrt{x + 5}$	$y = \frac{1}{x}$
Domain: any x because the graph continues forever both left and right Range: any y because the graph continues forever both up and down	Domain: $x \geq -5$ because the graph continues forever to the right starting at -5 Range: $y \geq 0$ because the graph continues forever up from the height of 0	Domain: $x \neq 0$ because the graph continues forever to the left and right but never touches $x = 0$ Range: $y \neq 0$ because the graph continues forever up and down but never has a height of $y = 0$

Evaluating Functions

Evaluating a function means to figure out what the output is when given a specific input. Let's look at the following function that shows the total cost at an amusement park, c, depending on the number of tickets bought, t, to ride the rides.

$$c = 2t + 3$$

We can evaluate this function for $t = 5$ by substituting into the equation as follows:

$$c = 2(5) + 3 = 10 + 3 = 13$$

So our output is $c = 13$ meaning it would cost \$13 to buy 5 tickets. Let's evaluate the same function for 10 tickets.

$$c = 2(10) + 3 = 20 + 3 = 23$$

Our output this time is $c = 23$.

Lesson 3.1

Determine if each of the following is a true function based on the equation or table. Explain how you know.

1. $y = x^2$

x	-2	-1	0	1	2
y	4	1	0	1	4

2. $x^2 + y^2 = 25$

x	-4	-3	0	3	4
y	± 3	± 4	± 5	± 4	± 3

3. $y = \sqrt{x+5}$

x	-5	-4	-1	4	11
y	0	1	2	3	4

4. $y = \frac{1}{4}x^3 - 5x$

x	-4	-2	0	2	4
y	4	8	0	-8	-4

5. $x^2 + y^2 = 100$

x	-8	-6	0	6	8
y	± 6	± 8	± 10	± 8	± 6

6. $y = 2x + 5$

x	-2	-1	0	1	2
y	1	3	5	7	9

7. $x = y^2$

x	0	1	4	9	25
y	0	± 1	± 2	± 3	± 5

8. $y = 2x^2 - 1$

x	-2	-1	0	1	2
y	7	1	-1	1	7

9. $x^2 - y^2 = 9$

x	-5	-3	3	5
y	± 4	0	0	± 4

10. $\frac{x^2}{4} + \frac{y^2}{4} = 1$

x	-2	0	2
y	0	± 4	0

11. $y = -\frac{1}{2}x$

x	-4	-2	0	2	4
y	2	1	0	-1	-2

12. $y = \frac{2}{x}$

x	-2	-1	1	2
y	-1	-2	2	1

13. Explain how to determine whether or not an equation models a function.

14. Explain how to determine whether or not a table models a function.

Determine if the following descriptions of relationships represent true functions. Explain why they do or why they do not.

15. Input: Time elapsed, Output: Distance run around the track.

16. Input: Store's name, Output: Number of letters in the name.

17. Input: Person's age, Output: Yearly salary.

18. Input: Amount of food eaten, Output: A dog's weight.

19. Input: Person's name/identity, Output: That person's birthday.

20. Input: Person's age, Output: Height.

21. Input: Name of a food, Output: Classification of that food (such as meat, dairy, grain, fruit, vegetable).

22. Input: Time studied for test, Output: Test score.

Evaluate the given function using the given input.

23. $a = 4b$, $b = -2$

24. $y = \frac{1}{2}x + 3$, $x = 10$

25. $g = h^2 + 2$, $h = -3$

26. $c = t + 75$, $t = 100$

27. $a = -4b$, $b = -3$

28. $y = \frac{1}{4}x - 3$, $x = -8$

29. $g = h^2 - 6$, $h = -2$

30. $c = t - 85$, $t = 40$

31. $a = 2b + 5$, $b = 5$

32. $y = -\frac{1}{3}x + 2$, $x = 9$

33. $g = 2h^2 + 1$, $h = 3$

34. $c = t + 55$, $t = 70$

3.2 Graphing Functions

We can graph functions to get a visual representation of the relationship between two quantities. We graph these on the coordinate plane, but we may not always use the variables x and y.

Input/Output Charts

To graph a function, we first need an input/output chart. This chart will give us the points we need to graph on the coordinate plane. Let's start by graphing the following function:

$$c = 2t + 3$$

For this function, notice that t is the input, or independent variable, and c is the output, or dependent variable. We'll now make a simple chart with five spaces to fill out as follows:

Input t					
Output c					

Sometimes input values will be given to us to plug in, other times we will need to make up our own. In this case, we are not given values for the input. Therefore, it is suggested to use the values from -2 to 2 to make sure we get a good picture of the function. It is not always necessary to find five points, but the more points we have, the better graph we will get.

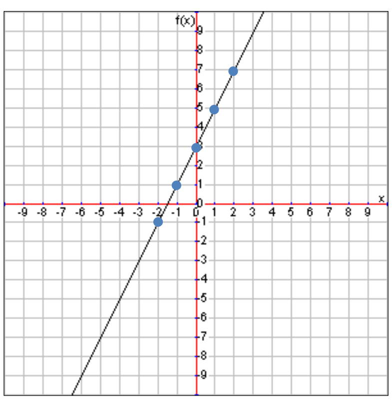
Input t	-2	-1	0	1	2
Output c					

Now we evaluate the function for each input. Let's look at the work for $t = -2$.

$$c = 2(-2) + 3 = -4 + 3 = -1$$

Input t	-2	-1	0	1	2
Output c	-1	1	3	5	7

Following this same process for each input value, we get the table at the right.



Now we plot each associated input and output as a point like this: $(input, output)$ or (t, c). Since t is the dependent variable, that takes the place of x and c will take the place of y. Graph each point and connect the points as we can see at the left.

In most cases the input/output chart only uses the variables as labels instead of "input" and "output". That would look like this:

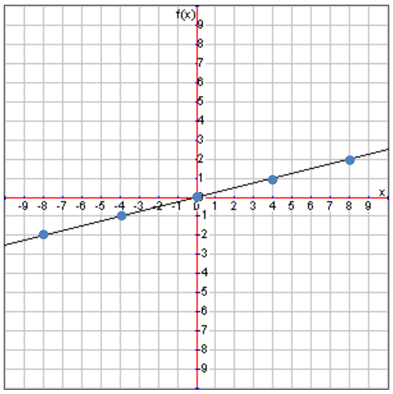
t	-2	-1	0	1	2
c	-1	1	3	5	7

Notice we plotted five points: $(-2,-1), (-1,1), (0,3), (1,5)$, and $(2,7)$.

Deciding on Appropriate Inputs

Since we are graphing by hand, it is easiest if we work with integer inputs and outputs. Some functions have fractions, decimals, or even square roots that make our choice of inputs critical. For example, consider the function $a = \frac{1}{4}b$.

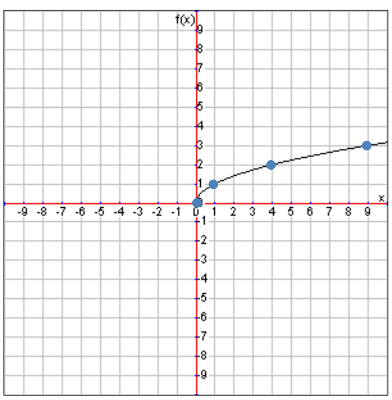
b	-8	-4	0	4	8
a	-2	-1	0	1	2



If we choose $b = 1$ as an input, we'll have to graph the point $\left(1, \frac{1}{4}\right)$ which is not convenient by hand. Therefore, we should choose values for b that we can multiply by $\frac{1}{4}$ and get integer outputs for a. Perhaps the input/output chart given to the left would work best yielding the graph below the chart.

Notice that choosing multiples of 4 for our inputs allowed integer outputs.

x	0	1	4	9
y	0	1	2	3



Let's look at the square root function $y = \sqrt{x}$. Since we can't take the square root of negative numbers, we won't use any negative inputs. Also, since the number 2 does not have an integer square root, we'll skip ahead to the inputs that do have integer square roots. Therefore we might use an input/output chart like the one to the left yielding the graph below the chart.

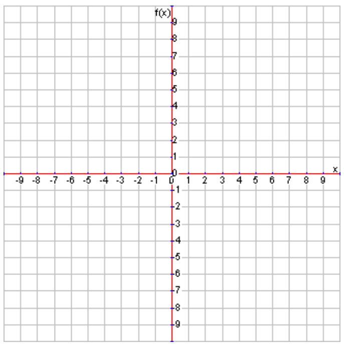
Notice that we only used four inputs instead of five since the next input yielding an integer output would be $x = 16$ and that x value would be off the coordinate plane we have which only goes up to $x = 10$.

Lesson 3.2

Graph the following functions by filling out the x/y chart using the given inputs (x values).

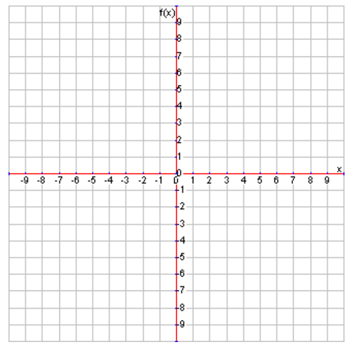
1. $y = x^2 - 7$

x	-2	-1	0	1	2
y					



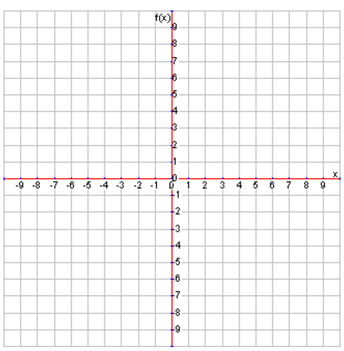
2. $y = \frac{1}{3}x + 2$

x	-6	-3	0	3	6
y					



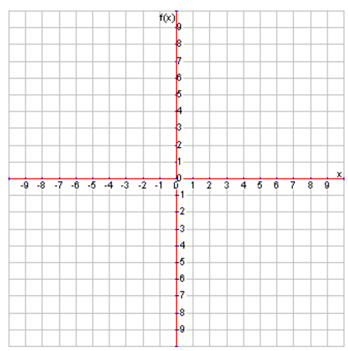
3. $y = \sqrt{x + 9}$

x	-9	-8	-5	0	7
y					



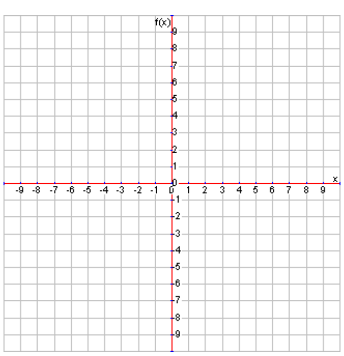
4. $y = 2x^2 - 1$

x	-2	-1	0	1	2
y					



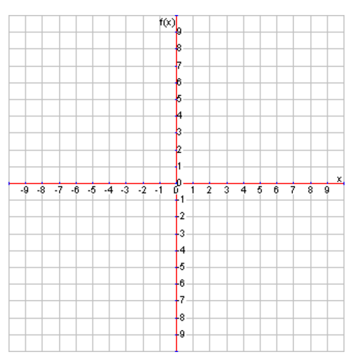
5. $y = \frac{1}{5}x + 2$

x	-10	-5	0	5	10
y					



6. $y = \sqrt{x + 7}$

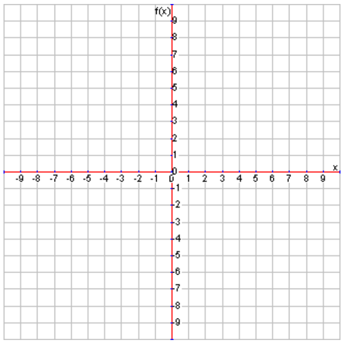
x	-7	-6	-3	2	9
y					



Graph the following functions by filling out the x/y chart using the inputs (x values) that you think are appropriate.

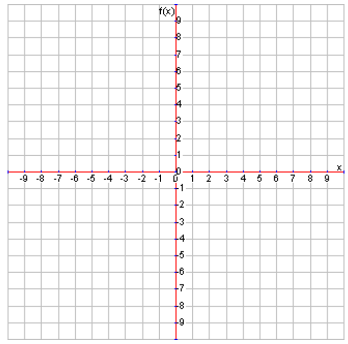
7. $y = 2x^2 - 8$

x					
y					



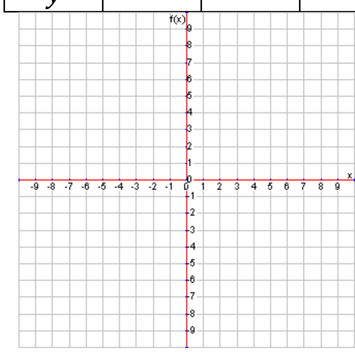
8. $y = \frac{2}{3}x - 4$

x					
y					



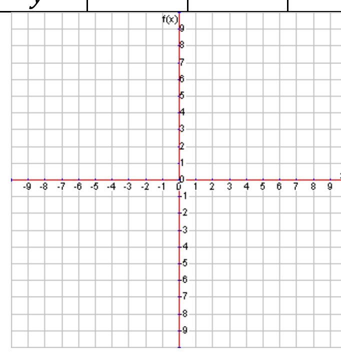
9. $y = \frac{1}{2}x - 4$

x					
y					



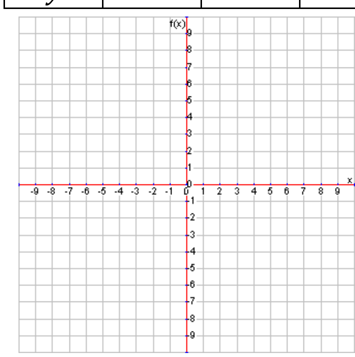
10. $y = \sqrt{x + 8}$

x					
y					



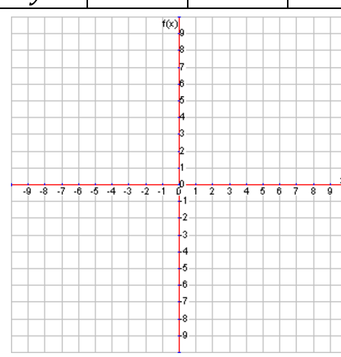
11. $y = -\sqrt{x + 7}$

x					
y					



12. $y = -x^2 + 4$

x					
y					



13. Explain why it would be beneficial to choose the inputs $-2, -1, 0, 1$, and 2 for the function $y = x^2 + 1$.

14. Explain why it would be beneficial to choose the inputs $-8, -4, 0, 4$, and 8 for the function $y = \frac{3}{4}x - 2$.

15. Explain why it would be beneficial to choose the inputs $-9, -8, -5, 0$, and 7 for the function $y = \sqrt{x + 9}$.

16. Explain how you would choose 5 different inputs for the function $y = \sqrt{x + 6}$. Explain why you feel these are the best input values for this function.

17. For problems 2, 5, 8, 9, describe a pattern in the change in the y values for each function.

18. For problems 2, 5, 8, 9, explain similarities and differences in the structure of the equations.

19. For problems 2, 5, 8, 9, explain similarities and differences in the graph of each function.

3.3 Linear and Non-Linear Functions

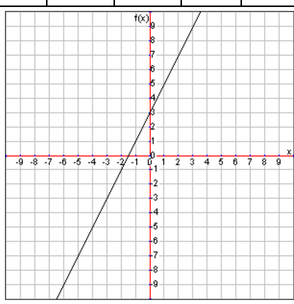
Most functions of cost are linear functions. If every cup of coffee costs \$1.60 then two cups would cost \$3.20, three cups would cost \$4.80, and so forth. Let's explore what makes a function linear.

Definition of a Linear Function

Here are some examples of linear functions in equation, table, graph, and story form.

$c = 2t + 3$

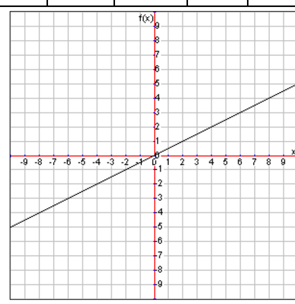
t	-2	-1	0	1	2
c	-1	1	3	5	7



A carnival charges \$3 to enter and \$2 per ticket for riding rides.

$s = \frac{1}{2}w$

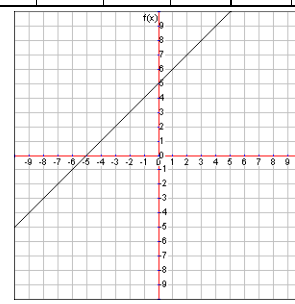
w	-2	-1	0	1	2
s	-1	$-.5$	0	$.5$	1



A company can make one sweater for every two pounds of wool it has.

$h = d + 5$

d	-2	-1	0	1	2
h	3	4	5	6	7



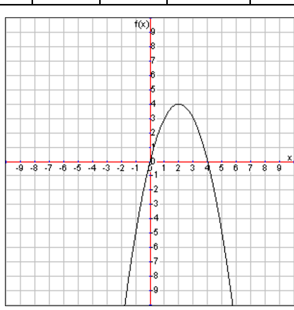
A plant today has a height of 5 mm and grows one mm each day.

What do you think makes these functions linear? Do you notice anything they have in common?

Here are some examples of **non-linear functions**. Can you define linear functions now?

$h = -(t-2)^2 + 4$

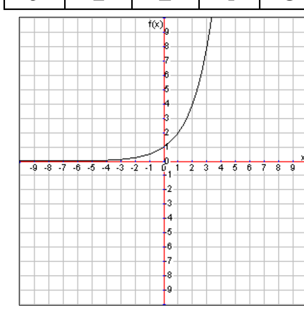
t	0	1	2	3	4
h	0	-5	-12	-21	-32



A rocket's height after t seconds is given by the above equation.

$c = 2^d$

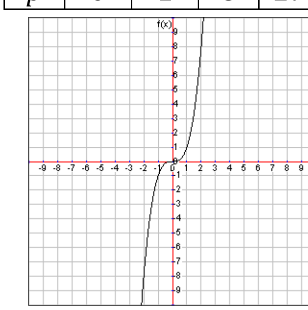
d	0	1	2	3	4
c	1	2	4	8	16



The number of cicadas double each day after the initial day one hatches.

$p = f^3$

f	0	1	2	3	4
p	0	1	8	27	64



Your dad pays you a number of pennies equal to how many flies you've killed cubed.

A **linear function** is a function that makes a straight line when graphed. Thus **non-linear** functions are any functions that are not linear. Graphing may be the quickest way to tell if a function is linear or non-linear, but we can also determine if a function is linear from its input/output table or equation.

This can be seen in the input/output table in that there is a constant difference in the dependent variable values. In other words, when the independent variable increases by one, the dependent variable will always increase or decrease the same amount. Let's look at our three linear tables again.

t	-2	-1	0	1	2
c	-1	1	3	5	7

$+2 \quad +2 \quad +2 \quad +2$

w	-2	-1	0	1	2
s	-1	$-.5$	0	$.5$	1

$+.5 \quad +.5 \quad +.5 \quad +.5$

d	-2	-1	0	1	2
h	3	4	5	6	7

$+1 \quad +1 \quad +1 \quad +1$

Notice that the first table has a constant difference of two, meaning that as t increases by one, c increases by two every time. The second table has a constant difference of half, and the third table has a constant difference of one.

Alternately, notice that all three equations have the independent variable only to the first power. Meaning there is no exponent showing with the variable. That also means they are linear functions.

$c = 2t + 3$ $\qquad$ $s = \frac{1}{2}w$ $\qquad$ $h = d + 5$

Check the non-linear functions given on the previous page and see that they are not a straight line when graphed, have no constant difference, and have exponents in their equation.

In general, anything of the form $\boldsymbol{y = mx + b}$ is considered a **linear function** where m is the slope and b is called the y-intercept. Notice that the x has an unwritten exponent of one with it.

There are times when a linear function is not given in slope-intercept form. (That's what we call $y = mx + b$.) Sometimes a linear function is given in standard form which is $Ax + By = C$. However, since the exponent on the x variable is still a one, we can get it in slope intercept form. For example, consider the following:

$$2x + 3y = 6$$

$$2x + 3y - 2x = 6 - 2x$$

$$3y = -2x + 6$$

$$\frac{3y}{3} = \frac{-2x + 6}{3}$$

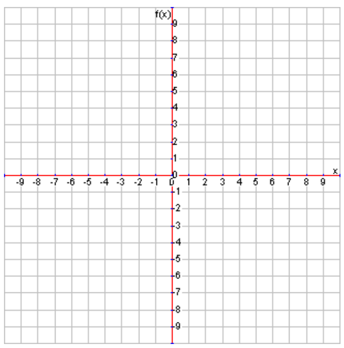
$$y = -\frac{2}{3}x + 2$$

Lesson 3.3

Determine whether the following functions are linear or non-linear and explain how you know. Blank x/y charts and coordinate planes have been given to graph the functions if that helps you.

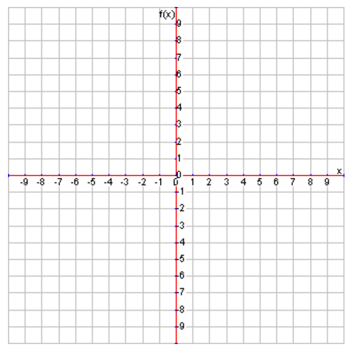
1. $y = x^2 - 2x$

x					
y					



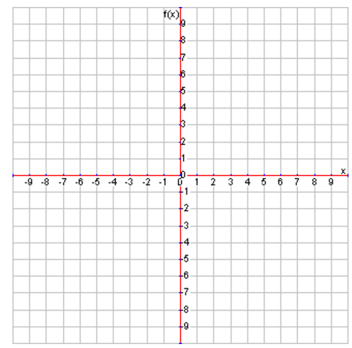
2. $y = \frac{1}{3}x - 2$

x					
y					



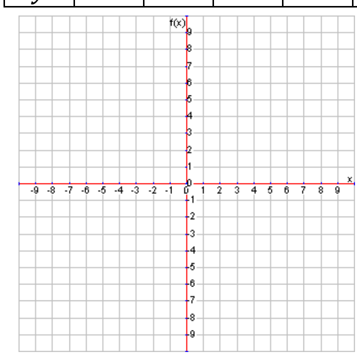
3. $y = -2x + 2$

x					
y					



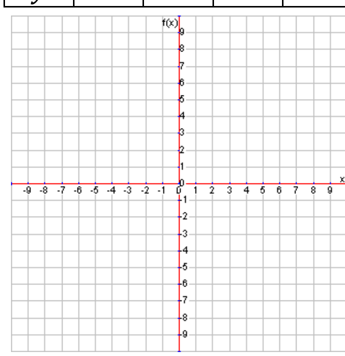
4. $y = x^2 + 3$

x					
y					



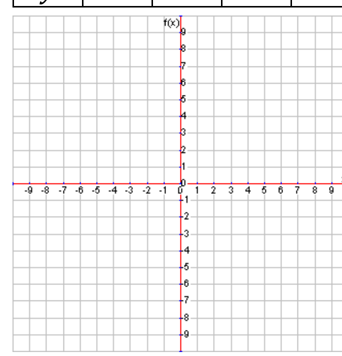
5. $5x + 3y = 0$

x					
y					



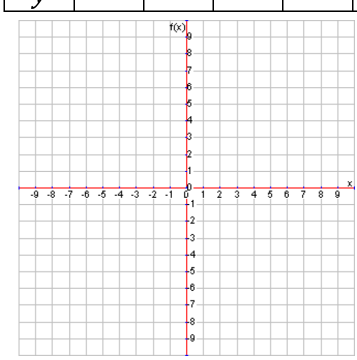
6. $y - 4x = -5$

x					
y					



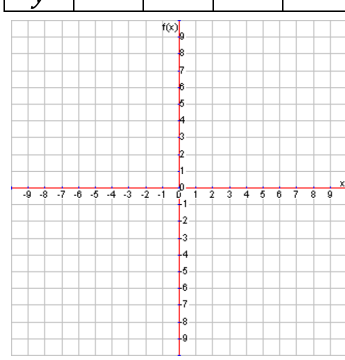
7. $y = \sqrt{x + 9}$

x					
y					



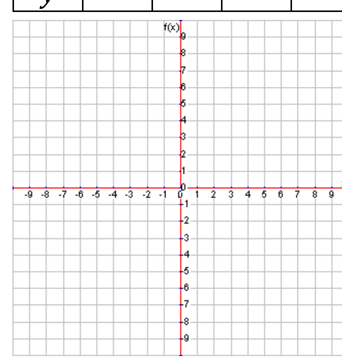
8. $y = 3^x - 2$

x					
y					



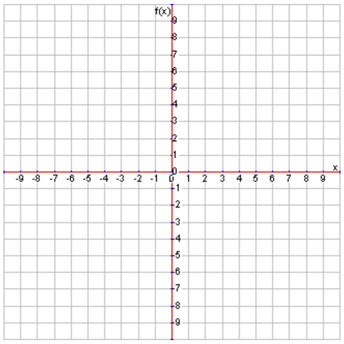
9. $y = x^3 - x^2$

x					
y					



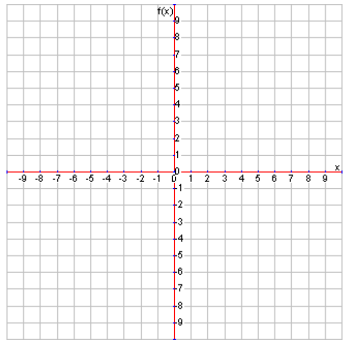
10. $y = x^3 - 7x$

x					
y					



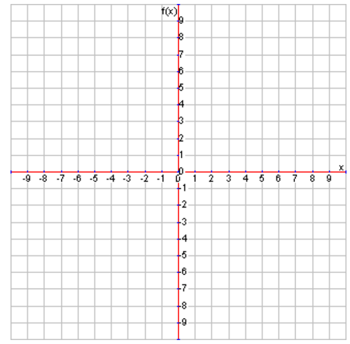
11. $y = 2^x + 3$

x					
y					



12. $y = \sqrt{x - 2}$

x					
y					

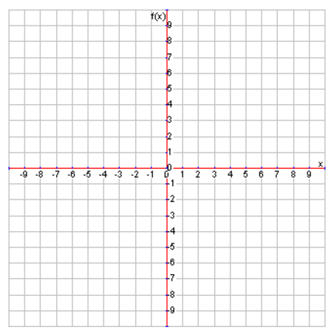


13. Give an example of a linear function in equation form.

14. Give an example of a linear function in table form.

x					
y					

15. Sketch an example of a linear function in graph form.

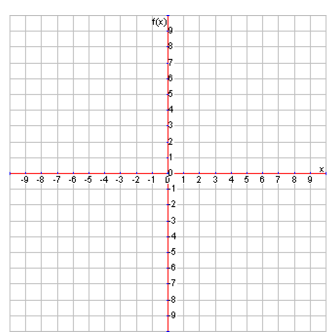


16. Give an example of a non-linear function in equation form.

17. Give an example of a non-linear function in table form.

x					
y					

18. Sketch an example of a non-linear function in graph form.



3.4 Exploring Linear Functions

Last section we defined a linear function in several different ways:

- A function whose graph is a straight line,
- A function whose rate of change is constant, or
- A function whose equation is of the form $y = mx + b$.

We'll now begin exploring the difference aspects of a linear function and how each piece affects the function as a whole.

Input (Independent Variable) and Output (Dependent Variable)

Since a function is a rule that assigns to each input exactly one output, it is crucial to identify and understand what the input and output are in the multiple forms of a function. Let's look at the following linear functions written in different forms.

Example 1

The number of grapes (g) depends on the number of branches (b) off the main vine and is represented by this equation:

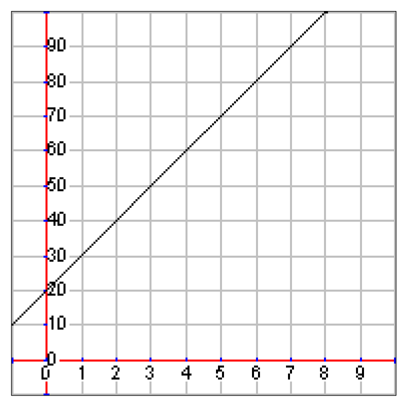
$$g = 100b + 4$$

Example 2

A dairy farmer can produce 25 gallons of milk $(g)$from 3 cows (c).

Example 3

The cost (c) for joining a gym includes a start-up fee plus dues every month (m).



In example 1, what is the input? What do we start with? What comes first? The branches of the vine. Once we have branches, those branches then produce grapes. So the input, the thing we start with, is the **independent variable** b.

The independent variable is the variable that could be anything (at least anything within the domain). We could have any number of branches we want and the branches produce, or output, the grapes. That means that g is the **dependent variable**. The number of grapes *depends* on the number of branches.

In example 2, what is the input and what is the output? A farmer puts cows *in* his barn and gets *out* milk. The cows are the input meaning that c is the independent variable. The milk is the output meaning that g is the dependent variable.

In example 3, what is the input and what is the output? What do we really want to know? The final cost. However, to find the cost we first have to know how many months you are going to be a member. That means that the number of months is the input, or independent variable. Once we input the number months into the rule (which happens to be times 15 and then plus 20), we output the cost, which is the dependent variable.

In the standard equation form of a linear function, $y = mx + b$, what is the input and output? Since we're talking about x and y on the coordinate plane, those are my input and output, but which is which? Generally speaking, but not always, the output is the variable by itself in any equation. In particular, in our generic form linear function, the variable y is the output. That makes x the input. If we plug in *(input)* an x value then we get out *(output)* a y value.

Slope or Rate of Change

Let's look at our examples again.

Example 1

The number of grapes (g) depends on the number of branches (b) off the main vine and is represented by this equation:

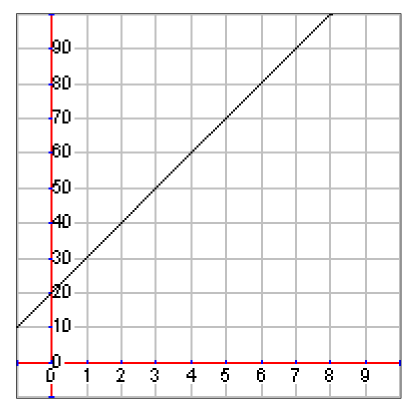
$$g = 100b + 4$$

Example 2

A dairy farmer can produce 25 gallons of milk $(g)$from 3 cows (c).

Example 3

The cost (c) for joining a gym includes a start-up fee plus dues every month (m).



In example one, the rate of change is given in the equation as the slope, or m in the equation $y = mx + b$. Note that the slope is 100 which means that the rate of change is also 100, or $\frac{100}{1}$ in fraction form. This means that 100 grapes grow for every one branch.

Perhaps it is easiest to see the rate of change (or slope) in the second example. How does the amount of milk change for the farmer? He gets 25 more gallons of milk for every 3 more cows he has, so we would write that rate of change as $\frac{25}{3}$.

In the third example, we'll need to find the slope by counting the rise and run. This is easiest to do from the y-intercept, the point where the line crosses the y-axis. Notice that the line crosses the y-axis at 20. The next nice point is at $(1, 30)$. To get to that point from the y-intercept, you have to go up ten and right one. That means the slope, or rate of change, is 10. This means that the gym charges 10 dollars for every one month of membership.

y-Intercept or Initial Value

Let's look at our examples one last time.

Example 1

The number of grapes (g) depends on the number of branches (b) off the main vine and is represented by this equation:

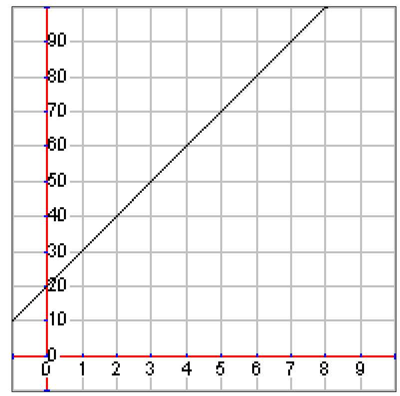
$$g = 100b + 4$$

Example 2

A dairy farmer can produce 25 gallons of milk $(g)$from 3 cows (c).

Example 3

The cost (c) for joining a gym includes a start-up fee plus dues every month (m).



In example one, the initial is given in the equation as the y-intercept, or b value in $y = mx + b$. Note that the initial value is 4. This means that only 4 grapes grow off the main vine no matter how many branches come off the main vine.

The second example may be confusing because it's hard to see an initial value. An initial value would mean the amount of milk that the farmer starts with. Well, without any cows he wouldn't have any milk, so the initial value is 0. That's why there is no other number mentioned.

We previously found the y-intercept for the third example to be 20. That means that no matter how many months you pay for membership to the gym, there will always be an additional \$20 fee to pay. The problem context describes it as a start-up fee. So if you pay for 3 months membership, you'll pay the \$20 fee on top of the price per month. If you pay for 85 months of membership, you'll still pay the same \$20 fee on top of the price per month.

Lesson 3.4

Identify the rate of change, initial value, independent variable, and dependent variable. Then describe what the rate of change and initial value mean in the context of each situation. Finally, write the equation of the linear function.

1. A 2.5 foot rocket's distance traveled in meters (d) based on time in seconds (t) is modeled by the following function: $d = 5t + 2$.

Rate of Change:____________

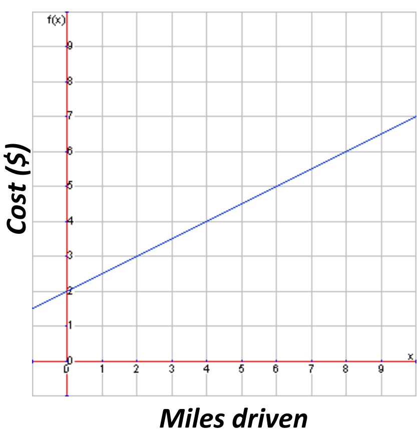
Initial Value:______________

Independent Variable:______

Dependent Variable:________

Contextual Description of Rate of Change	Contextual Description of Initial Value

2. The cost for 6 people to travel in a taxi in New York (c) based on the number of miles driven (m) is shown by the following graph:



Rate of Change:___________

Initial Value:______________

Independent Variable:______

Dependent Variable:________

EQ of Line:________________

Contextual Description of Rate of Change	Contextual Description of Initial Value

3. Planet Wiener receives $2.25 for every hotdog sold. They spend $105 for 25 packages of hot dogs and 10 packages of buns. Think of the linear function that demonstrates the profit (p) based on the number of hotdogs sold (h).

Rate of Change:___________

Initial Value:______________

Independent Variable:______

Dependent Variable:________

EQ of Line:________________

Contextual Description of Rate of Change	Contextual Description of Initial Value

4. The weight (in pounds) of a $20'$ x $10''$ x $12''$ aquarium tank (w) based on the number of gallons of water inside (g) is modeled by the following function: $w = 8.5g + 20$.

Rate of Change:____________

Initial Value:_______________

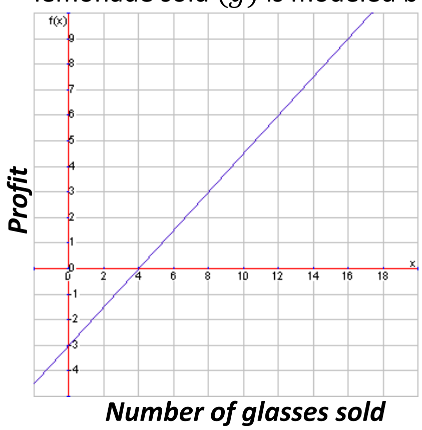
Independent Variable:______

Dependent Variable:________

Contextual Description of Rate of Change

Contextual Description of Initial Value

5. The amount of profit of the lemonade stand on 120 W Main Street (p) based on the number of glasses of lemonade sold (g) is modeled by the following graph:



Rate of Change:___________

Initial Value:______________

Independent Variable:______

Dependent Variable:________

EQ of Line:________________

Contextual Description of Rate of Change

Contextual Description of Initial Value

6. A candle starts at a height of 5 inches and diameter of 3 inches and burns 1 inch every 2 hours. Think of the linear function that demonstrates the height of the candle (h) in terms of the time it has been burning (t).

Rate of Change:___________

Initial Value:______________

Independent Variable:______

Dependent Variable:________

EQ of Line:________________

Contextual Description of Rate of Change

Contextual Description of Initial Value

7. The cost (c) to stay in a 4 star hotel each night (n) is modeled by the following function: $c = 104n + 15$

Rate of Change:______________

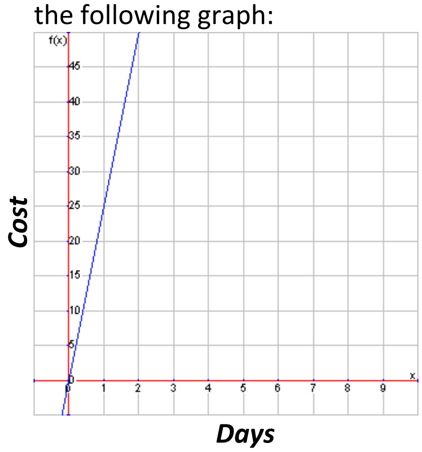
Initial Value:_________________

Independent Variable:_________

Dependent Variable:___________

Contextual Description of Rate of Change	Contextual Description of Initial Value

8. The cost (c) to attend a sports clinic 37 miles away based on the number of days attended (d) is modeled by the following graph:



Rate of Change:_____________

Initial Value:________________

Independent Variable:________

Dependent Variable:__________

EQ of Line:__________________

Contextual Description of Rate of Change	Contextual Description of Initial Value

9. A dog kennel charges \$40 for each night the dog stays in the kennel. Each day includes a 2 hour play time and 1 hour etiquette training. The kennel also charges a \$10 bathing fee for a bath before the dog returns home. Think of the linear function that demonstrates the cost of putting a dog in the kennel (c) in terms of the number of nights (n).

Rate of Change:______________

Initial Value:_________________

Independent Variable:_______

Dependent Variable:__________

EQ of Line:__________________

Contextual Description of Rate of Change	Contextual Description of Initial Value

10. The number of gallons of gas in your 15 gallon gas tank (g) based on the number of miles traveled (m) is modeled y the following function: $g = -\frac{1}{25}m + 12$.

Rate of Change:_______________

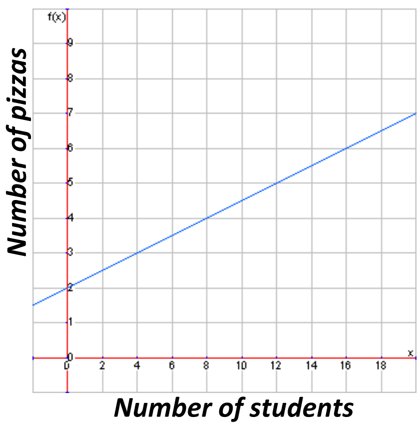
Initial Value:__________________

Independent Variable:_________

Dependent Variable:___________

Contextual Description of Rate of Change	Contextual Description of Initial Value

11. The number of pizzas ordered for 8th grade night (p) based on the number of students (s) is shown by the following graph:



Rate of Change:______________

Initial Value:__________________

Independent Variable:_________

Dependent Variable:__________

EQ of Line:__________________

Contextual Description of Rate of Change	Contextual Description of Initial Value

12. It costs \$5.50 to mail a large package to New Zealand. The post office will weigh your package and charge you an extra \$0.30 per pound. The delivery takes 2 weeks. Think of the linear function that demonstrates the cost to mail a large package to New Zealand (c) based on the number pounds it weighs (p).

Rate of Change:______________

Initial Value:__________________

Independent Variable:_________

Dependent Variable:__________

EQ of Line:___________________

Contextual Description of Rate of Change	Contextual Description of Initial Value

13. An author wrote an 876-page book. The amount of profit (p) based on the number books sold (b) is modeled by the following function: $p = 7b + 1050$.

Rate of Change:_______________

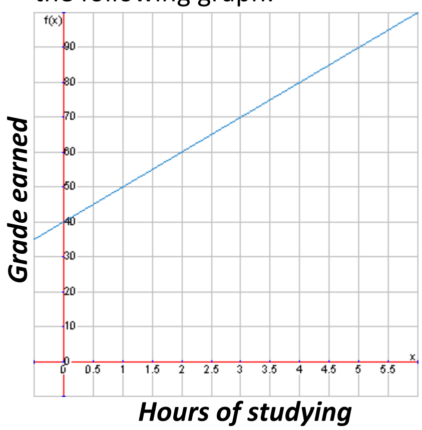
Initial Value:_________________

Independent Variable:_________

Dependent Variable:___________

Contextual Description of Rate of Change	Contextual Description of Initial Value

14. The average grade earned on the Unit 3 test (g) based on the number of hours of studying (h) is modeled by the following graph:



Rate of Change:______________

Initial Value:________________

Independent Variable:________

Dependent Variable:__________

EQ of Line:___________________

Contextual Description of Rate of Change	Contextual Description of Initial Value

15. Kiley invited 32 people to her 13th birthday party at the bowling alley. She hopes most people can come! It costs \$40 to reserve the bowling alley. It will cost an additional \$2 per friend to bowl. Think of the linear function that demonstrates the cost of the birthday party (c) in terms of the number of friends who attend and bowl (f).

Rate of Change:______________

Initial Value:_________________

Independent Variable:_________

Dependent Variable:__________

EQ of Line:__________________

Contextual Description of Rate of Change	Contextual Description of Initial Value

16. You started a mowing business so you could buy a 2015 Chevy Camaro when you turn 16. The amount of money (m) in your bank account based on the number of yards you mow (y) is modeled by the following function: $m = 30y$.

Rate of Change:______________

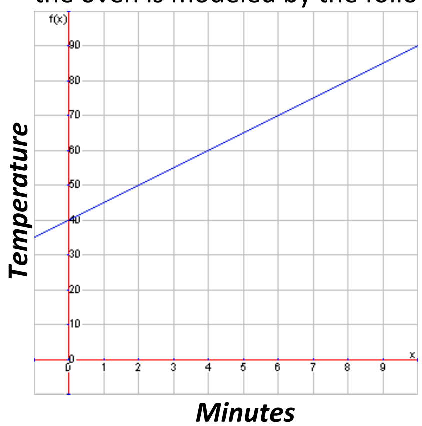
Initial Value:________________

Independent Variable:________

Dependent Variable:_________

Contextual Description of Rate of Change	Contextual Description of Initial Value

17. When an oven is set at $350°\mathrm{F}$, the internal temperature (t) of a chicken breast after every minute (m) it's in the oven is modeled by the following graph:



Rate of Change:_____________

Initial Value:________________

Independent Variable:________

Dependent Variable:__________

EQ of Line:_________________

Contextual Description of Rate of Change	Contextual Description of Initial Value

18. Walter's Water Adventures charges $34 to enter. This fee helps pay for maintenance and lifeguards. They always have 3 lifeguards at each slide plus 2 watching the wave pool. Think of the linear function that demonstrates the number of lifeguards on duty (l) based on the number of slides open (s) on a given day.

Rate of Change:_____________

Initial Value:________________

Independent Variable:________

Dependent Variable:_________

EQ of Line:_________________

Contextual Description of Rate of Change	Contextual Description of Initial Value

3.5 Increasing, Decreasing, Max, and Min

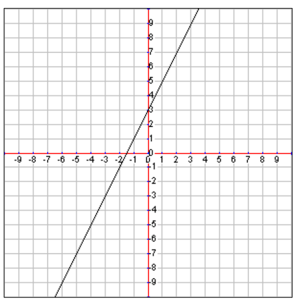
So far we have been describing graphs using quantitative information. That's just a fancy way to say that we've been using numbers. Specifically, we have described linear function graphs using the rate of change and initial value. Both are numerical data, however, at times it is more beneficial to describe functions in a qualitative manner. This means describing the qualities, non-numerical characteristics, of the function.

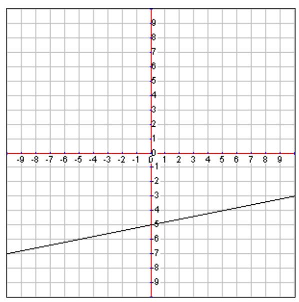
Qualitative Data: Increasing or Decreasing

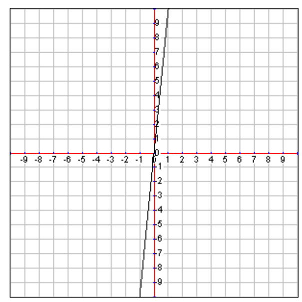
Let's start with the idea of an increasing or decreasing function. An **increasing function** roughly speaking is one that is going up when you look at it from left to right. This means that a **decreasing function** is one that is going down when you look at it from left to right. Let's start by looking at linear functions.

Linear functions

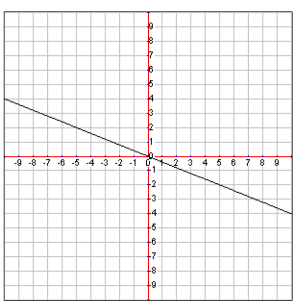
Increasing Linear Functions

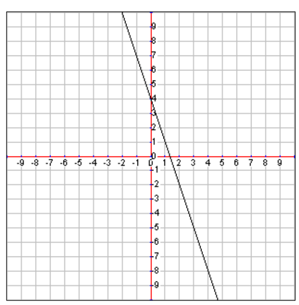


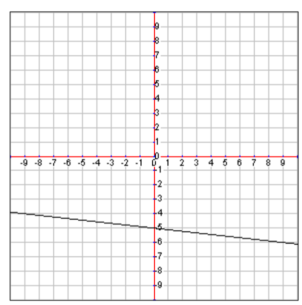




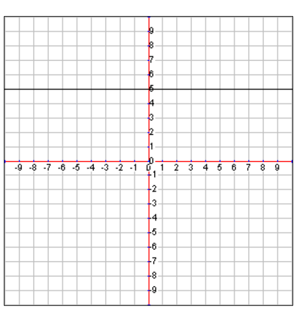
Decreasing Linear Functions







It is worth noting that in linear functions, whether it is increasing or decreasing is dependent on the slope of the line. Notice that those with positive slopes are increasing and those with negative slopes are decreasing.

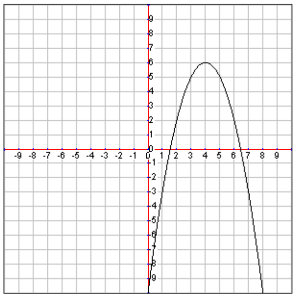


However, how would we classify the graph to the left? It is possible to have a third option rather than just increasing or decreasing. It could be neither. The graph to the left is called **constant** because it stays the same from left to right.

It would be natural to then ask about a vertical line, but remember that a vertical line would not pass the vertical line test and therefore would not be a true function.

Non-linear functions

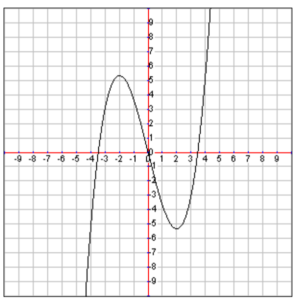
Linear functions are easy to identify as increasing, decreasing, or constant because they are a straight line. Non-linear functions might be both increasing and decreasing at different points on the graph. Consider the following graph of the function $y = -x^2 + 8x - 10$.



Looking at this graph from left to right, we see that the graph starts out increasing, but then it reaches a high point and start decreasing. So how do we classify this function?

Let's see if we can isolate where the function is increasing and just look at that piece of the graph. Notice that the graph is increasing until it reaches the point at $x = 4$. That means we can say that the function is increasing when $x < 4$. In other words, whenever x is less than 4, the graph is increasing.

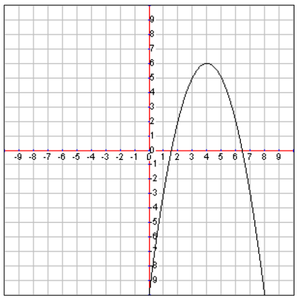
We can similarly see where the graph is decreasing which when $x > 4$. Notice that we don't use greater than or equal to, just greater than because at the exact point $x = 4$ the graph is at the high point and neither increasing or decreasing. We'll get to what that point is called in just a moment.



To the left is the graph of the function $y = \frac{1}{3}x^3 - 4x$. Notice that it's increasing, then decreasing, then increasing again. So we would say it is increasing when $x < -2$ and $x > 2$. It is decreasing in the interval $-2 < x < 2$.

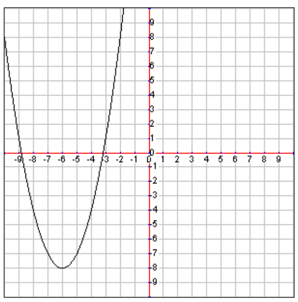
Qualitative Data: Maximum and Minimum

Another way that we can qualitatively describe a function graph is by identifying any maximum or minimum y value that is achieved. For linear graphs this doesn't make sense because the line will never have a maximum or minimum height. We'll look at non-linear functions only for a max or min.



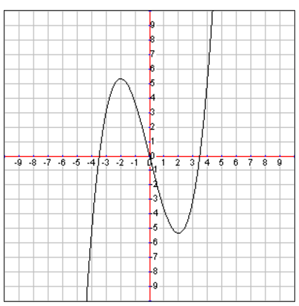
We say there is a **maximum** at x (input) when the function has a higher y value (output) than at any other input. Looking at our original non-linear example there is a maximum of $y = 6$. For this particular graph we can see that the maximum occurs at $x = 4$, but we generally list the output, or y value, for a maximum.

Max: $y = 6$



We can do the same thing for a minimum. We say there is a **minimum** at x (input) when the function has a lower y value (output) than at any other input. In the example to left we see that it has a minimum of $y = -8$. It might also be worth noting that the minimum occurs at $x = -6$, but we list the minimum by its y value.

Min: $y = -8$



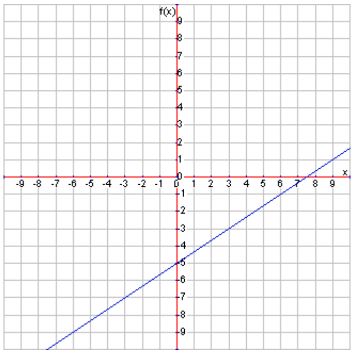
Sometimes we have what are called **local** maximums or minimums. For example, in the graph to the left there is a local max of $y \approx 5$ at $x = -2$ and a local min of $y \approx -5$ at $x = 2$. They are called local because they are not the max or min for the whole graph, just the max or min in a small area.

Local max: $y \approx 5$ Local min: $y \approx -5$

Lesson 3.5

For each linear graph tell whether it is increasing, decreasing, or constant.

1.



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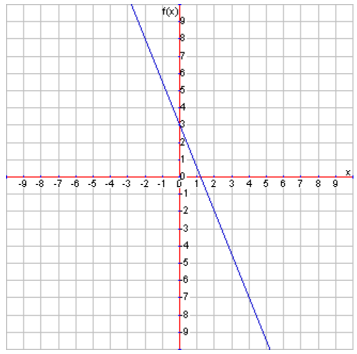

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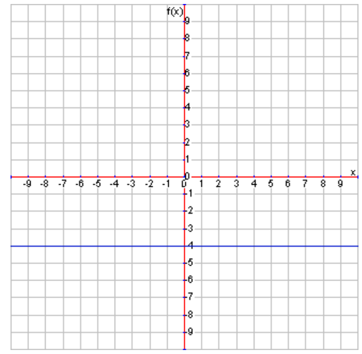

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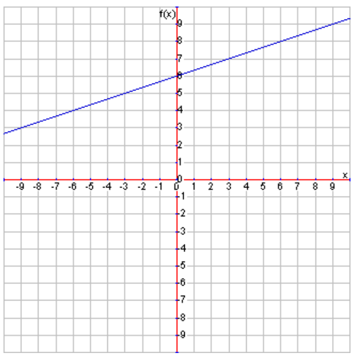

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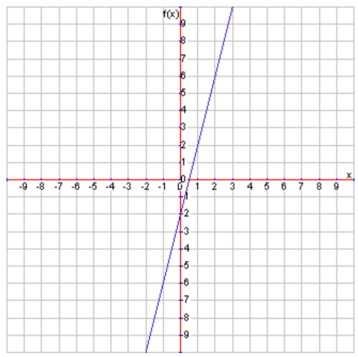

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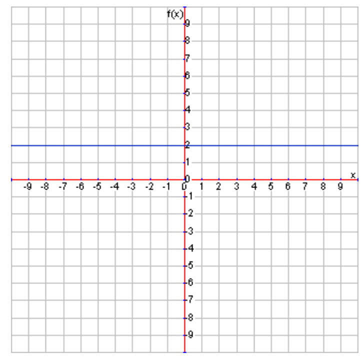

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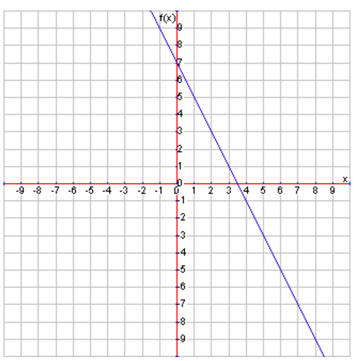

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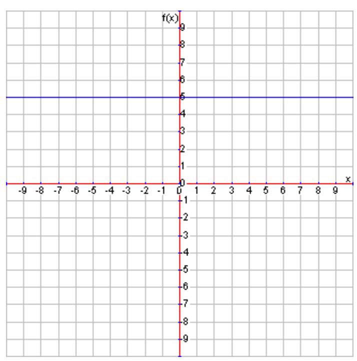

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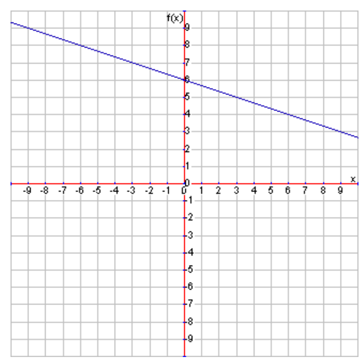

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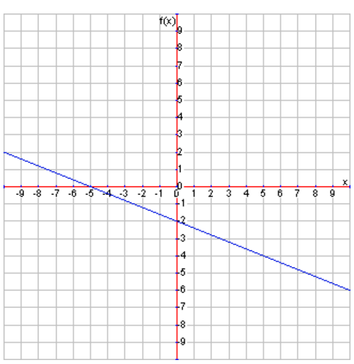

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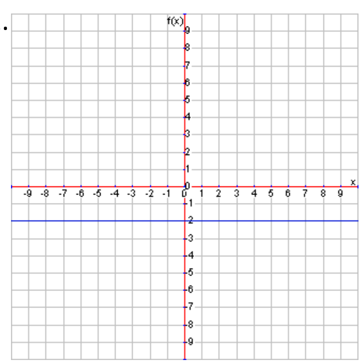
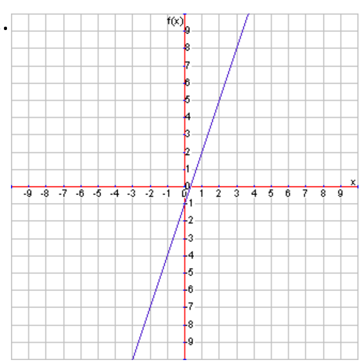
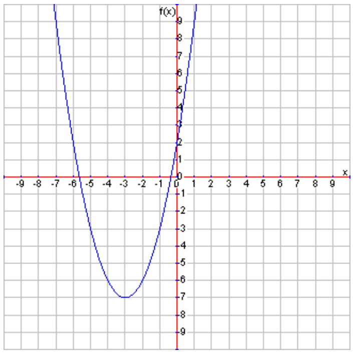


12.



For each non-linear graph tell where it is increasing and decreasing and identify any maximum, minimum, local maximum, or local minimum.

13.



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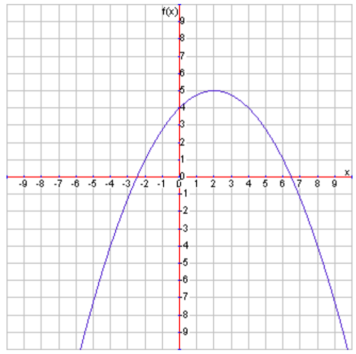

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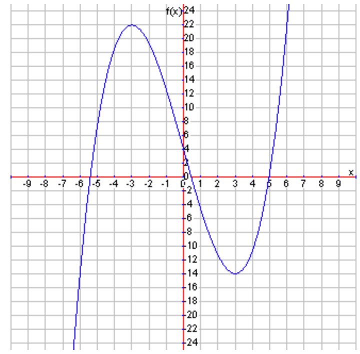

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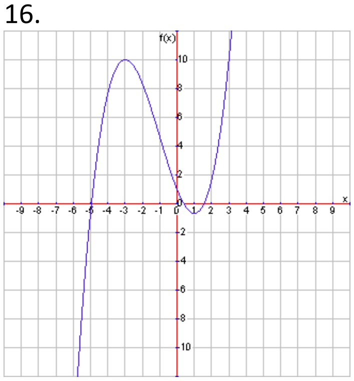

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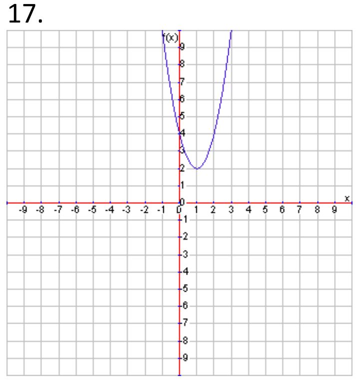

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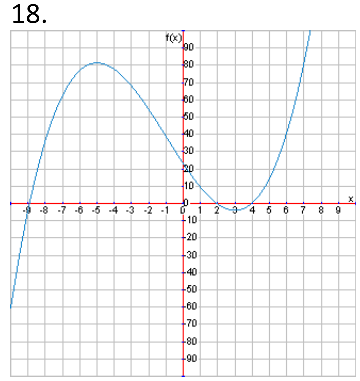

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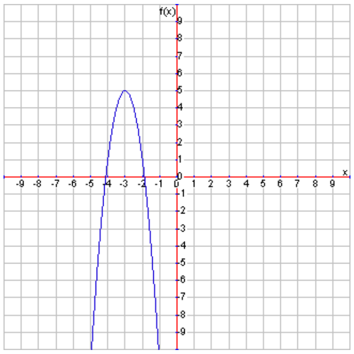

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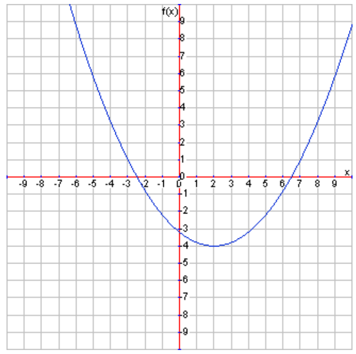

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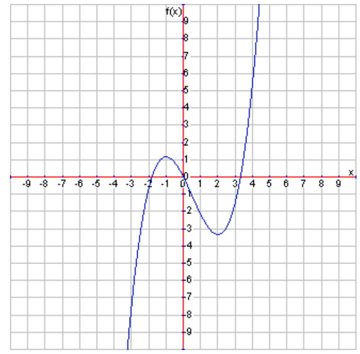

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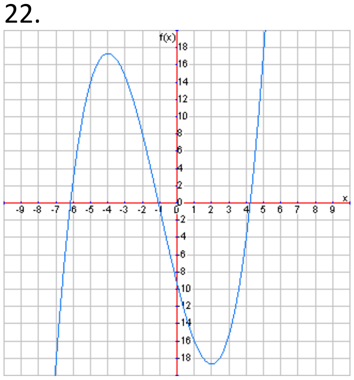

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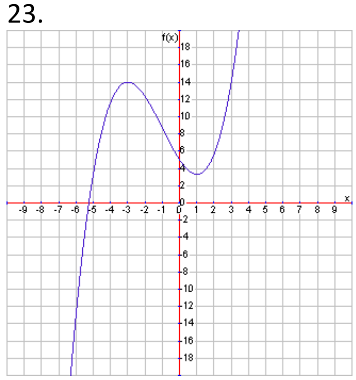

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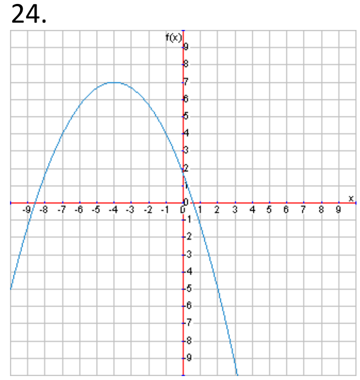
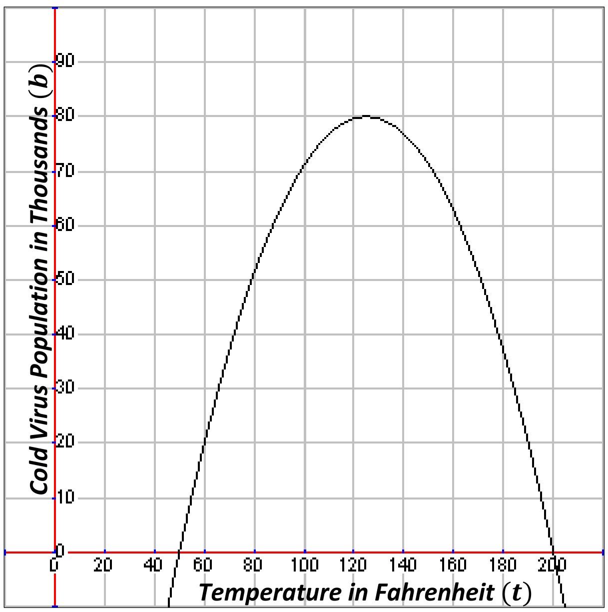


3.6 Contextualizing Function Qualities

Now that we know all the parts of functions, specifically non-linear functions, let's put them in the context of a problem. We want to make sure we understand what each piece of a function in the words.

The Common Cold

Most common cold viruses thrive in normal every day temperatures. It's when those temperatures get cold (like in a doctor's office) or hot (like in boiling water) that the virus finally dies. Let's pretend that if we were to graph how many thousands of the common cold viruses can live in one thousand cubic feet (b) based on the temperature of that space (t) we get something like this:



What inputs make sense in this context? The inputs that make sense are temperatures between $50°\,F$ and $200°\,F$. Outside of those temperatures there are negative amounts of virus, which doesn't make sense.

What are all the possible population sizes? There could be anywhere from zero to around $80{,}000$ viruses living depending on the temperature.

What is the highest population? This is the maximum. The highest point is around the $80{,}000$ viruses.

What temperatures give the highest population? This is the temperature where the maximum takes place. The highest population occurs around $125°\,F$. Notice this is just an estimate that we get from the graph.

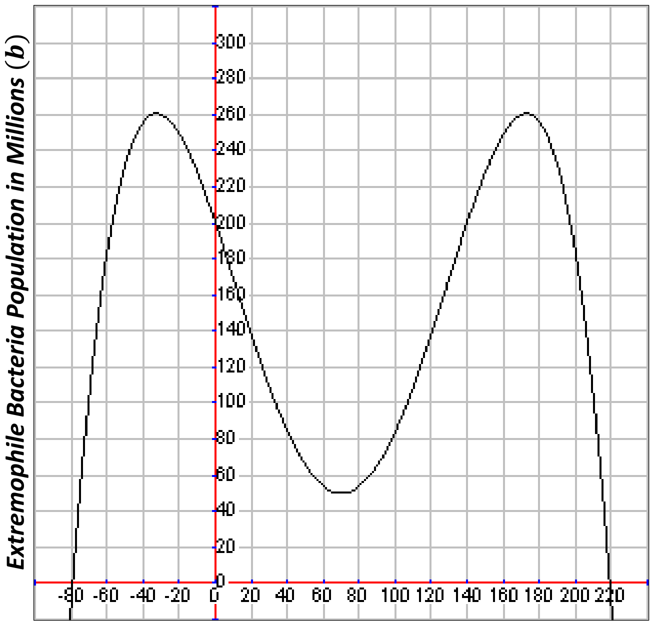
At what temperature range are the viruses growing bigger and bigger populations? This question is asking where the function is increasing. This would happen from $50°\,F$ to $125°\,F$.

At what temperature range are the virus populations growing smaller and smaller? This question is asking where the function is decreasing. This would happen from $125°\,F$ to $200°\,F$. Above $200°\,F$ the viruses all die, so we don't care beyond that.

The chance of a the common cold spreading through airborne contact is much less when the population in a thousand cubic foot zone is less than $\mathbf{30,000}$. Why are doctor's offices so cold? In order to keep the population of the common cold virus at acceptable levels, the doctor's offices must be around $65°\,F$ or lower. That's chilly!

Extremophile Bacteria

There is a certain type of bacteria called extremophiles that thrive in extreme temperatures. These bacteria like really super cold or insanely hot conditions. Pretend that if we were to graph the function of the population in millions of extremophile bacteria living in one thousand cubic feet (b) at a given Fahrenheit temperature (t), the graph might look like this:



Temperature in Fahrenheit (t)

What inputs would make sense in this context? The temperatures between $-80°\ F$ and $220°\ F$ are where the bacteria live. Outside of those temperatures there are less than zero bacteria, which doesn't make sense. We can't have negative bacteria.

What are all the possible population sizes? There could be anywhere from zero to around $260{,}000{,}000$ bacteria living depending on the temperature.

What is the highest population that extremophile bacteria will have? These are the maximums. The highest point looks be around the $260{,}000{,}000$ mark at both points.

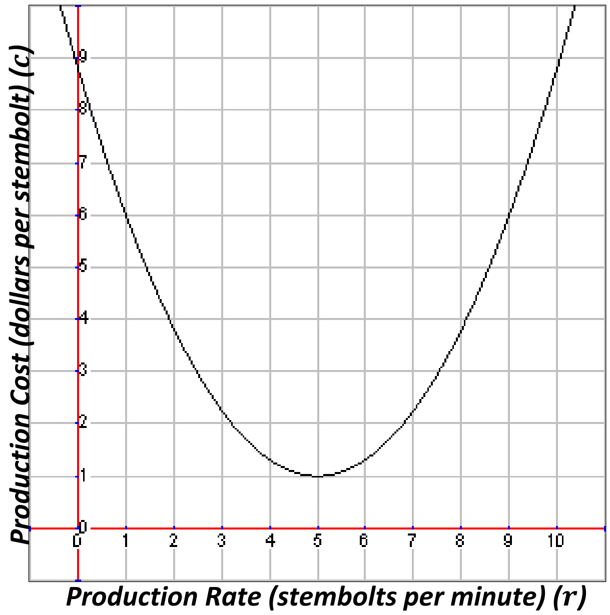
What temperatures give the highest population? These are the temps where the maximums take place. The high populations occur around $-32°\ F$ and $172°\ F$.

What is the lowest population we can get of extremophile bacteria within comfortable living temperatures? This is basically asking for the local min which is about $50{,}000{,}000$ at a temperature of around $70°\ F$. Again these are estimates from the graph.

At what temperature range are the bacteria growing bigger and bigger populations? This question is asking where the function is increasing. This would happen in two places: from $-80°\ F$ to $-32°\ F$ is the first and from $70°\ F$ to $172°\ F$.

At what temperature range are the bacteria growing smaller and smaller populations? This question is asking where the function is decreasing. This would happen in two places: from $-32°\ F$ to $70°\ F$ is the first and from $172°\ F$ to $220°\ F$. Above $220°\ F$ the bacteria all die, so we don't care beyond that.

Use the following graph showing a function modeling the production cost per stembolt (c) a factory gets in terms of the production rate of how many stembolts it produces per minute (r) to answer the questions.



1. If the possible inputs for this function are between one and nine, what does that mean in the context of this problem?

2. Within those inputs, what are all the different costs per stembolt that the company could have?

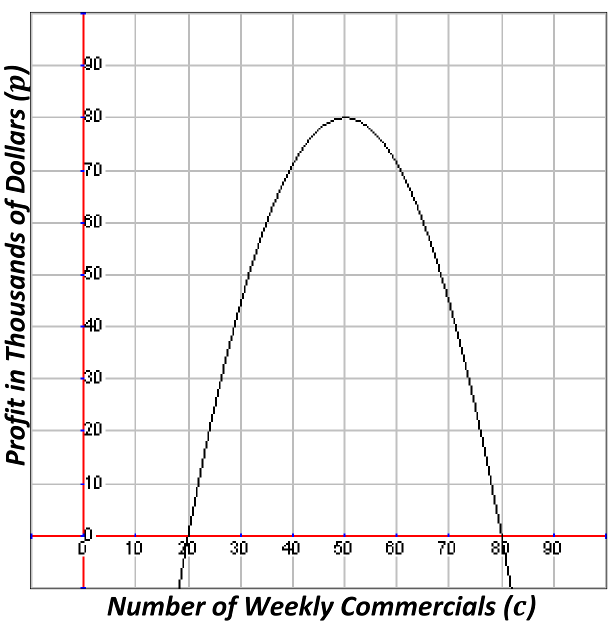
3. At what production rate does the company get the cheapest production cost?

4. What is the cheapest production cost?

5. Between what production rates does the company get cheaper and cheaper production costs?

6. Between what production rates does the company get higher and higher production costs?

Use the following graph showing a function modeling the company's weekly profit in thousands of dollars (p) in terms of the number of weekly commercials it airs (c) to answer the questions.



7. What inputs make sense in the context of this problem?

8. What are all the different profits that the company could have?

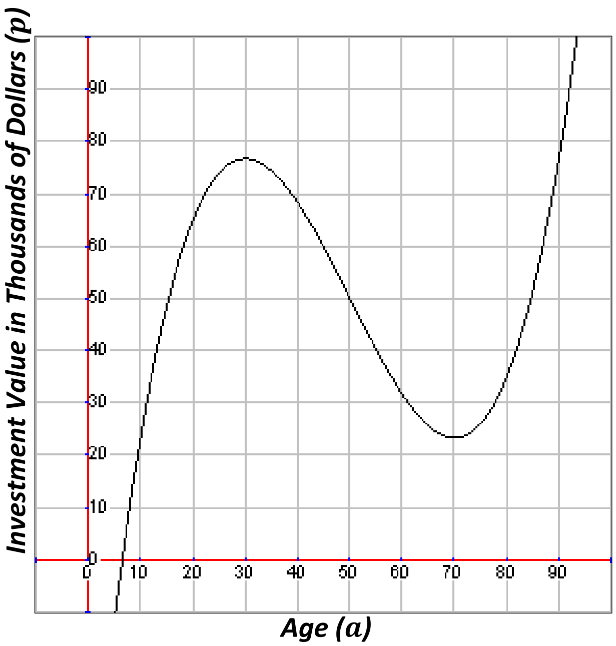
9. How many weekly commercials gives the best profit for the company?

10. What is the best profit the company can expect?

11. Between how many weekly commercials does the company get better and better profits?

12. Between how many weekly commercials does the company get worse and worse profits?

Use the following graph showing a function modeling a man's stock market investment value in thousands of dollars (v) in terms of his age (a) to answer the questions.



13. If the man began investing at 20 years old and retired at the age of 80 (at which point he sold all his stocks), what inputs make sense in the context of this problem?

14. What are all the different investment values the man had during the time he was investing?

15. At what age was his investment value the highest? How high was it?

16. At what age was his investment value the lowest? How low was it?

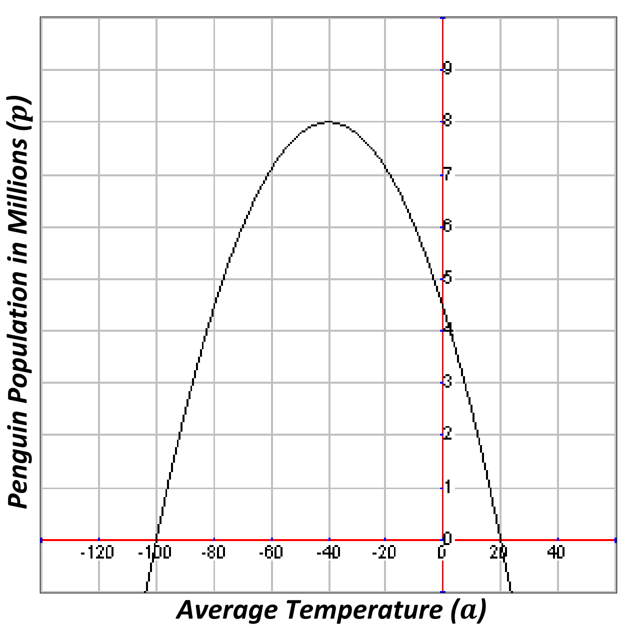
17. Between what ages was his investment growing in value?

18. Between what ages was his investment losing value?

19. Overall, since he started investing at 20 years old and retired at 80 years old, did he make or lose money? How much?

20. What appears to be the earliest age he should have retired (after 80 years old) in order to have at least broken even on his investments?

Use the following graph showing a function modeling the penguin population in millions (p) in terms of average temperature of the Antarctic in degrees Fahrenheit (t) to answer the questions.



21. What inputs make sense in the context of this problem?

22. What are all the different populations that the penguins could have?

23. What average temperature gives the highest penguin population?

24. What is the highest population of the penguins?

25. Between what temperatures does the population grow?

26. Between what temperatures does the population shrink?

3.7 Sketching a Piecewise Function

Now that we understand qualitative descriptions of graphs, we can use that information to sketch graphs of a function or give a verbal description of an already sketched graph. For these graphs, we won't have any numeric reference points to go by. Instead we'll just use Quadrant I of the coordinate plane and give approximate graphs that represent the described situation.

The term **piecewise** means that the function may have different qualities at different intervals. For example, the graph may start off constant, then increase and finally decrease. It could start increasing linearly and then increase in a non-linear fashion. So we generally sketch the graph a piece at a time.

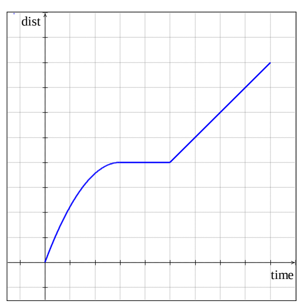
Matching Description and Graph

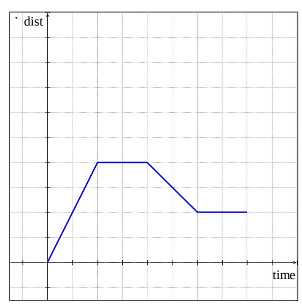
Distance versus time

We'll begin by reading a description of a situation and then decide which graph best fits the data. Here is our situation:

> George started at his friend's house and began walking home. After a few blocks, he realized he forget his cell phone and hurried back to his friend's house to pick it up. After grabbing his phone, he immediately began running back home because he was afraid he was going to be late. Unfortunately he got stuck for a little while trying to cross the busy street. After crossing the busy street, he decided to walk the rest of the way home instead of running. Which graph shows George's distance from home in terms of time?

A. 

B. 

C. 

In this case the choice may be obvious because only one of the three graphs starts at non-zero distance. That means that graph A must be the correct graph for George. Notice how the line segments are steeper when he goes back for his cell phone and heads back home again. That's because he was running during that time so more distance was being covered in less time. Also note the little flat line segment that shows us when George was waiting at the busy street. There he traveled no distance because he was waiting.

Using the same above graphs, match this situation with its graph:

Joanne started running a marathon as fast as she could. During the first few minutes she gradually slowed down until she stopped at about the half-way point of the marathon. She had to take a long break sitting on a bench because she had run too fast. She then ran at a slower constant speed until the end of the marathon where she promptly collapsed. Which graph shows Joanne's distance run in terms of time?

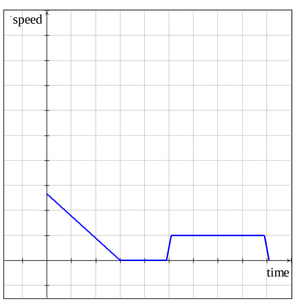
For this situation, we're looking for a graph starting off very steep (to show the fast speed) but slowing getting less steep (to show the slowing down). Then there should be a segment where the distance does not change over time (representing a speed of zero). Afterwards we should see the distance getting bigger because Joanne completes the marathon. This must be graph B.

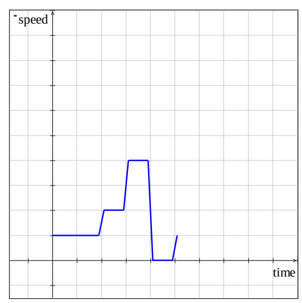
What situation might graph C represent?

Speed versus time

Examining these same situations in terms of speed would offer us different graphs. Let's look at George's situation first:

George started at his friend's house and began walking home. After a few blocks, he realized he forget his cell phone and hurried back to his friend's house to pick it up. After grabbing his phone, he immediately began running back home because he was afraid he was going to be late. Unfortunately he got stuck for a little while trying to cross the busy street. After crossing the busy street, he decided to walk the rest of the way home instead of running. Which graph shows George's speed in terms of time?

D. 

E. 

The function showing George's speed in terms of time is represented by graph E. Notice that the speed starts off greater than zero, increases, and then increases again before going down to zero (where he stopped at the busy street) and finishing at a constant speed. More importantly notice the differences between this graph and George's graph showing distance as a function of time.

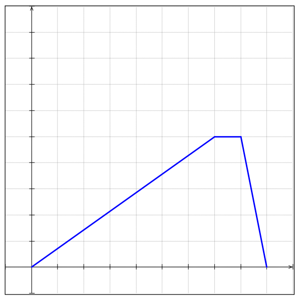
Now look at graph D. This graph represents Joanne's situation except it shows her speed instead of distance. Why is this true?

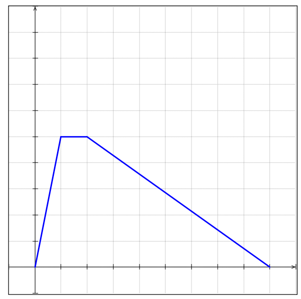
Speed or distance?

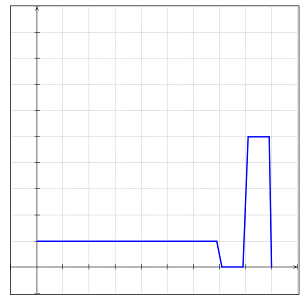
To get one last look at how different variables (speed and distance in this case) can drastically change the graph of a function, consider the following situation:

A child climbed slowly up a slide, sat at the top for a little while, and then quickly slid down.

Which of the following graphs shows height off the ground (which is a distance) versus time and which shows speed versus time?

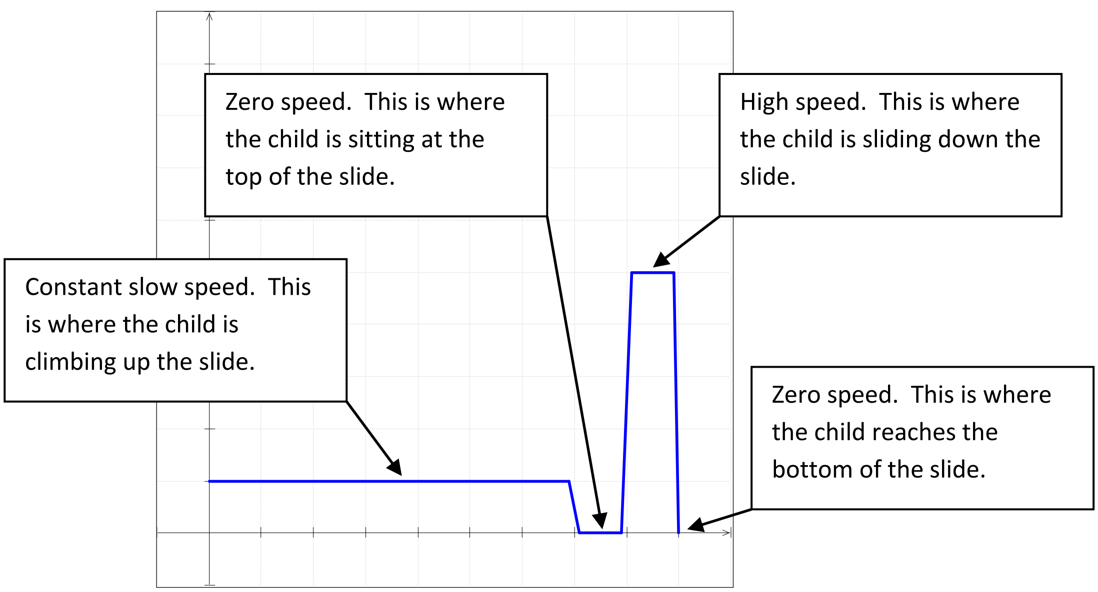
F. 

G. 

H. 

While graph G is tempting to choose as showing the height versus time because it looks like a slide, that is incorrect. Graph F shows the height as a function of time. Notice how the child takes a lot of time (horizontal distance) to get to the top of the slide and then takes far less time to have a height of zero.

Graph H is a graph of speed versus time. Let's look at why this is true.

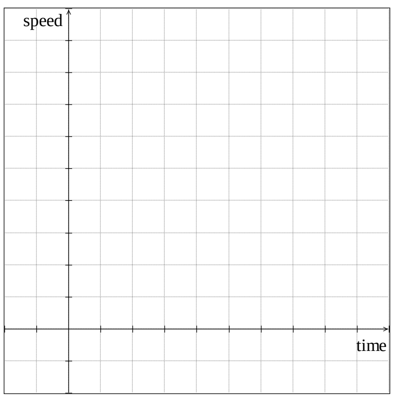


Sketching the Graph

Now let's try sketching a graph given a verbal description of a situation.

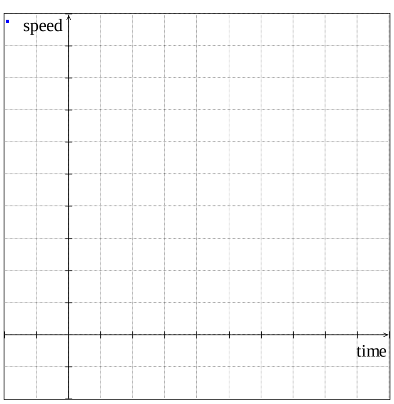
A cat is sitting on a pillow across the room watching the cicadas climb up the sliding glass door. The cat sits perfectly still for several moments before quickly charging towards the sliding glass door where she slams into it coming to halt. After pausing a moment to realize the cicadas were scared away, the cat slowly slinks to the middle of the room to wait on the next cicada to show up. Sketch a graph modeling the function of the cat's speed in terms of time.

The first thing to do is identify our two variables we are comparing and determine which is the dependent (going on the y-axis) and which is the independent (going on the x-axis). In this problem we are looking at speed in terms of time. That means the speed is dependent on the time, and therefore speed is our dependent variable which will go on the y-axis. So we might begin our graph by labeling our axes like this:

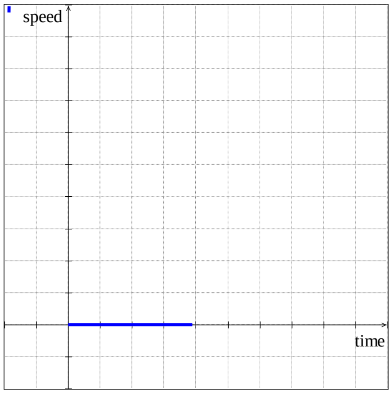


Notice we are only using Quadrant I because it doesn't make sense in this context to talk about negative time or speed.

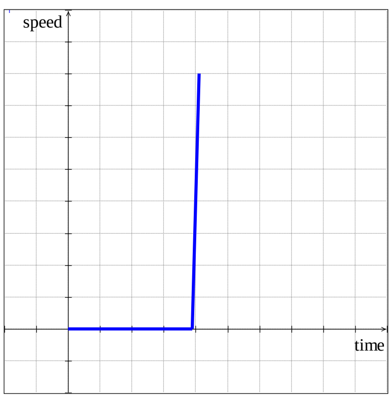
Next we start at the beginning of the problem. What was the cat's initial speed? It was sitting perfectly still for several moments. That means for a good chunk of the time on our time, the speed will be zero. So we'll use a flat line at height zero starting at the origin to represent this like so:



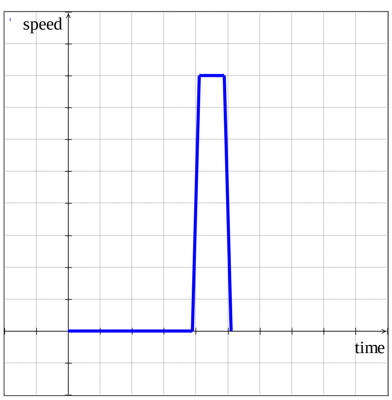
Now that we have the part of the graph representing the cat sitting still, we move to the next part of the problem. The next thing the cat does is quickly charges at the sliding glass door. That means the speed is going to be high, so we'll need a line going up to a relatively high speed.



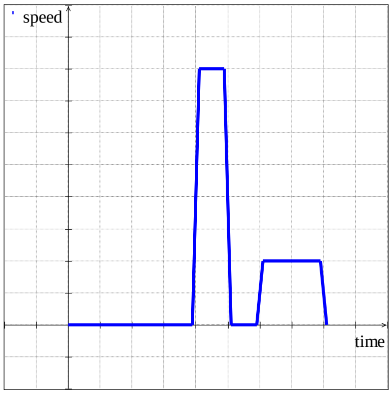
However, we also need to consider how long the cat stayed at this speed. Since it was only across the room, the cat probably did not spend a lot of time at a high speed. We'll represent this by only having the speed stay constantly high for a short amount in the x direction (horizontally).



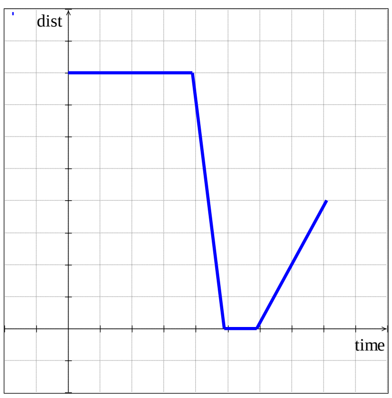
Next the cat slams into the glass door bringing its speed to zero. It also paused a moment, meaning its speed was zero just for a little bit.



Finally the cat slowly walks back to its pillow (low speed) before sitting back down (zero speed) in the middle of the room. So our final graph may look like this:

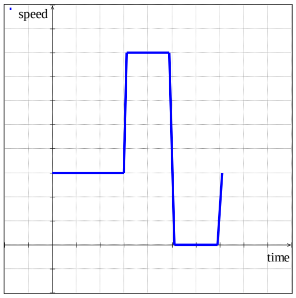


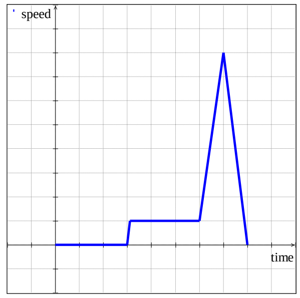
Now let's think about making a graph to represent a function of the cat's distance from the sliding glass door in terms of time. Why might it look something like this?

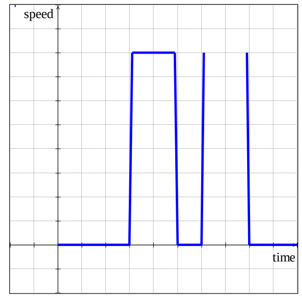


Lesson 3.7

Match each description with its function graph showing speed in terms of time.

A. 

B. 

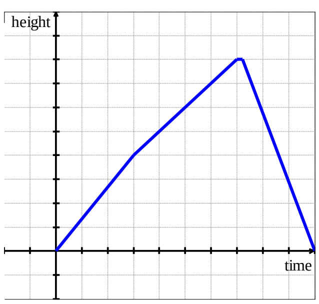
C. 

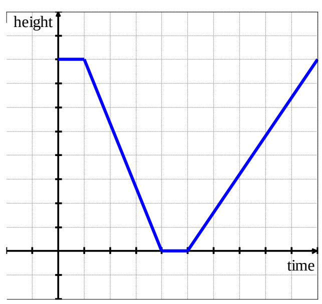
1. A squirrel chews on an acorn for a little while before hearing a car coming down the street. It then runs quickly to the base of a nearby tree where it sits for a second listening again for the car. Still hearing the car, the squirrel climbs up the tree quickly and sits very still on a high branch.

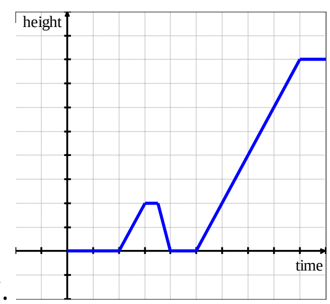
2. A possum is slowly walking through a backyard when a noise scares it causing it to hurry to a hiding place. It waits at the hiding place for a little while to make sure it's safe and then continues its slow walk through the backyard.

3. A frog is waiting quietly in a pond for a fly. Noticing a dragonfly landing on the water nearby, the frog slowly creeps its way to within striking distance. Once the frog is in range, it explodes into action quickly jumping towards the dragonfly and latching onto with its tongue. The frog then settles down to enjoy its meal.

Match each description with its function graph showing height in terms of time.

D. 

E. 

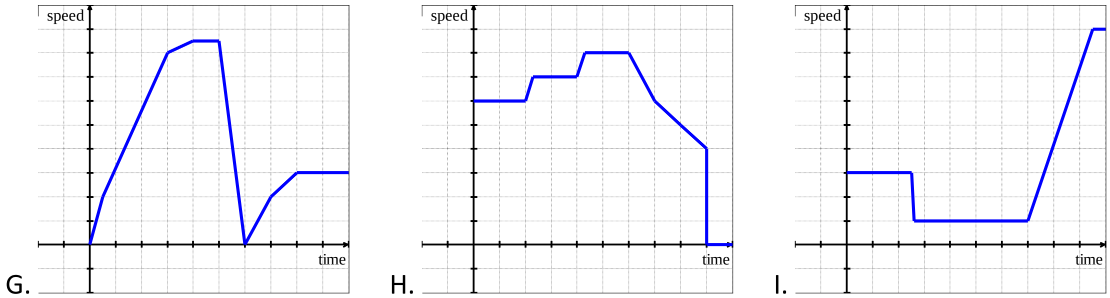
F. 

4. Sean starts to bike up a long steep hill. Half way up, he gets off his bike to walk the rest of the hill. When he makes it to the top, he races down the other side until he makes it to the bottom.

5. Micah is racing down a flat road. He comes to a small hill and charges up as fast as he can. Coming down the other side, Micah gains speed for the big hill ahead. Micah climbs the hill to the top, and hops off his bike to stretch.

6. Jerika hops on her bike as she comes out of her garage which sits at the top of a large hill. She coasts down the hill and starts pedaling as the road flattens. She realizes she forgot something, so she rides back up to her house.

Match each description with its function graph showing speed in terms of time.



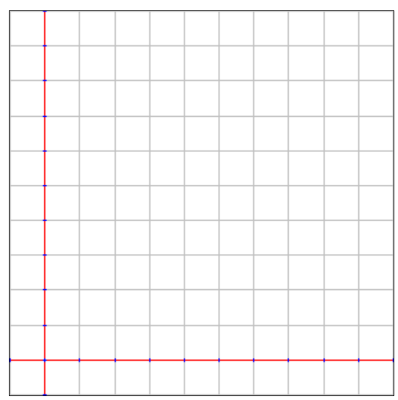
7. Sean starts to bike up a long steep hill. Half way up, he gets off his bike to walk the rest of the hill. When he makes it to the top, he races down the other side until he makes it to the bottom.

8. Micah is racing down a flat road. He comes to a small hill and charges up as fast as he can. Coming down the other side, Micah gains speed for the big hill ahead. Micah climbs the hill to the top, and hops off his bike to stretch.

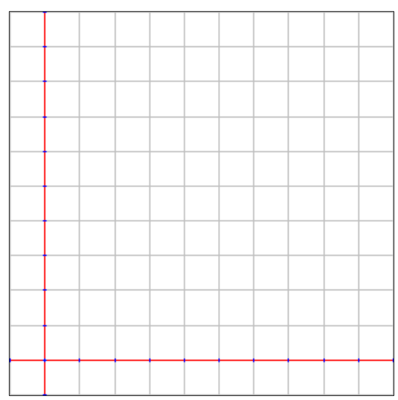
9. Jerika hops on her bike as she comes out of her garage which sits at the top of a large hill. She coasts down the hill and starts pedaling as the road flattens. She realizes she forgot something, so she turns around and rides back up to her house.

Sketch a graph modeling a function for the following situations.

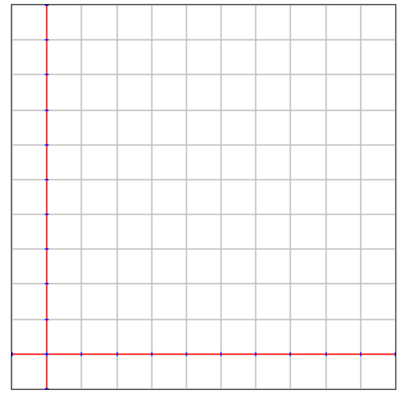
10. A runner starts off her day running at an average speed down her street. At the end of a street is a slight hill going down so she runs even faster down the hill. At the bottom of the hill she has to go back up to the level of her street and has to slow way down. Sketch a graph of a function of runner's speed in terms of time.



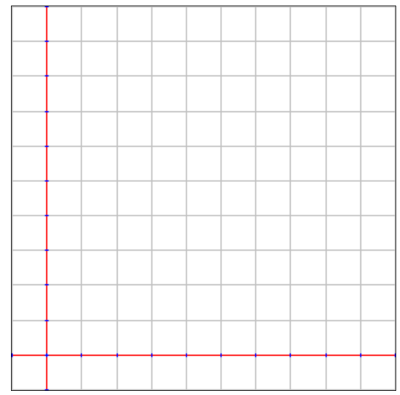
11. A runner starts off her day running at an average speed down her street. At the end of a street is a big hill going down, so she runs very fast down the hill. At the bottom of the hill she runs on flat ground at an average speed for a while before going back up another hill where she slows way down. Sketch a graph of a function of runner's height in terms of time.



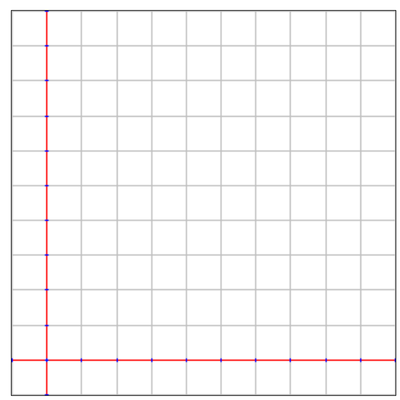
12. A fish swims casually with her friends. All of a sudden, she hears a boat, so she darts down toward the bottom of the ocean and hides motionlessly behind the coral. She remains still until she hears the boat pass. When the coast is clear, she goes back to swimming with her friends. Sketch a graph of a function of the fish's speed in terms of time.



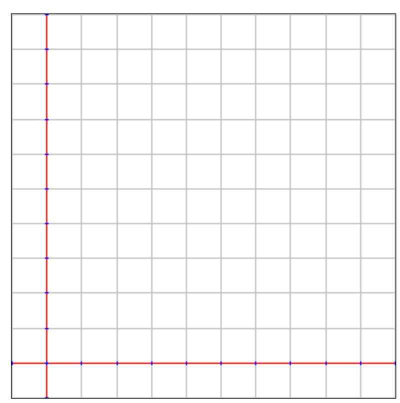
13. My dad drove me to school this morning. We started off by pulling out of the driveway and getting on the ramp for the interstate. It wasn't long before my dad saw a police car, so he slowed down. The police car pulled us over, so we sat on the side of the road until the cop finished talking to my dad. Sketch a graph of a function of the car's speed in terms of time.



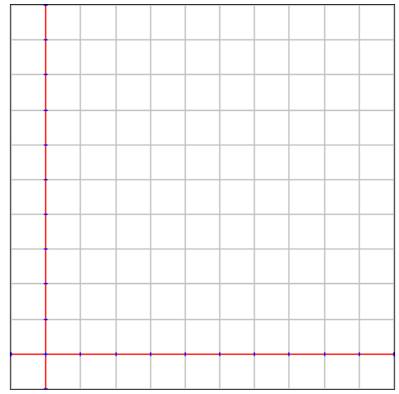
14. Rashid starts on the top of a snow-covered hill. He sleds down and coasts on flat ground for a few feet. Tickled with excitement, Rashid runs up the hill for another invigorating race. About half way up the hill, he recognizes a friend of his has fallen off his sled. Rashid stops to help his friend and begins slowly pulling his friend back up the hill. Tired, Rashid and his friend finally make it to the top of the hill. Sketch a graph of a function of Rashid's height in terms of time.



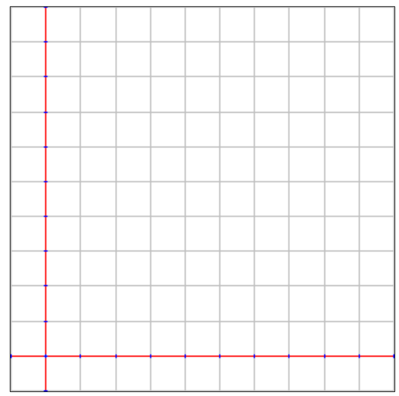
15. Roller coaster cars start out by slowly going up a hill. When all of the cars reach the top of the hill, the cars speed down the other side. Next, the cars are pulled up another, but smaller, hill. Racing down the other side, the cars race through a tunnel and come to a screeching halt where passengers are unloaded. Sketch a graph of a function of the roller coaster cars' speed in terms of time.



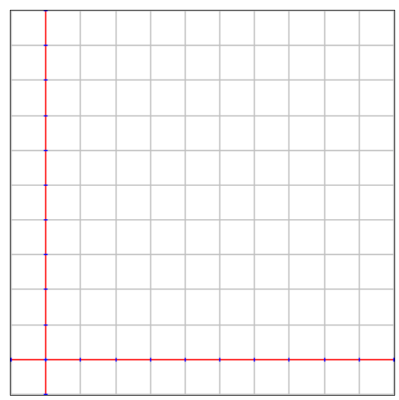
16. A dog is sitting on his owner's lap. When the owner throws the ball, the dog sprints after the ball and catches it mid-air. The dog trots back and plops back on the owner's lap. The owner throws the ball again; tired, the dog jogs over to the ball and lies down next to it. Sketch a graph of a function of the dog's speed in terms of time.



17. A function starts out increasing slowly then it increases faster and faster before hitting a maximum spike about halfway through the graph. From the spike it decreases quickly and then decreases slower and slower before finally leveling out toward the end of the graph.



18. A function starts off very high and stays level for a little while. It then drops quickly to about the halfway mark and stays level again for a little while. It then drops very close to the bottom and stays level after that.



Review Unit 3: Functions

No calculator necessary. Please do not use a calculator.

Unit 3 Goals

- *Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. (8.F.1)*
- *Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationships or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or table of values. (8.F.4)*
- *Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationships or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or table of values. (8.F.4)*
- *Describe qualitatively the functional relationship between two quantities by analyzing a graph. Sketch a graph that exhibits the qualitative features of a function that has been described verbally. (8.F.5)*

You may not use a calculator.

Determine if each of the following is a true function based on the equation or table. Explain how you know.

1. $x^2 + y^2 = 100$

x	-8	-6	0	6	8
y	± 6	± 8	± 10	± 8	± 6

2. $y = \frac{1}{2}x^2 - 6$

x	-4	-2	0	2	4
y	2	-4	-6	-4	2

Evaluate the given function using the given value as inputs.

3. $k = \frac{1}{2}j - 8$
   $j = 8$

4. $y = x^2 + 6$
   $x = 4$

5. $a = b - 47$
   $b = 100$

6. $g = -4h + 10$
   $h = 3$

Answer the following question in complete sentences.

7. Give a definition of a function in your own words.

8. Determine if the following describes a true function or not. Explain why or why not.

Input: Number of candy bars purchased, Output: The amount of money spent

9. Determine if the following describes a true function or not. Explain why or why not.

Input: Age of a person, Output: The number of hours spent playing video games

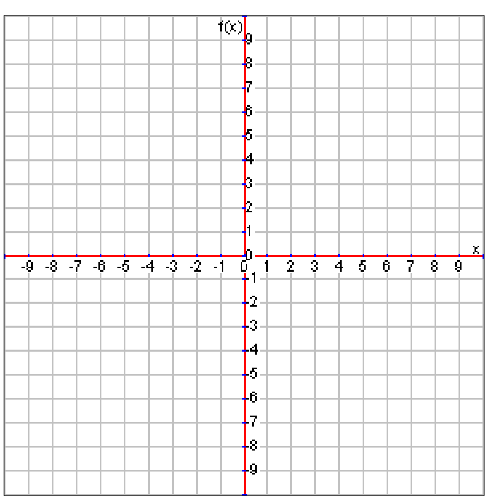
10. Determine if the following describes a true function or not. Explain why or why not.

Input: Number of students in a class, Output: Number of birthdays in the class

Graph the following functions by filling out the x/y ***chart using the given inputs (*** x ***values) or choosing inputs that you think are appropriate.***

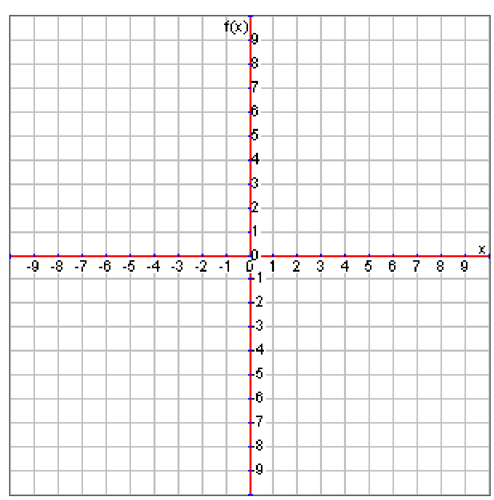
11. $y = \sqrt{x+8}$

x	-8	-7	-4	1	8
y					



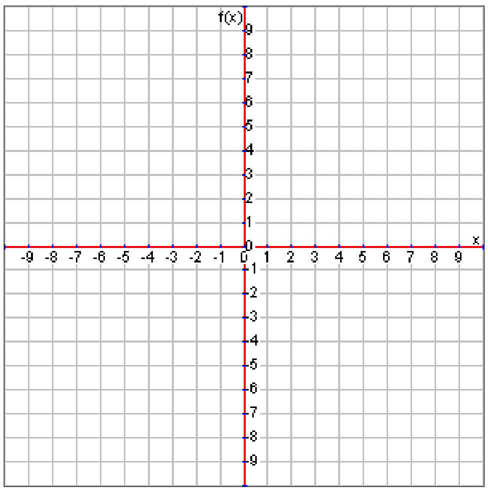
12. $y = x^2 - 5$

x					
y					



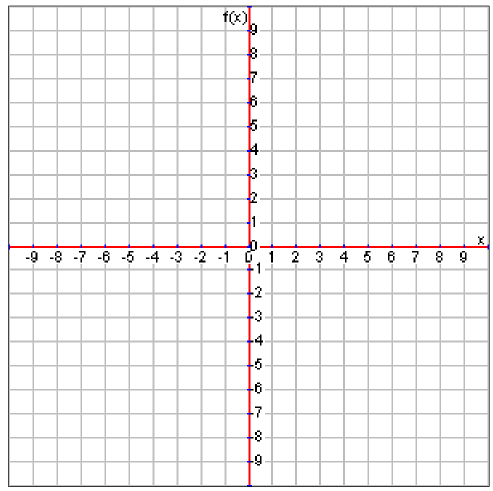
13. $y = \frac{2}{3}x - 3$

x					
y					



14. $y = 2x - 5$

x					
y					



Determine whether the following functions are linear or non-linear and explain how you know.

15. $y = 2(x - 4) + 3$

16. $y = 2(x - 4)^2 + 3$

17. $y = \frac{3}{4}x^4$

Answer the following questions about different types of functions.

18. Give an example of a linear function in equation form and explain how you know it is linear.

19. Give an example of a non-linear function in equation form and explain how you know it is non-linear.

Identify the rate of change, initial value, independent variable, and dependent variable. Then describe what the rate of change and initial value mean in the context of each situation. Finally, give the equation of the line.

20. The amount of money in dollars a mailman gets paid (p) to deliver mail to houses (h) is modeled by the following function: $p = 4h + 125$.

Rate of Change:___________

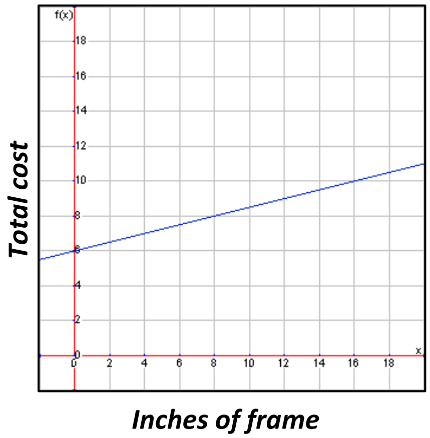
Initial Value:______________

Independent Variable:_____

Dependent Variable:______

Contextual Description of Rate of Change	Contextual Description of Initial Value

21. The function relating the cost of framing (c) to how many inches of frame around a picture (i) is shown by the following graph:



Rate of Change:___________

Initial Value:______________

Independent Variable:_____

Dependent Variable:_______

EQ of Line:________________

Contextual Description of Rate of Change	Contextual Description of Initial Value

22. Imagine you saved $4000 to spend winter break visiting your relatives in New York. It costs $1200 for the plane ticket and $300 per night for the hotel. Think of the function that demonstrates the cost (c) based on the number of nights (n) you spend.

Rate of Change:___________

Initial Value:______________

Independent Variable:_____

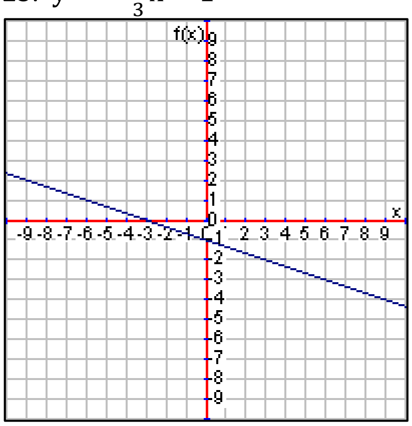
Dependent Variable:______

EQ of Line:_______________

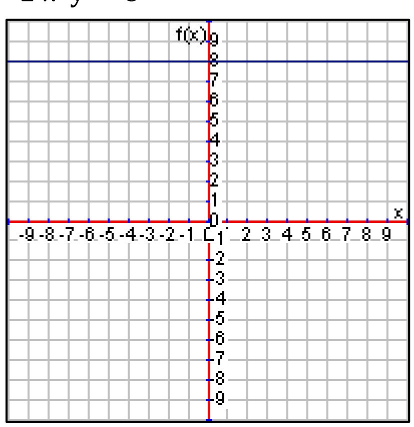
Contextual Description of Rate of Change	Contextual Description of Initial Value

Tell whether the following linear functions are increasing, decreasing, or constant.

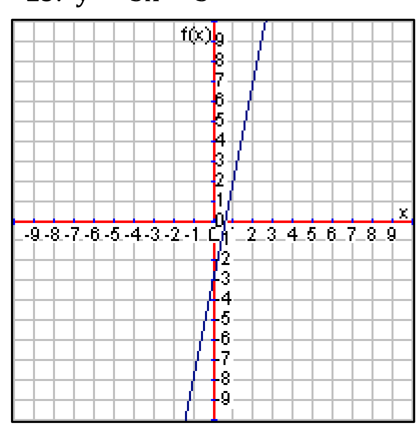
23. $y = -\frac{1}{3}x - 1$



24. $y = 8$

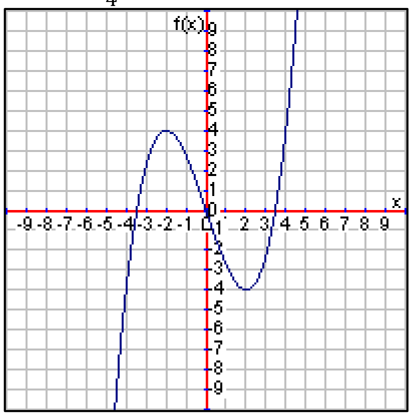


25. $y = 5x - 3$

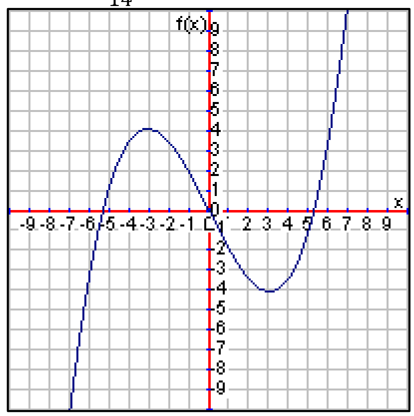


For the following functions tell where they are increasing and where they are decreasing. Then give the max or min of the function.

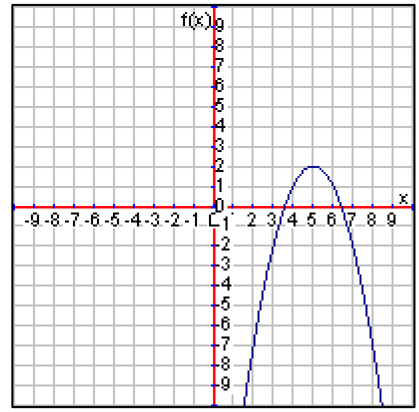
26. $y = \frac{1}{4}x^3 - 3x$



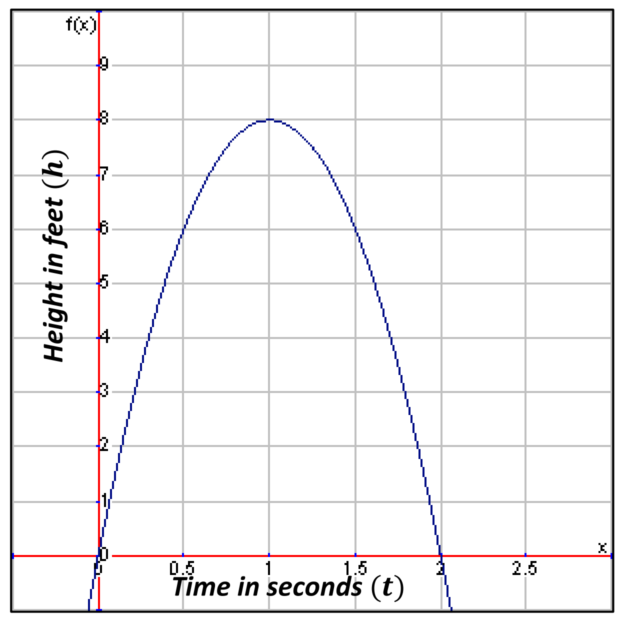
27. $y = \frac{1}{14}x^3 - 2x$



28. $y = -(x - 5)^2 + 2$



Use the following graph showing a function modeling the height (h) of an angry bird that is thrown in terms of time (t) in seconds to answer the questions.



29. What is the maximum height of the angry bird and when does the bird reach its max height?

30. What inputs makes sense in this context?

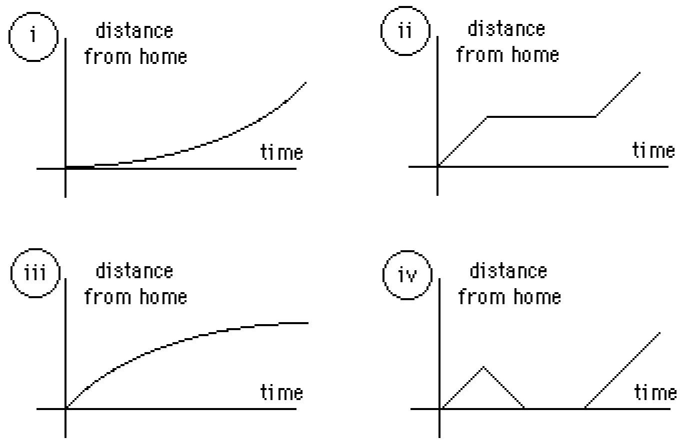
31. During what times is the bird's height increasing?

32. During what times is the bird's height decreasing?

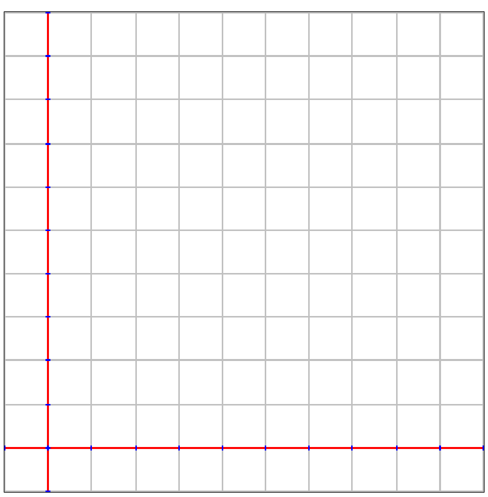
33. What are all the different heights the bird reaches?

Determine which graph matches the story and explain why.

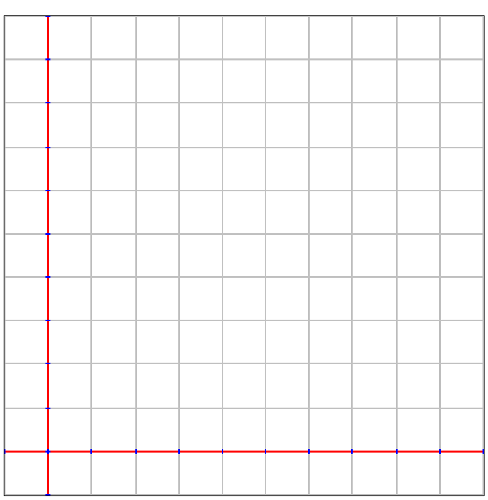
34. A young boy decided he was fed up with his parents and wanted to join the circus. After gathering his belongings in a hobo-style bag on a stick, he started running away from home very fast. He continually slowed down the longer he ran until he finally stopped about halfway to the circus when he realized he was being irrational.



Sketch a graph modeling a function for the following situations.



35. A seed was planted in the early spring. A sprout appeared and grew rapidly in the rainy spring. Growth nearly stopped during the dry and hot summer. In the middle of the summer, a rabbit ate the plant. Sketch a graph of a function of the plant's height in terms of time.



36. A child starts walking home from school. He stops at his friend's house on the way home to play video games. Around dinner time, his mom comes to pick him up and drive him home. Sketch a graph of a function of the child's distance from home in terms of time.

Unit 4: Linear Functions

Pre-Test Unit 4: Linear Functions

You may use a calculator.

Define variables and create equations for each of the following linear situations. *(5 pts; 1 pt for each variable definition, 3 pts for correct equation)*

1. You make the best cup of coffee when you use 1 teaspoon of sugar for every 4 ounces of coffee.

2. You start the day with $\$22$ in your pocket and then sell your produce at $\$5$ for 4 ears of corn.

Use the given equation to solve the linear questions. *(5 pts; 1 pt for correct substitution, 2 pts for correct solution method, 2 pts for correct answer)*

3. If you made a huge vat of tea that contained 288 ounces of tea (t), how many teaspoons of sugar (s) would you need if you followed the equation $s = \frac{1}{4}t$?

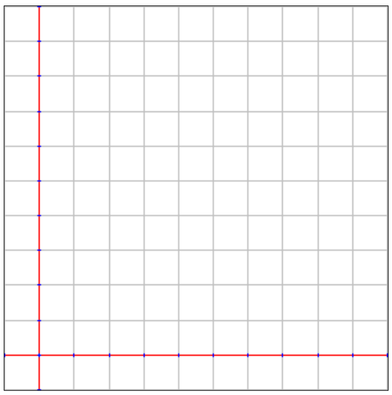
4. If you made $\$24$ per day as a waitress plus $\$3$ for every table you served (t), you would make a total amount of money (m) based on the following equation $m = 3t + 24$. How many tables would you have to serve to make $\$150$?

Create an equation to solve the linear question. *(5 pts; 3 pts for equation, 2 pts for correct answer)*

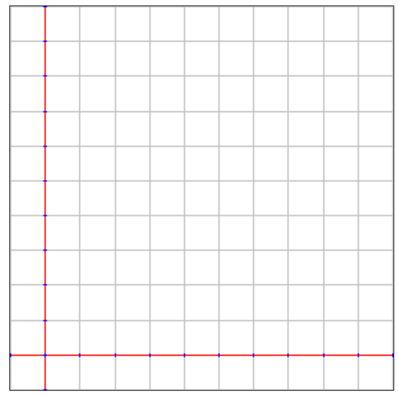
5. A man has $\$500$ in the bank when he started working at McDonald's making $\$8$ an hour. How many hours will he have to work to have a total of $\$2500$?

Create a graph for each of the following linear situations or equations. *(5 pts; 1 pt for each axis label, 3 pts for correct line drawn)*

6. Mrs. Buttersworth makes pancake syrup by mixing together 2 teaspoons of sugar (t) for every 5 ounces of maple syrup (m).

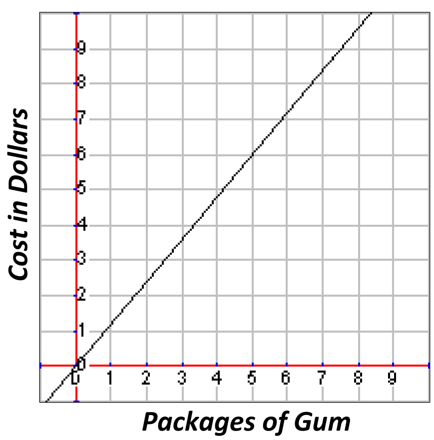


7. A race car gets a 2 meter head start and then travels at a speed of 1 meter (m) every 3 seconds (s).

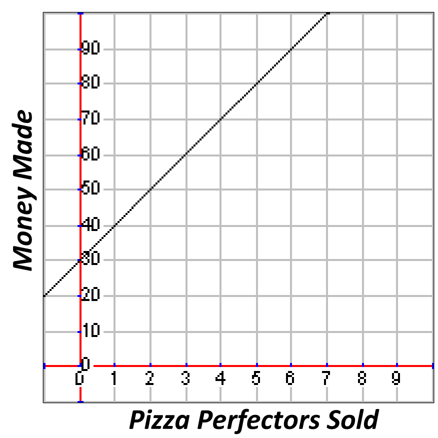


Use the given graph to solve the linear questions. *(5 pts; 3 pts if within $\frac{1}{2}$ a unit)*

8. How much would it cost for five packages of gum?

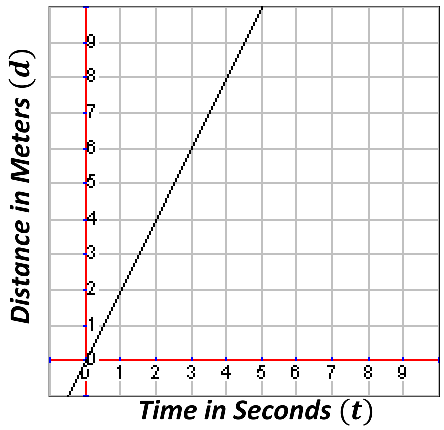


9. How many Pizzaz Pizza Pie Perfectors does the salesman have sell to make $100?

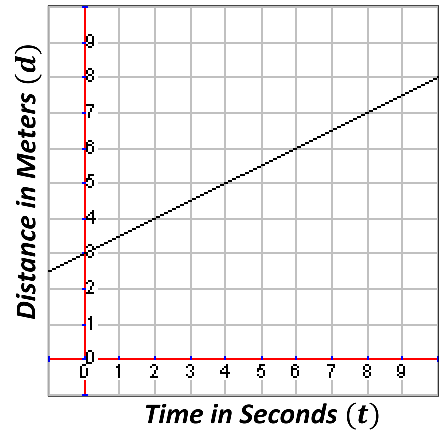


Give the equation for the following linear graph or table. *(5 pts; 2 pts for variables in correct place, 2 pts for correct slope, 1 pt for correct* y*-intercept)*

10. A remote control car's distance in meters (d) over time in seconds (t).



12. A remote control boat's distance down river in meters (d) over time in seconds (t).



11. The total cost (c) to buy llamas (l).

l	2	4	6	8	10
c	\$434	\$868	\$1302	\$1736	\$2170

13. The total cost (c) to buy dingos (d).

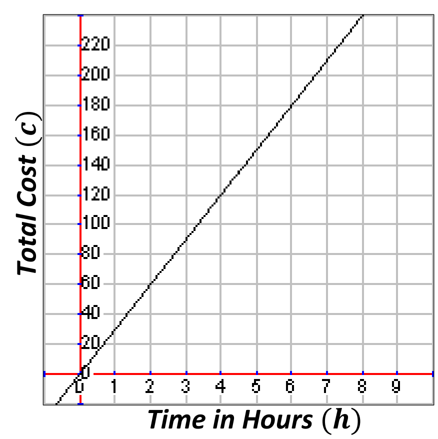
d	2	4	6	8	10
c	\$350	\$650	\$950	\$1250	\$1550

Fill out the table for each of the following linear situations, equations or graphs. *(5 pts; 1 pt for each)*

14. Herbert sells vacuum cleaner brushes where the total cost (c) for customers is \$25 for each set of 3 brushes (b) plus there is a \$15 account activation fee with the vacuum cleaner brush company.

b	3		9		15
c		\$65		\$115	

15. The graph shows how much it costs (c) to pay a Goat Whisperer to work with your goat herd by the hour (h).



h	2		6		10
c		\$120		\$240	

Use the given table to solve the linear questions. *(5 pts; partial credit at teacher discretion)*

16. How many grapes (g) would you need in a fruit salad with 8 strawberries (s)?

s	2	3	4	5	6
g	8	12	16	20	24

17. How many Bully Be Gone armbands (b) did the Janet sell if she made \$42 (m) today?

b	5	10	15	20	25
m	\$30	\$40	\$50	\$60	\$70

Answer the following questions comparing linear function equations and descriptions. *(5 pts; 2 pts for correct answer only)*

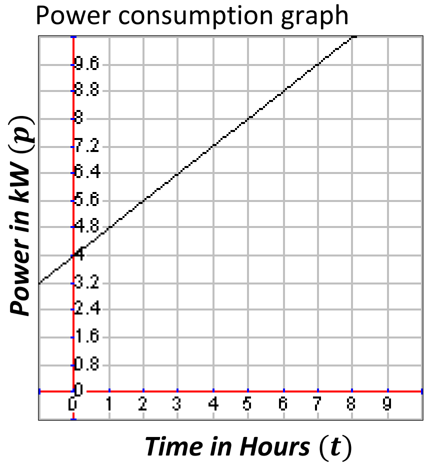
NASA is testing a series of new rockets to decide which one to use for the upcoming Moving to Mars Mission. Here is the information about the power consumption (p) in kW of electricity in terms of time (t) in hours of each rocket.

Rocket A: Power consumption is modeled by the equation $p = 1.6t$

Rocket B: Power consumption is modeled in the following table

t	2	4	6	8	10
p	5	8	11	14	17

Rocket C: Power consumption graph



Rocket D: Consumes approximately 14 kW in 10 hours plus an initial 3 kW at lift off

18. Which rocket uses the least power per hour and how do you know?

19. Which rocket uses the least power at lift off (initially) and how do you know?

20. Which rocket would use the least total power if the trip only took 5 hours and how do you know?

4.1 Equations of Linear Functions

In this section we'll still be using the slope-intercept form of proportional function equations. Let's take another look at defining the variables so we don't confuse the initial value for one of the variables we are comparing in the proportional relationship.

Defining the Variables and Writing Equations

We want to make sure we can correctly identify the initial value in each situation so that we define our variables correctly. The key to equations is finding the initial value first. This tells us what our dependent variable is. Then we look for the lowest terms proportion ratio or rate of change to use for our slope.

Example 1: *A company gained value at the rate of* $\$200{,}000$ *per day after its IPO (initial public offering) of one million dollars.*

Notice that the initial public offering is where the company started at. That is the initial value. As such, the $\$1{,}000{,}000$ tells us that the value of the company in dollars is going to be our independent variable. We can go ahead and define that now.

$$v = value\ of\ company\ in\ dollars$$

Now we ask, what is that dollar amount being compared to? The hint is in the word "rate" which compares the value in dollars and the number of days after the company's opening to the public stock market. That means we could define the independent variable as follows:

$$d = days\ since\ the\ IPO$$

Finally, we put it all together to make our equation. We know the value depends on the number of days since the initial public offering, so we substitute our slope and initial value into the generic form of a linear functions which is $y = mx + b$ as follows.

$$v = 200{,}000d + 1{,}000{,}000$$

Example 2: *To be a member at a gym you must pay a one-time* $\$25$ *entry fee plus* $\$10$ *per month.*

First, what is the initial value? There is a one-time fee of $\$25$ that is the initial value. This means that dollars, in this case the cost, is our dependent variable. We can define it as follows:

$$c = cost\ of\ gym\ membership$$

Next, think of the slope or rate of change in this problem. How does the rate change? It changes by $\$10$ per month. The entry fee does not change and is not part of the rate of change. What changes is how much you pay when compared to the number of months you sign up for at the gym. That means we are comparing the amount you pay and the number of months making the number of months the independent variable. We can define the independent variable and write the equation based on that as follows:

$$m = number\ of\ months\ for\ membership \qquad\qquad c = 10m + 25$$

What do we do if there is no initial value? How can we tell which is the independent and which is the dependent variable? Consider the following example.

Example 3: *To buy an all access pass at the local concert hall you must pay* $\$200$ *for every* 3 *months.*

First, what is the initial value? We don't have one, but we do have a rate of change that tells us we are comparing money (or cost) to the number of months you buy a pass for. Without an initial value, we could officially let either of those be the independent and dependent variable, but it does make more sense that the total cost would depend on the number of months you buy the pass for. Therefore, we'll define the variables and equation as follows:

$$c = cost\ of\ all\ access\ pass$$

$$m = \#\ of\ months\ to\ use\ all\ access\ pass$$

$$c = \frac{200}{3}m$$

Again, without the initial value, we could set up the equation in the opposite way as $m = \frac{3}{200}c$, but that intuitively may feel backwards.

One last warning: there could be extra information in the problems that are not necessary. Just because a number is there, doesn't mean it has to be used. Think about whether or not it fits the problem and has to do with what you are trying to solve.

Solving Linear Equations

Let's solve some proportional function problems that have initial values using equations. Consider the gym membership situation from above. We could ask how much it would cost to be a member for one year. That would mean that $m = 12$ since one year is twelve months. We'll substitute and solve as follows:

$$c = 10m + 25$$

$$c = 10(12) + 25$$

$$c = 120 + 25$$

$$c = 145$$

This means that the cost for one year would be $145 total. We could also solve this problem if we knew the cost and not the number of months. For example, how many months of membership could you buy if you had $365? That would mean that $c = 365$, so we'll substitute and solve as follows:

$$c = 10m + 25$$

$$365 = 10m + 25$$

$$365 - 25 = 10m + 25 - 25$$

$$365 - 25 = 10m \cancel{+ 25 - 25}$$

$$340 = 10m$$

$$\frac{340}{10} = \frac{10m}{10}$$

$$\frac{340}{10} = \frac{\cancel{10}m}{\cancel{10}}$$

$$34 = m$$

So we could buy 34 months of membership with \$365.

Finding the Initial Value (y-intercept) or Rate of Change (Slope)

Pretend that a company gained value by \$100 per day and after 7 days was worth \$850. What was the initial value of the company? Let's start by defining the variables and writing an equation as best as we can. We know we are comparing the value of the company and the number of days. So we might do this:

$$v = value\ of\ company\ in\ dollars$$

$$d = days\ since\ the\ IPO$$

$$v = 100d + b$$

Notice that we don't know the initial value because that's what the question asked us to find. Therefore we'll just leave the variable b in the equation as a place holder for our initial value. However, we do know that after 7 days, or when $d = 7$, the company was worth \$850, or $v = 850$. Let's substitute those values in and solve for the initial value, or b.

$$v = 100d + b$$

$$850 = 100(7) + b$$

$$850 = 700 + b$$

$$850 - 700 = 700 + b - 700$$

$$850 - 700 = \cancel{700} + b \cancel{- 700}$$

$$150 = b$$

Therefore the initial value of the company was \$150.

Similarly we could solve for the slope, or rate of change, in a problem. Consider an appliance salesman who gets paid \$50 every day plus some unknown amount for every appliance they sell that day. Let's say the appliance salesman made \$475 after selling 17 appliances. How much does he get paid per appliance? Again, we'll define our variables and write the equation first.

$$p = total\ pay$$

$$a = number\ of\ appliances\ sold \qquad\qquad p = ma + 50$$

Notice that the \$50 every day is an initial value for the pay each day. That never changes. We also left the m in the equation because we don't know the rate he gets per appliance. So let's substitute what know, which is that when $a = 17$ then $p = 475$, and solve for m.

$$p = ma + 50$$

$$475 = m * 17 + 50$$

$$475 = 17m + 50$$

$$475 - 50 = 17m + 50 - 50$$

$$475 - 50 = 17m \cancel{+ 50 - 50}$$

$$425 = 17m$$

$$\frac{425}{17} = \frac{17m}{17}$$

$$\frac{425}{17} = \frac{\cancel{17}m}{\cancel{17}}$$

$$m = 25$$

This means that the salesman gets paid \$25 per appliance that he sells.

Comparing Linear Equations

Now we have two values to compare in equations, the rate of change and the initial value. Let's say that a restaurant wants to buy paper plates in bulk so they want to join a wholesale store (like Sam's Club). There are three stores they could join:

Store 1: Charges a membership fee of \$100 and charges \$25 per bulk package of paper plates.

Store 2: Charges a membership fee of \$50 and charges \$30 per bulk package of paper plates.

Store 3: Charges a membership fee of \$200 and charges \$20 per bulk package of paper plates.

Let's start by defining variables and writing equations for each store.

$$c = total\ cost$$

$$p = number\ of\ bulk\ packages\ of\ paper\ plates\ purchased$$

Store 1: $c = 25p + 100$ **Store 2:** $c = 30p + 50$ **Store 3:** $c = 20p + 200$

Now it would be difficult to ask which store is cheapest because that depends on how many packages of paper plates you buy. However, we can ask which store has the cheapest rate for packages of paper plates. Which one does? Yes, Store 3 with the price of \$20 per package.

Which store has the highest membership fee? That would be Store 3 as well since it charges \$200.

Which store would give the overall cheapest price if the restaurant needed to buy two packages of paper plates? What if it were ten packages? What if it were twenty packages? Substitute each of those values in for p and solve for c.

Two packages

Store 1: $c = 25(2) + 100$

$c = 150$

Store 2: $c = 30(2) + 50$

$c = 110$

Store 3: $c = 20(2) + 200$

$c = 240$

In this case Store 2 is the best store to buy from.

Ten packages

Store 1: $c = 25(10) + 100$

$c = 350$

Store 2: $c = 30(10) + 50$

$c = 350$

Store 3: $c = 20(10) + 200$

$c = 400$

In this case there is a tie between Store 1 and Store 2.

Twenty packages

Store 1: $c = 25(20) + 100$

$c = 600$

Store 2: $c = 30(20) + 50$

$c = 650$

Store 3: $c = 20(20) + 200$

$c = 600$

In this case there is a tie between Store 1 and Store 3. You should be able to extrapolate (think ahead) and see that if the restaurant needs to buy more than twenty packages of paper plates in bulk, Store 3 will always be cheaper. Can you explain why?

Proportion = Linear without Initial Value

We should note that linear functions can also be proportional functions. You may recall that a proportion typically looks like one fraction equal to another fraction: $\frac{a}{b} = \frac{c}{d}$. You might also note that the two fractions there look nothing like our slope-intercept form of a linear equation which is $y = mx + b$. Let's see if we can reconcile that fact first.

First think of a proportional relationship such as the farmer being able to gather 25 gallons of milk daily for every 3 cows. That means if there are zero cows, there is zero milk. In other words the initial value is zero. Last unit we could have written the equation for this relationship as: $g = \frac{25}{3}c$ where g is the number of gallons of milk the farmer gathers daily and c is the number of dairy cows he has. That equation is definitely not a fraction equals a fraction which is typically how we think of proportions.

However, the point of a proportion is to have two equivalent ratios where each ratio is a comparison of two values. In other words, we typically have a proportion that reads something like, "The number of gallons of milk gathered daily (g) compared to the number of dairy cows (c) is 25 to 3." We might write that proportion as follows: $\frac{g}{c} = \frac{25}{3}$. Now the question is whether or not the two equations are the same. Let's see if we can transform the proportion into the linear equation.

$$\frac{g}{c} = \frac{25}{3}$$

$$c * \frac{g}{c} = \frac{25}{3} * c$$

$$\cancel{c} * \frac{g}{\cancel{c}} = \frac{25}{3} * c$$

$$g = \frac{25}{3}c$$

Thus we see that every proportional function can be written as a linear function. Since this is true, we may as well stick with our familiar linear function form, $y = mx + b$, except that for proportional functions the b value will be zero. This means that the lowest terms proportion ratio is really just the slope of linear function.

Is it Proportional?

If a proportional function is a linear function with an initial value of zero, we can now test if an equation is proportional as well as linear by looking for that initial value. Consider the following three equations and decide if each is proportional as well as linear. In each equation, c is the cost of fixing a leaking ridge cap on a roof based on f, the length in feet that needs to be fixed.

Company 1	**Company 2**	**Company 3**
$c = 12f$	$c = 10f + 50$	$c = 15f$

Which of these have no initial value? It looks like Company 1 and Company 3 have no initial value thereby making them proportional. In other words the ratio of cost to feet fixed is 12 to 1 for Company 1 and 15 to 1 for Company 3.

Company 2 is not proportional because there is an initial value (perhaps a labor fee) of 50. It's still linear because it is in the form $y = mx + b$, where the slope is $\frac{10}{1}$, but the initial value makes it not proportional.

Lesson 4.1

Define variables and create equations for each of the following linear situations.

1. It will cost $\$45$ to replace the chain on your bicycle plus $\$15$ per hour of labor.

2. It takes you 11 minutes for every 2 miles you run.

3. You spend 5 minutes on every 2 questions on the test.

4. It takes you 9 minutes for every 2 toilets to clean, and you have already spent 45 minutes cleaning the house.

5. Your parents pay you $\$5$ for every hour you babysit.

6. The musical cast started with $\$1200$ in donations and earns $\$45$ for every 6 tickets sold.

7. At the beginning of the year, you receive 20 free participation points. You can lose 2 participation points every time you forget to bring your supplies to class.

8. Student council ordered one pizza for every 4 students that are attending the after school dance.

9. Which of the problems above are proportional and how do you know?

Use the given equation to solve the linear questions.

10. If a roller coaster starts 12 meters above the ground and climbs 2 meters every second (s), the roller coaster's height (h) would be based on the equation $h = 2s + 12$. How long would it take to reach the top of the hill that is 80 meters above the ground?

11. If it is going to cost you $\$525$ dollars to start a lawn care business with your friend, but you will earn an average of $\$73$ for every 4 yards (y), your profit (p) is based on the equation $p = \frac{73}{4}y - 525$. How much profit would you make if you were scheduled to mow 48 yards the first summer?

12. It was raining at a rate of 1 inch every 3 hours. If it rained at that constant rate for 6 hours (h), how many inches of rain (r) would there be if you followed the equation $r = \frac{1}{3}h$?

13. It costs \$550 for buses to transport students to the C. A. N. D. L. E. S. Holocaust Museum in Terre Haute, Indiana. If the museum charges \$5 for every 2 students (s), the total cost (c) of the trip is based on the equation $c = \frac{5}{2}s + 550$. How much would it cost to bring 196 students?

14. School policy states that there must be one teacher for every 24 students. If there are 120 students attending the field trip, how many teachers would be necessary to chaperone if you followed the equation $t = \frac{1}{24}s$?

15. If you spent $\$10.35$ total (t) purchasing songs online for $\$1.15$ each, how many songs did you buy (s) if you followed the equation $t = 1.15s$?

16. If a rose bush is planted when it is 14 inches tall and it grows three inches every five days (d), it's height (h) is based on the equation $h = \frac{3}{5}d + 14$. How many days would it take for the rose bush to be 50 inches tall?

17. The average computer consumes 130 watts of power per hour. If your energy bill shows you used a total of 845 watts (w) for a single day, for how many hours (h) was your computer running if you followed the equation $w = 130h$.

18. Which of the above problems represent proportional situations and how do you know?

Create an equation to solve the following questions.

19. A mama bird must gather 5 worms for every 2 baby birds in order to provide them with adequate nutrition. If she has 6 baby birds, how many worms must she find?

20. At the Charleston Bowling Lanes, it costs \$2 to rent shoes plus $\$1.50$ per game of bowling. How many games would you be able to bowl for $\$11$?

21. In the Tour De France Lance Armstrong pedaled at an average pace of 49 kilometers per hour. If the race is 3479 kilometers long, how long did Lance spend cycling?

22. You've been working on your math homework for 25 minutes already. If it takes about 10 minutes for every 3 problems, how long will you have spent on homework if you only have 6 problems left?

23. A famous fashion designer spent $\$9.5$ million on fabric for her new spring line. If she earns approximately $\$1.2$ million for each dress she sells, how many dresses will she have to sell to make a profit of $\$14.5$ million?

24. Your parents put a down payment on your car, but they are requiring you to pay the monthly payment of $\$85$. If you will have to pay a total of $\$2125$ for the car, how long will it take you to pay it off?

25. The Pick Your Burgers restaurant spent $\$98{,}145$ on food, utilities, and labor this month. If the average table of 4 customers spent $\$73$, how much profit did the restaurant make if they served 7092 customers?

26. On average, it takes 5 bales of hay to feed 2 horses. If you have 9 horses, how many bales of hay will you have to purchase?

Answer the following questions comparing linear equations and descriptions.

You are deciding which gas company to choose as you travel across the country on a long vacation with your family. Here is the information about the cost (c) for gallons of gas (g) for each company.

Gas Up Charges $\$4.01$ for each gallon of gas	**Automart** Charges $\$81$ for 20 gallons of gas
The Fuel Shop Cost is modeled by the equation $c = 4.03g$	**Full Tank** Cost is modeled by the equation $c = 4.10g$

27. Which company charges the most per gallon of gas? How do you know?

28. Which company charges the least per gallon of gas? How do you know?

29. How much would each company charge you for 12 gallons of gas? Which is the cheapest?

30. If you had $\$100$ to spend on gas, how many gallons could you buy from each gas station?

Dr. Kal is studying how age and gender affect calorie expenditure. Here is the information about the number of calories burned (c) based on the number of miles (m) walked in a day.

Paul (25) Burns 1390 calories plus 1040 calories from walking 10 miles	**Ishmael (58)** Burns 1305 calories plus 220 calories from walking 2 miles
Jerika (31) Calorie expenditure is based on the equation $c = 98m + 1225$	**Pamela (62)** Calorie expenditure is based on the equation $c = \frac{205}{2}m + 1189$

31. Who burns the most calories per mile, and how do you know?

32. Who burns the least calories per mile, and how do you know?

33. Who burns the most calories without walking, and how do you know?

34. How far would each person have to walk (to the nearest hundredth) to burn 2000 calories?

35. If each person walks 10 miles, who burns the most calories for that day?

4.2 Graphs of Linear Functions

In Unit 3 we graphed functions, so we already know how to graph linear (and proportional) functions. After a review of that, we'll discuss solving with a graph and comparing graphs.

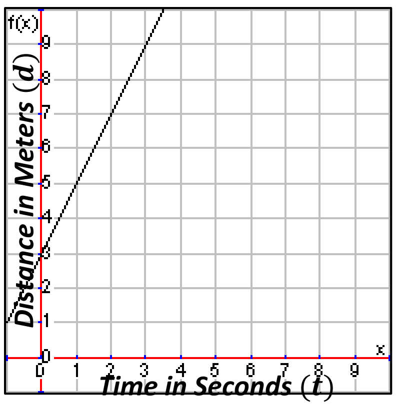
Graphing Linear Functions with Initial Values

If a linear function is given to us as an equation, we simply graph it using an x/y chart like we did previously in Unit 3.

Example 1: *Adam was given a 3 meter head start and runs at 2 m/s which is the equation* $d = 2t + 3$.

Notice in this case we don't have an x and y as the variable, but we know that the variable d is equivalent to the variable y since it is the dependent variable or output and the variable t is equivalent to the variable x since it is the independent variable or input. Thus we can graph as follows:

t	0	1	2	3
d	3	5	7	9



Example 2: *A company uses* 2 *bottles of ink to print* 3 *t-shirts and the machine uses one bottle to warm up.*

Let's start by defining the variables and writing an equation:

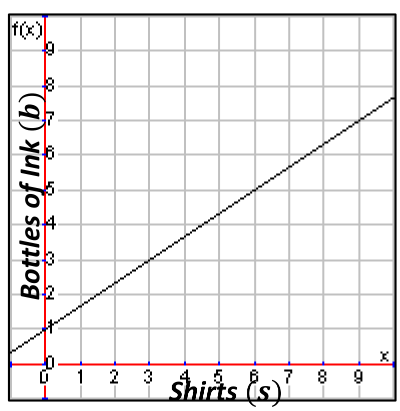
$b = bottles\ of\ ink\ used$

$s = shirts\ printed$

$b = \frac{2}{3}s + 1$

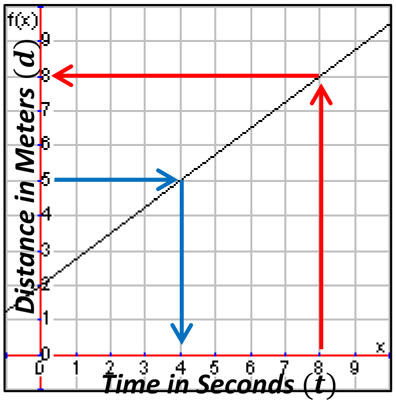
Now graph with an x/y chart as follows:

s	0	3	6
b	1	3	5



Solving Linear Situations with Graphs

If we are given a graph with an initial value, we can still solve linear problems. Consider the following graph and answer the question, "How far did the toy car travel after 8 seconds?"

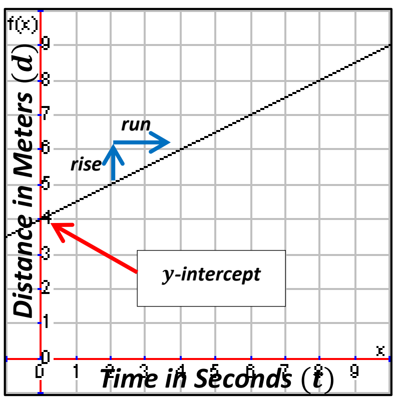


To solve using the graph, simply go to 8 seconds and move up to the line. How far did the car travel? It looks like the toy car has traveled 8 meters at that point.

Similarly we could ask a question such as, "How long did it take the toy car to travel 5 meters?" Go up to five meters and move over to the line. Looking straight down from that point we see that it took the toy car 4 seconds.

Getting Equations from Graphs with Initial Values

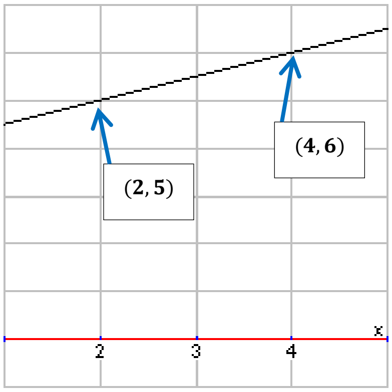
We can still get equations from graphs with initial values, but it will take the extra step of looking for the y-intercept on the graph.



First, find the slope using two points from the graph. Notice that on this graph we have a point on the line at $(2,5)$ and $(4,6)$. Those are not the only two nice integer value points on this line, but we only need two. Since the slope can be thought of as $\frac{rise}{run}$, we simply look for how far the line rises and runs between those two points.

It goes from 5 meters to 6 meters, so that is a rise of 1. It goes from 2 seconds to 4 seconds which means a run of 2. Therefore the rise over the run is $\frac{1}{2}$. This is our slope, rate of change, or lowest terms proportion ratio (whatever we want to call it).

We are still missing the y-intercept, or initial value, but we can clearly see it on the graph. The line crosses the y-axis at 4, so our initial value is 4. Now we can write the equation: $d = \frac{1}{2}t + 4$. What if we didn't have the whole graph, but only those two points to work with? Let's zoom in and find out.



We still have our two points and therefore can find the slope, but we can't see the initial value. However, since we know the slope and a point on the line, we can substitute those values in to find the initial value.

$$d = mt + b$$

Substitute the slope

$$d = \frac{1}{2}t + b$$

Substitute one point

$$6 = \frac{1}{2}(4) + b$$

$$6 = 2 + b$$

Solve for initial value

$$6 - 2 = 2 + b - 2$$

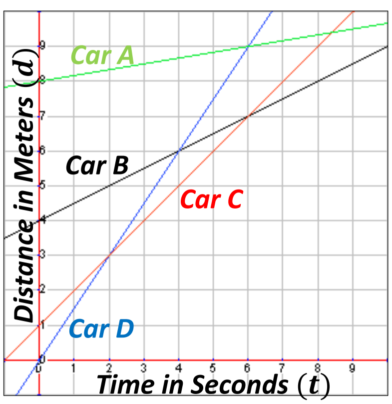
$$4 = b$$

Write the equation

$$d = \frac{1}{2}t + 4$$

Comparing Linear Function Graphs

We may need to compare graphs to other graphs or compare graphs to equations. Just as with equations, one of the main comparison points is the slope of the graph. For example, looking at the following graph, we could ask which toy race car is faster.



To compare these race cars, we need to know their speeds. The speed is a measure of distance over time, which is the slope or steepness of each line. So let's find the slope for each race car. After finding the rise and run for each car, we should come to the following values:

Car A: $\frac{1}{6}\ m/s$ **Car B:** $\frac{1}{2}\ m/s$ **Car C:** $1\ m/s$ **Car D:** $\frac{3}{2}\ m/s$

Based on this information, we see that Car D is the fastest car. We also know that Car A is the slowest. The curious thing is that Car A, even though it is the slowest, stays above most of the lines for the majority of the graph. This is because of the initial value for each car.

The initial value effectively is a head start for each of the cars. Notice the following initial values:

Car A: $8\ m$ **Car B:** $4\ m$ **Car C:** $1\ m$ **Car D:** $0\ m$

The reason the slowest car is ahead in the race for most of the graph is because of its tremendous head start. It received a whopping 8 meter head start! Also note that Car D did not have a head start and is therefore proportional.

We could also ask lots of other interesting questions now. For example, if the race were only 5 seconds long, which car would travel the farthest? Go to 5 seconds on the graph and move up. Which car is highest? Car A (again, due to the head start).

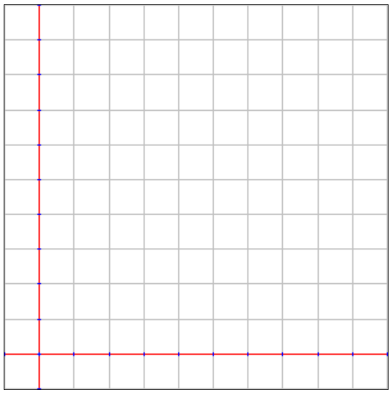
If the race were 10 meters long, which car would get to the finish line first? Go up to 10 meters and look over. Which car do you hit first? Car D finishes somewhere between 6 and 7 seconds.

If the race were 10 meters long, what order would the cars finish in? Go up to 10 meters and look over. Car D finishes first and then Car C, but we're not sure when Car A and B finish. It turns out that Car A and B would tie for last place at 12 seconds. Can you prove it?

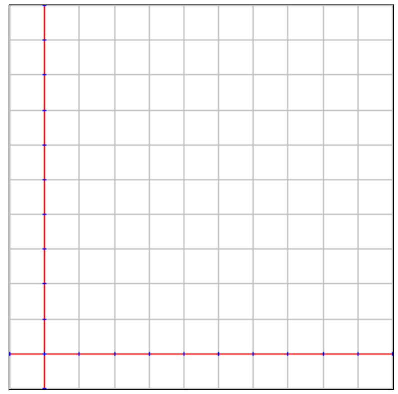
Lesson 4.2

Create a graph for each of the following linear situations or equations.

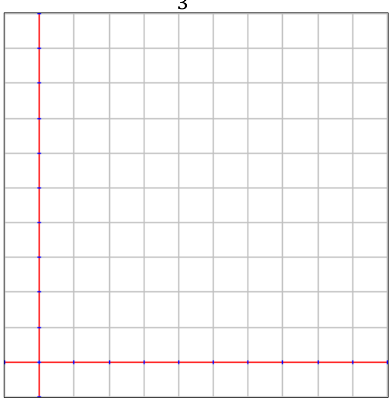
1. At Pizza Hut It costs \$8 for each large pizza plus \$5 for a delivery tip.



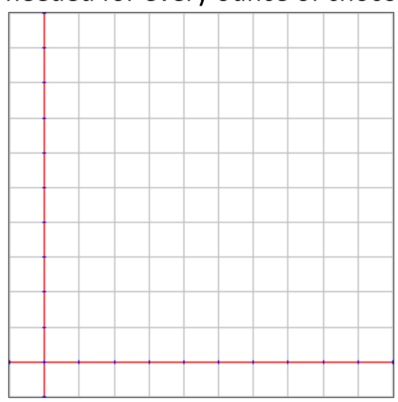
2. It costs \$40 per ticket to Six Flags plus \$100 for gas there and back.



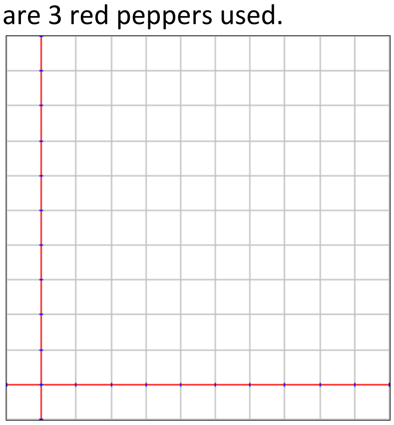
3. The number of lives (l) based on the number of levels completed (c) is determined by the following equation: $l = \frac{1}{3}c + 4$



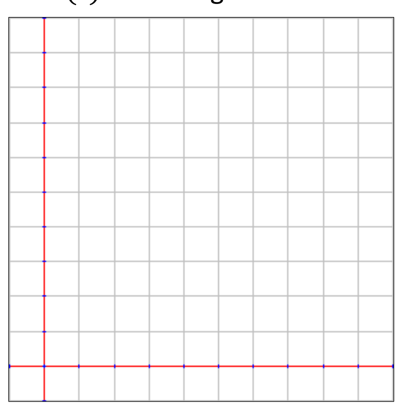
4. When making fudge, four ounces of sugar are needed for every ounce of chocolate.



5. For every 2 green peppers used in a salsa there are 3 red peppers used.

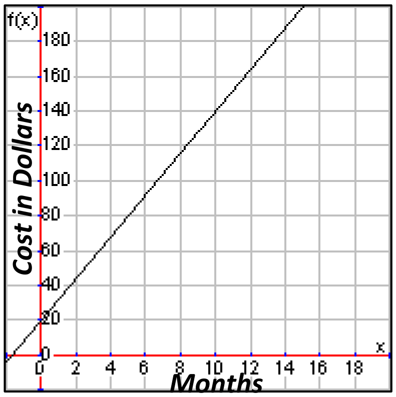


6. Mutant alien frogs from Zappax have a number of feet (f) based on the number of toes they are born with (t) according to the following equation: $f = \frac{1}{7}t$.

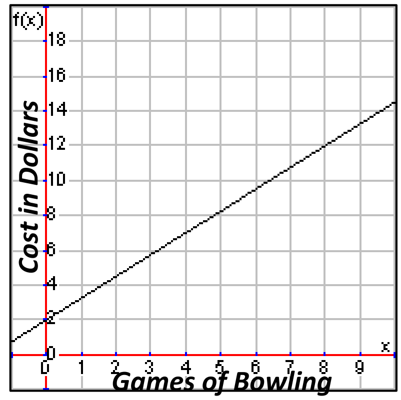


Use the given graph to solve the linear questions.

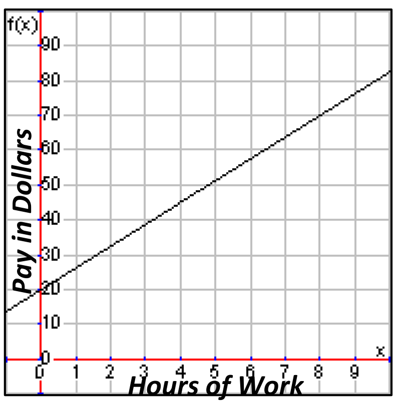
7. How much will it cost for ten months of internet service?



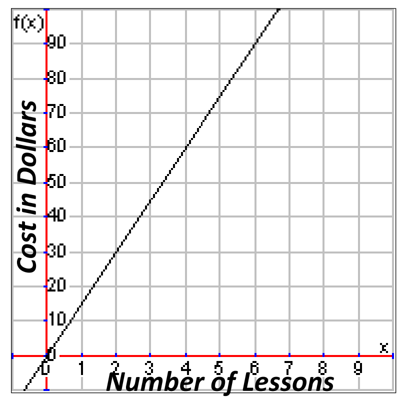
8. How many games of bowling can you play if you can spend $12?



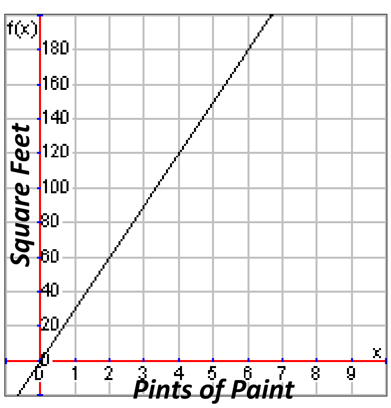
9. How many hours would you have to work to earn $70?



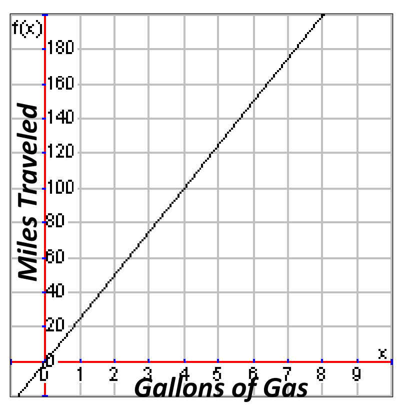
10. How much would it cost for four lessons?



11. How many pints of paint should you buy if you have to paint 120 square feet?



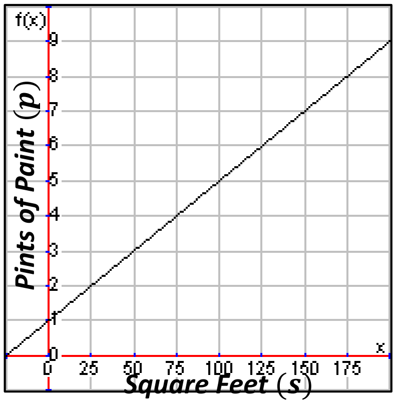
12. How many miles can you travel if you have four gallons of gas left in your tank?



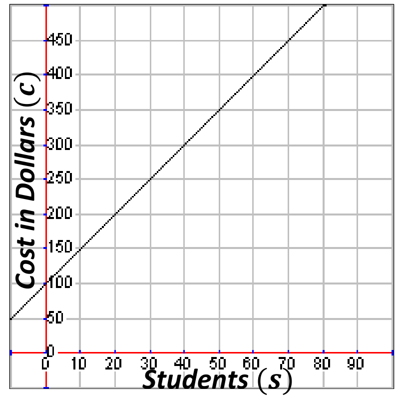
13. Which of the above graphs are proportional situations and how do you know?

Create an equation for the following linear graphs.

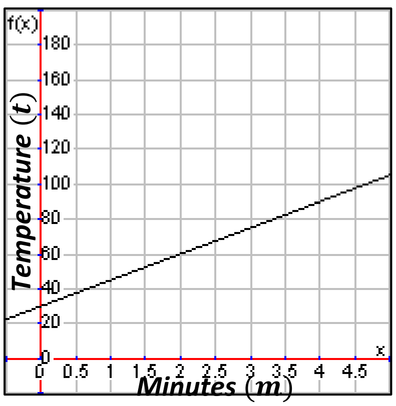
14. Number of pints of paint (p) needed for a certain number of square feet (s)



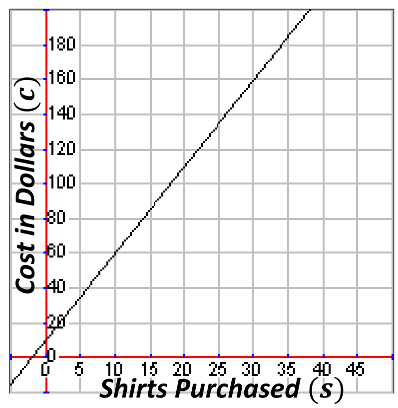
15. The cost (c) of a field trip based on the number of students (s) attending



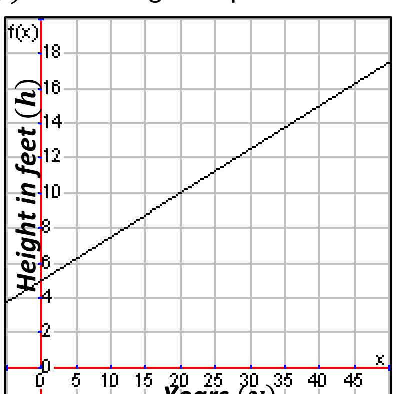
16. Temperature (t) of water per minute (m) of time on the stove



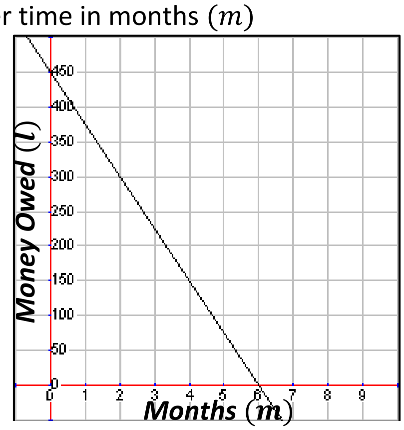
17. Cost (c) of an order depending on the number of shirts (s) purchased



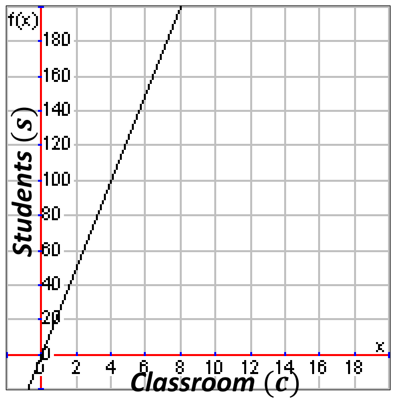
18. A tree's height (h) based on the number of years (y) since being transplanted



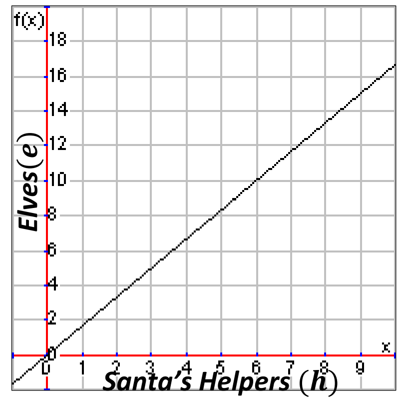
19. Money you owe on your loan (l) for your first car over time in months (m)



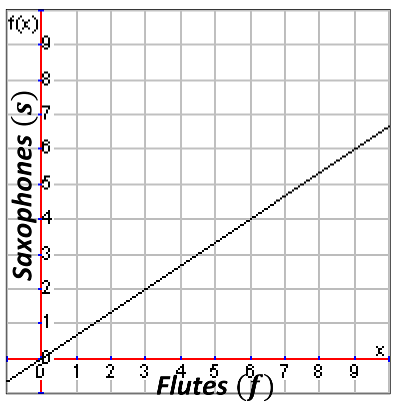
20. Number of students (s) in every classroom (c)



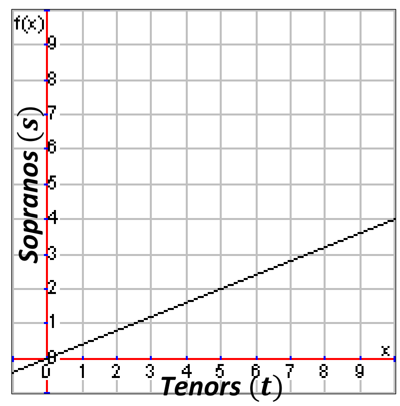
21. Number of elves (e) for Santa's helpers (h)



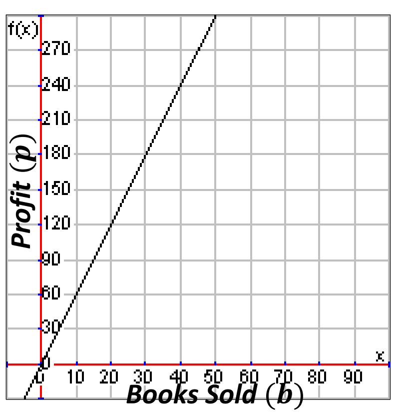
22. Number of saxophones (s) compared to the number of flutes (f) in an orchestra



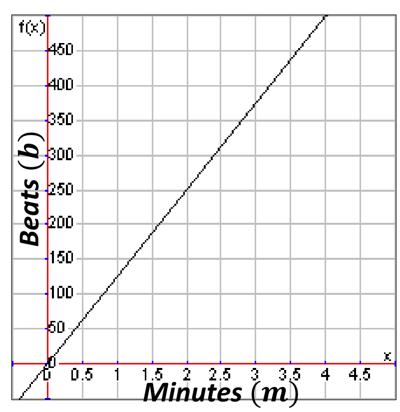
23. Number of sopranos (s) compared to the number of tenors (t) in a choir.



24. Amount of profit (p) based on the number of books sold (b)



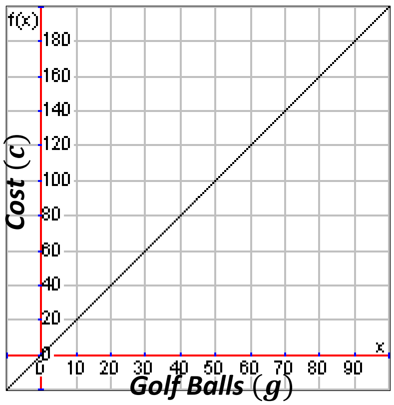
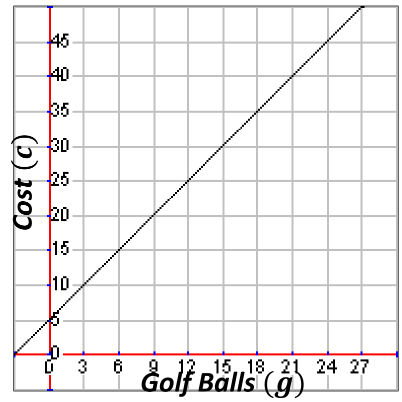
25. Number of beats (b) per minute (m) in a hip-hop song



26. Which of the graphs from problems 14 to 25 are proportional and how do you know?

Answer the following questions comparing linear function equations, graphs and descriptions.

Various golf ball manufacturers offer deals for packs of golf balls. Here is the information about the total cost (c) *for golf balls* (g) *including shipping costs.*

Callaway Charges a fee of $\$10$ for shipping and $\$5$ for 3 golf balls	**Nike** Cost is modeled by the equation $c = \frac{5}{2}g + 5$
Titleist 	**Top-Flight** 

27. Which manufacturer has the cheapest cost per golf ball, and how do you know?

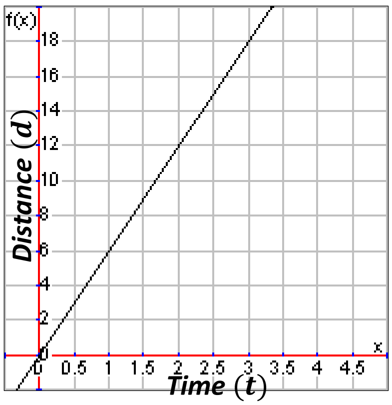
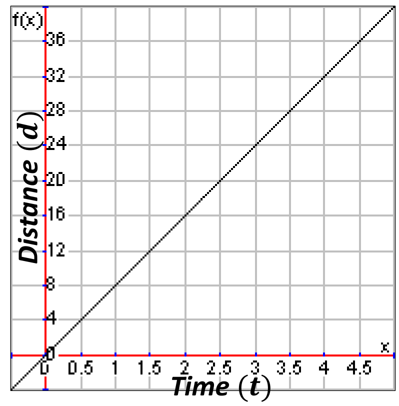
28. Which manufacturer has the cheapest shipping fee, and how do you know?

29. How many golf balls could you buy at each company for $\$200$? Which manufacture would give you the most golf balls for that amount of money?

30. Which manufacturer would be the cheapest if you wanted to buy 30 golf balls?

Answer the following questions comparing proportional function equations, graphs and descriptions.

Scientists are studying how location affects the speed of a bottlenose dolphin. Here is the information about the distance (d) in kilometers a dolphin traveled in terms of time (t) in hours.

Dolphin in Gulf of Mexico Swims 11 kilometers in 2 hours	**Dolphin in Mediterranean Sea** Distance is modeled by the equation $d = \frac{35}{4}t$
Dolphin in Indian Ocean 	**Dolphin in North Atlantic Ocean** 

31. Which location has the fastest dolphin?

32. Which location has the slowest dolphin?

33. How far could each dolphin travel in 4 hours? Which location has the dolphin that went the farthest?

34. How long would it take each dolphin to swim 100 kilometers? Which location has the dolphin that finished in the shortest amount of time?

4.3 Tables of Linear Functions

The final concept we'll cover this unit is the table form of linear functions. Just like in the previous sections, some of these linear functions may also be proportional which means there will be no initial value. We'll start by assuming there is an initial value..

Filling Out a Table from Equations and Graphs

Perhaps the simplest thing that we can do is fill out a table based on an equation or a graph. Since the table is designed to look at specific input/output pairings, we may need to pick appropriate inputs just like we did when graphing functions. Let's fill out the tables for the following examples.

Example 1: $d = 5t - 3$

t	0		2		4
d		2		12	

For this table, note that some of the values have been given for us. If we are given an input (t in this case), then simply input that into the equation to find the output. For example, we have $t = 0$ as an input, so substitute in as follows: $d = 5(0) - 3$. Multiplying five by zero and then subtracting three gives us $d = -3$.

t	0		2		4
d	-3	2		12	

If we are given an output (d in this case), substitute that into the equation and solve for the input. For example, note that we are given $d = 2$ as an output. Substitute into the equation and solve as follows:

$$d = 5t - 3$$

$$2 = 5t - 3$$

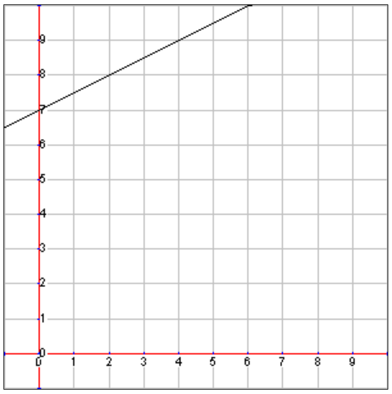
Adding three to both sides and then dividing by five gives us that $t = 1$.

t	0	1	2		4
d	-3	2		12	

It also works to look for patterns. For example, we see that the d values are going up by fives, so the next gap for d should be 7, then the 12 is given to us, and the last should be 17. We similarly know that the missing t value is 3. So our final table filled out (with our solutions in red) should look like this:

t	0	1	2	3	4
d	-3	2	7	12	17

Example 2:



In this example, we aren't given axis labels, so we'll use the standard x and y variables. Here's the table we want to fill out:

x	2		6		10
y		9		11	

For the x values (the input), we can guess from the pattern that we're counting by twos. Can you think of a reason for this based on the graph?

x	2	4	6	8	10
y		9		11	

Now that we have all the inputs, let's get the outputs (y values). At an input of 2, the output on the graph is 8. At an input of 6, follow the graph up to see that the output is 10. However, the input of 10 is off the graph. What will we do? Look for a pattern! Notice that the y values are going up by 1 for every 2 in the x direction. Following this pattern, we know that the last output should be 12.

x	2	4	6	8	10
y	8	9	10	11	12

Writing the Equation of a Table

To get a fuller picture of a function, we may want to look at the equation of the function. Since we are currently talking about linear functions with an initial value, we are dealing with equations that will be in the form $y = mx + b$ where m is the slope, rate of change, or lowest terms proportion ratio and b is the initial value or y-intercept. To get the slope, we need the rise (the change in the output) and the run (the change in the input). Examine the following table to see if you can identify the slope.

$In\ (q)$	2	4	6	8	10
$Out\ (z)$	2	7	12	17	22

Check the difference between each of the adjacent outputs (outputs that are next to each other) for the rise. Check the difference between each of the adjacent inputs for the run.

$+2$ $+2$ $+2$ $+2$ Run

$In\ (q)$	2	4	6	8	10
$Out\ (z)$	2	7	12	17	22

$+5$ $+5$ $+5$ $+5$ Rise

Now we see that the rise is five and the run is two. That means that our slope is $\frac{5}{2}$ and our equation is $z = \frac{5}{2}q + b$ so far, but we still need the initial value.

To find the initial value in this table, we need to find what the output is when the input is zero. We can either extend the table backwards following the pattern or solve the equation. Let's first extend the table backwards. Since the run is $+2$, to move backwards we'll go -2 for the input. Since the rise is $+5$, to move backwards on the table, we'll go -5 for the output. That will look like this:

Run

-2 $+2$

$In\ (q)$	0	2	4	6	8	10
$Out\ (z)$	-3	2	7	12	17	22

-5 $+5$

Rise

Work backwards

This shows us that the output at input zero (or initial value) is -3. Now we can finish our equation meaning that $z = 5q - 3$.

There may be times when it would take too long to count backwards on the table. For those times, just use the slope and a single input/output pair. Substitute all those values into the generic linear form $y = mx + b$ and solve for b. For example, we know the slope is $\frac{5}{2}$ and the output is $z = 2$ for an input of $q = 2$. Plug all those values (using the proper variables as input and output) in as follows:

$$z = mq + b$$

$$2 = \frac{5}{2}(2) + b$$

$$2 = 5 + b$$

Subtracting five from both sides of the equation shows us that $b = -3$ just like we found earlier.

Remember that in some cases the initial value will be zero making it a proportion. If you work backwards in a table and find an initial value of zero, then you'll know it is a proportion.

Solving Table Problems

Sometimes the answer is right there in the table, and other times we'll have to do some digging to find the answer. Consider the following table that shows the total cost (c) when buying hairless wildebeests (w) after paying the registration fee with the federal government to own exotic pets. How much was the registration fee? To answer this question we'll want to first figure out how much each hairless wildebeest costs by finding the rate of change (or slope). Find the rise and run like normal.

w	12	14	16	18	20
c	\$1900	\$2200	\$2500	\$2800	\$3100

Did you find that the rise was \$300 for a run of 2 wildebeests? This reduces to \$150 per wildebeest. Now we could extend the table all the way back to zero wildebeests to find the registration fee (which is the initial value), but it's probably easier to plug everything we know into an equation as follows:

$$c = mw + b$$

We know it's \$150 per wildebeest, that that is our m value. We also have an input of $w = 12$ giving us an output of $c = \$1900$. Now we'll substitute and solve:

$$1900 = 150(12) + b$$

$$1900 = 1800 + b$$

From here we can see that $b = \$100$ by subtracting 1800 from both sides of the equation. That means that the initial value, or registration fee in this case, was \$100.

Comparing Tables with Initial Values

To compare linear functions in table form, we need their slope and initial values. Consider the following three tables each representing stores that sell fire-breathing beetle wings and decide which store sells beetle wings (w) for the cheapest price (p). Other store charges an entry fee as well.

Hogwarts

w	2	4	6	8	10
p	\$18	\$21	\$24	\$27	\$30

Diagon Ally

w	1	2	3	4	5
p	\$7	\$9	\$11	\$13	\$15

Your Mom's Shop

w	3	6	9	12	15
p	\$13	\$16	\$19	\$22	\$25

Finding the rise and run for each store gives us that Hogwarts charges \$1.50 per beetle wing, Diagon Ally charges \$2 per beetle wing, and Your Mom's Shop charges \$1 per beetle wing.

By either extending the table backward or using the equation method you should find that Hogwarts charges an entry fee of \$15, Diagon Ally charges \$5, and Your Mom's Shop charges \$10 for entry.

If you had to buy 20 beetle wings, which store would be cheapest? Find the equations or extend the tables to see that the Hogwarts total price would be \$45, it would be \$45 at Diagon Ally, and would only be \$30 at Your Mom's Shop.

Lesson 4.3

Create a table for each of the following linear situations, equations or graphs.

1. Game Start charges $45 (c) for 2 video games (g) purchased.

g	4		12		20
c		$180		$360	

2. Susie's hair is 16 inches long and it grows 2 inches in length (l) every 3 months (m)

m					
l					

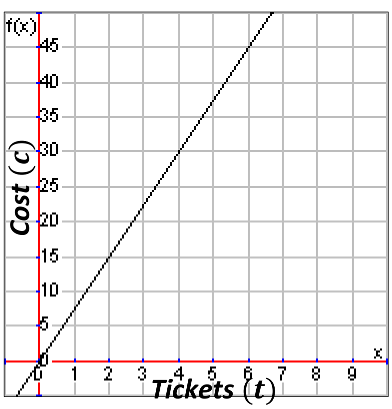
3. The height (h) of a tree in feet after a number of years (y) is determined by the following equation: $h = \frac{1}{3}y + 7$.

y	3		9		15
h		9		11	

4. At Peter's Pizza Palace the total cost (c) for pizzas (p) can be determined by using the following equation: $c = \frac{9}{2}p$.

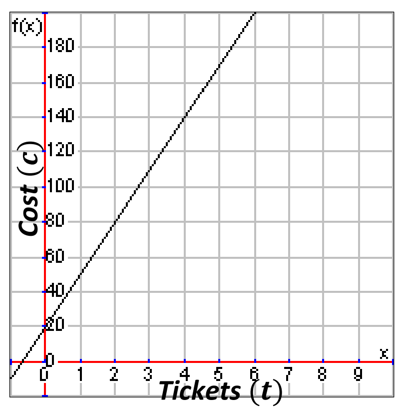
p					
c					

5. The graph shows the total cost (c) per ticket (t) for a student to go to the movies.



t	2		6		10
c		$30		$60	

6. The graph shows the cost (c) for tickets (t) to see Taylor Swift in concert if you order tickets online.



t					
c					

Create an equation for the following linear tables.

7. The total cost (c) for miles (m) traveled in a taxi.

m	2	4	6	8	10
c	\$4.50	\$6	\$7.50	\$9	\$10.50

8. The total cost (c) per tournament (t)

t	2	4	6	8	10
c	\$225	\$400	\$575	\$750	\$925

9. The total cost (c) to buy guitar picks (p).

p	5	10	15	20	25
c	\$2	\$4	\$6	\$8	\$10

10. The money earned (m) in a number of weeks (w).

w	2	4	6	8	10
m	\$10	\$20	\$30	\$40	\$50

11. The number of frogs (f) ordered for students (s) in science class.

s	9	15	21	27	33
f	7	9	11	13	15

12. The total cost (c) per hole of golf (g).

g	9	18	27	36	45
c	\$15	\$30	\$45	\$60	\$75

13. The distance traveled (d) in time in hours (h).

h	2	3	4	5	6
d	14	21	28	35	42

14. The amount of profit (p) of a stand selling lemon shake-ups (l).

l	250	300	350	400	450
p	\$50	\$200	\$350	\$500	\$650

15. The total weight of an aquarium (a) holding gallons (g) of water.

g	100	110	120	130	140
a	930	1015	1100	1185	1270

16. The length (l) of a bungee cord that is stretched depending on the weight (w) of the jumper.

w	100	110	120	130	140
l	80	83	86	89	92

17. Which of problems 8 to 16 represent proportions and how do you know?

Use the given tables to solve the linear questions.

18. How many calories (c) would you burn in a day if you walked 2 miles (m)?

m	4	5	6	7	8
c	1800	1900	2000	2100	2200

19. How many minutes (m) would it take for a pot of water to reach a temperature (t) of 210°F?

m	1	2	3	4	5
t	85	110	135	160	185

20. How many cups of cheese (c) would you need for an 18-inch pizza (p)?

p	8	12	16	20	24
c	2	3	4	5	6

21. How many months (m) could you afford the cost (c) of your own cell phone if you have $190?

m	2	4	6	8	10
c	$34	$58	$82	$106	$130

22. How much would it cost (c) to buy 13 shirts (s) at Kohl's?

s	2	4	6	8	10
c	$10	$30	$50	$70	$90

23. How many songs (s) could your purchase for $45 ($c$)?

s	4	6	8	10	12
c	$6	$9	$12	$15	$18

24. How many CDs (c) would an artist need to sell in order to make a profit (p) of $3,000?

c	50	60	70	80	90
p	$2250	$2300	$2350	$2400	$2450

25. How much profit (p) would Harry's Hot Dogs make if they sold 400 hot dogs (h) in a month?

h	200	225	250	275	300
p	100	150	200	250	300

26. Which of problems 18 to 25 are proportional and how do you know?

Answer the following questions comparing linear function equations, graphs, tables and descriptions.

Your neighborhood friends have decided to have a running race down the street. Here is the information about the distance (d) (including a head start in some cases) in terms of time (t) in seconds.

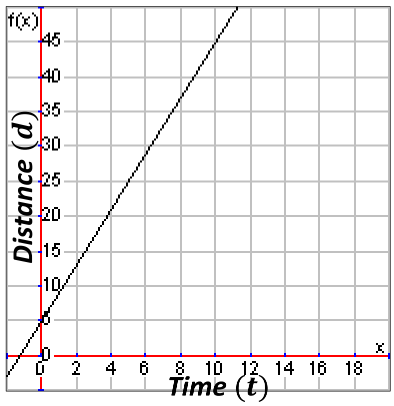
Mitchell

Runs 5 meters in 2 seconds and has a 10 meter head start

Kyra

Distance is modeled by the equation $d = \frac{9}{2}t + 3$

Gloria



Hashim

t	20	22	24	26	28
d	77	84	91	98	105

27. Which runner has the fastest pace, and how do you know?

28. Which runner has the biggest head start, and how do you know?

29. How far could each runner go in 10 seconds? Who would go the farthest?

30. Who would win the race if the race was 15 meters long?

Answer the following questions comparing proportional function equations, graphs, tables and descriptions.

Your family is deciding which activity to participate in while on your vacation in San Diego. Here is the information about the cost (c) for admission for all of your family members (f).

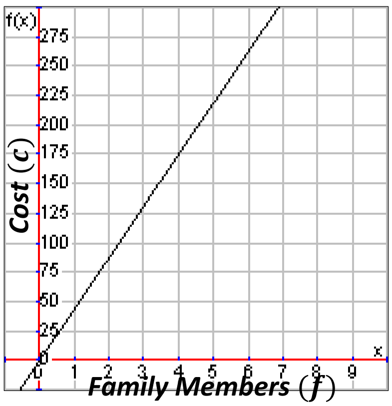
City Tour

Charges $30 per family member

San Diego Zoo

Cost is modeled by the equation $c = \frac{75}{2}f$

SeaWorld



Kayaking

f	2	4	6	8	10
c	65	130	195	260	325

31. Which activity is the cheapest per family member, and how do you know?

32. Which activity is the most expensive per family member, and how do you know?

33. How many people could you bring to each activity if you budgeted $\$400$? Which activity allows you to bring the most people for that amount of money?

34. How much would it cost at each activity to bring a family of 4? Which activity is the cheapest for that many people?

Review Unit 4: Linear Functions

You may use a calculator.

Unit 4 Goals

- *Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. (8.EE.5)*
- *Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation* $\mathrm{y} = \mathrm{mx}$ *for a line through the origin and the equation* $\mathrm{y} = \mathrm{mx} + \mathrm{b}$ *for a line intercepting the vertical axis at b. (8.EE.6)*
- *Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions. (8.F.2)*
- *Interpret the equation* $\mathrm{y} = \mathrm{mx} + \mathrm{b}$ *as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. (8.F.3)*
- *Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationships or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or table of values. (8.F.4)*

You may use a calculator.

Define variables and create equations for each of the following linear situations.

1. You spend \$10 on each gift for a family member.

2. There are 2 teachers for every 35 students on the field trip.

3. You have \$4000 in your bank account, and you pay \$450 for every 12 months of cell phone service.

4. You owe your parents \$75 for the dish you broke. You earn \$15 every 2 weeks in allowance.

Use the given equation to solve the linear questions.

5. If you collected 1400 Mario coins (c), how many stars (s) would you receive if you followed the equation $s = \frac{1}{100}c$?

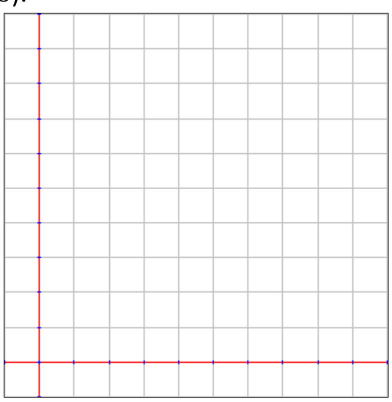
6. How many cars can you service (c) if you have 120 tires (t) and you followed the equation $t = 4c$?

7. If it costs \$36 to enter a carnival plus \$2 for each ticket to play games (g), you would spend a total amount of money (m) based on the equation $m = 2g + 36$. How many games can you play if you have \$50 to spend?

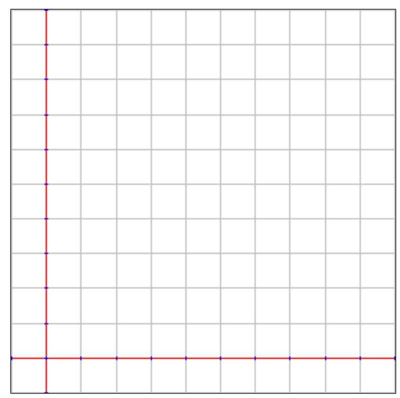
8. If a recipe for lasagna calls for $\frac{1}{2}$ cup cheese for every layer (l) plus 1 cup of cheese on top, you would use a total amount of cheese (c) based on the equation $c = \frac{1}{2}l + 1$. How many cups of cheese would you need if you wanted to make 4 layers?

Create a graph for each of the following linear situations or equations.

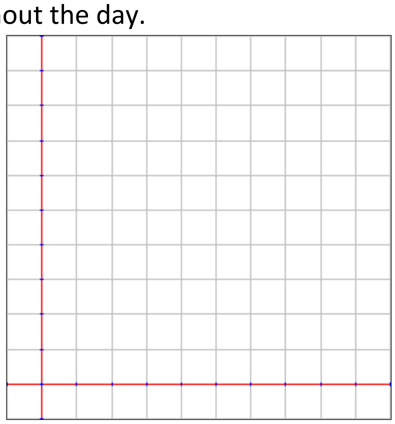
9. Grandma makes cinnamon butter by mixing 6 tablespoons of cinnamon (c) for every 2 pounds of butter (b).



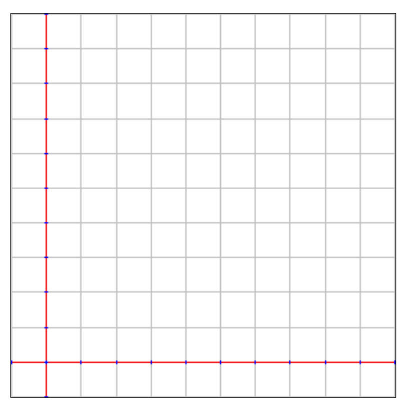
10. A snail travels at a speed of 1 foot (d) for every 2 minutes (m).



11. You drink 2 cups of water when you wake up and 3 cups of water (w) every 2 hours (h) throughout the day.

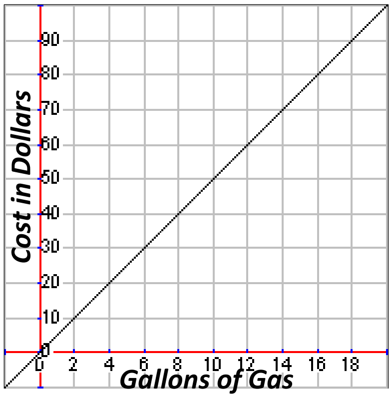


12. A runner gets a 4 meter head start and then travels at a speed of 3 meter (m) every 8 seconds (s).

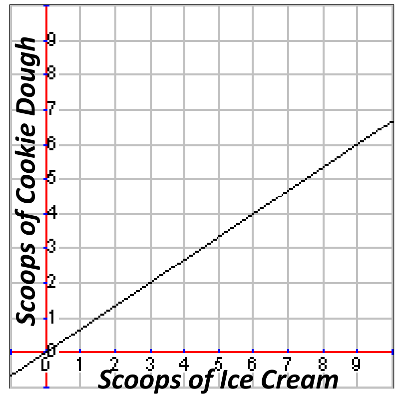


Use the given graph to solve the linear questions.

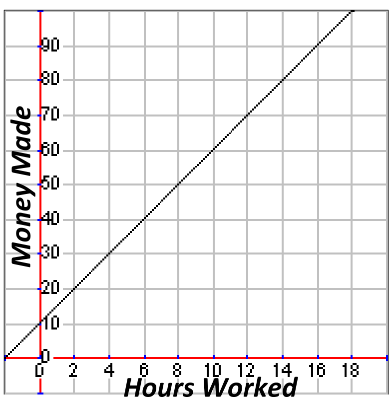
13. How much would it cost for 14 gallons of gas?



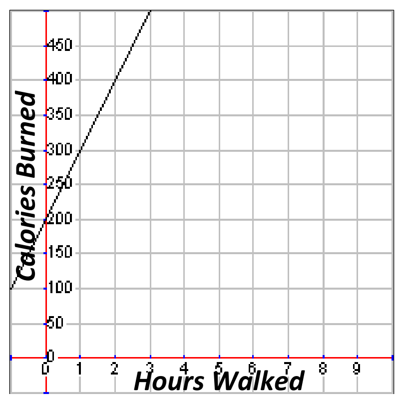
14. How many scoops of ice cream would you need with 4 scoops of cookie dough?



15. How many hours would you have to work to earn $50?

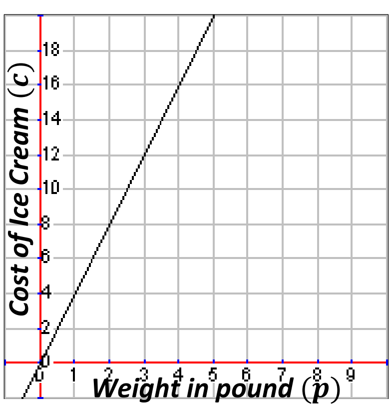


16. How many calories would you burn if you walked for 2 hours?

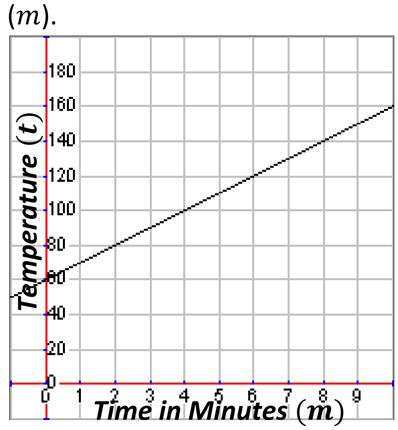


Give the equation for the following linear graph or table.

17. The cost of ice cream (c) per pound (p).



18. Temperature (t) of water on a stove over time in minutes (m).



19. The amount of money (m) for doing chores each week (w).

w	3	5	7	9	11
m	\$59	\$75	\$91	\$107	\$123

20. The total cost (c) to buy candy bars (b).

b	2	4	6	8	10
c	\$4	\$6	\$8	\$10	\$12

Fill out the table for each of the following linear situations, equations or graphs.

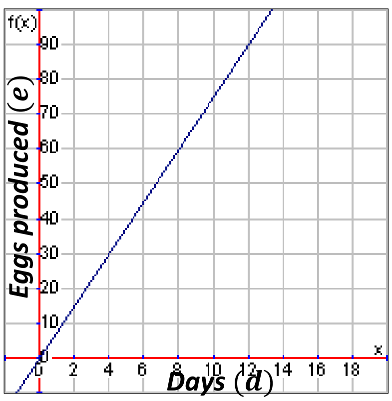
21. Monty sells used video games where total cost (c) for customers is \$50 for each set of 3 games (g) plus there is a \$10 warranty fee.

g	3		9		15
c		\$110		\$210	

22. The amount of trumpets (t) compared to flutes (f) is based on the following equation: $t = \frac{1}{3}f$.

f	9		15		21
t		4		6	

23. The graph shows the average number of eggs (e) produced by a herd chickens in a day (d).



e	30		60		90
d		6		10	

Use the given table to solve the linear questions.

24. How many teaspoons of sugar (s) would you need in 16 cups of coffee (c)?

s	1	2	3	4	5
c	2	4	6	8	10

25. How many nights (n) could you stay in a hotel if you can spend \$2000 (m)?

n	1	3	5	7	9
m	\$500	\$1100	\$1700	\$2300	\$2900

Answer the following questions comparing linear function equations and descriptions.

Carla's Cookies is looking for a new machine to make their cookies. Here is the information about the amount of cookies made (c) in terms of time (t) in minutes and power consumption (p) in watts of electricity in terms of time (t) in minutes for the machines.

Machine A: Cookies made is modeled by the equation $c = 23t$
Power consumption is modeled by the equation $p = 3.5t + 4$

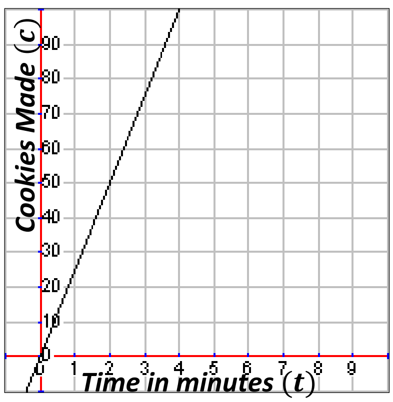
Machine B: Cookies made is modeled in the following table

t	10	20	30	40	50
c	200	400	600	800	1000

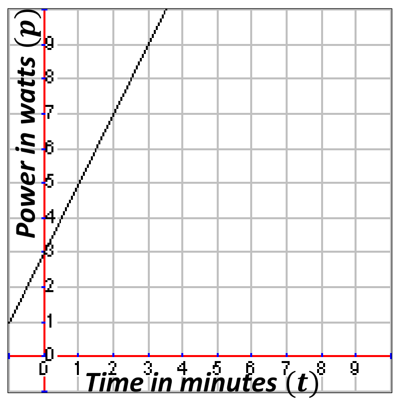
Power consumption is modeled in the following table

t	10	20	30	40	50
p	16	27	38	49	60

Machine C: Cookies graph



Power consumption graph



Machine D: Makes approximately 690 cookies in 30 minutes
Consumes approximately 90 watts in 30 minutes plus an initial 3 watts to power up the machine

26. Which machine makes cookies the fastest and how do you know?

27. Which machine makes cookies the slowest and how do you know?

28. Which machine uses the most power per minute and how do you know?

29. Which machine uses the least power per minute and how do you know?

30. Which machine uses the least power to turn on (initially) and how do you know?

31. Which machine would use the least total power if it ran for 30 minutes?

32. Which machine would use the least total power if it ran for 10 minutes?

2nd Quarter Exam Review

No calculator necessary. Please do not use a calculator.

1. What is $(5^3)(5^9)$ as a number to a single power?

2. Evaluate 3^{-2}.

3. Determine appropriate exponent to make the equation true: $\left(8^{\boxed{?}}\right)^6 = (8^2)^{-9}$.

4. Determine appropriate exponent to make the equation true: $\frac{5^{\boxed{?}}}{5^2} = (5^5)(5^{-9})$.

5. The population of New Zealand is approximately 4.4×10^6. If the population of China is approximately 2.2×10^9, about how many times bigger is the population of China than New Zealand?

6. Write 5,300,000,000 in scientific notation.

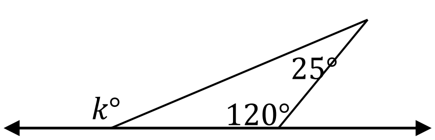
7. Write 4.6×10^{-5} in standard form.

8. What is the best unit of measurement for the amount a plant grows in the spring that grows approximately 3.9×10^{-2} meters in height, centimeters, meters, or kilometers?

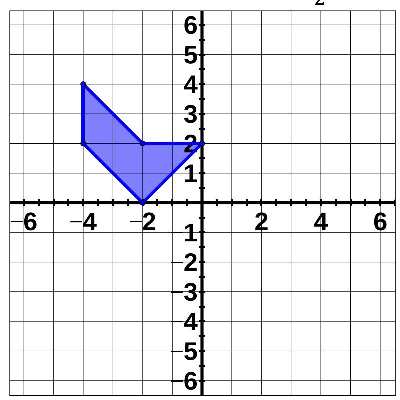
9. What is $(2.1 \times 10^{-9})(3 \times 10^{15})$ in scientific notation?

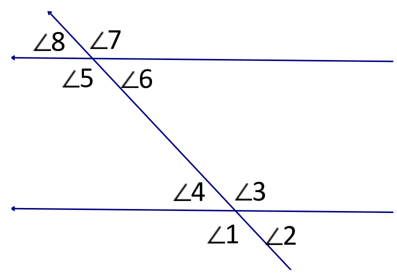
10. What is $4.19 \times 10^6 + 7 \times 10^5$ in sci. not.?

11. What is the value of k?



12. Rotate the given pre-image by $90°$ and then dilate by a scale factor of $\frac{1}{2}$.





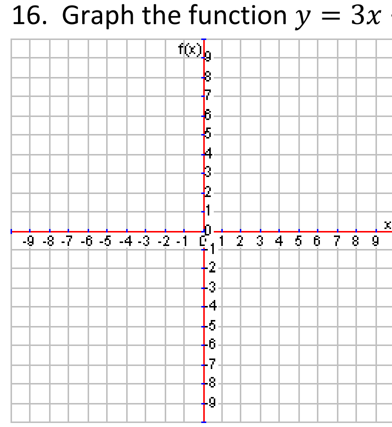
13. Name a pair of vertical angles and a pair of alternate interior angles.

14. Is the following a true function? Explain.

$$x^2 + y^2 = 25$$

15. The linear function $c = 7p + 30$ represents the cost of renting out a movie theatre (c) depending on how many people attend the movie (p). How much will the theatre cost to rent for a birthday party when there will be 100 people attending the party?

16. Graph the function $y = 3x - 2$.



17. Which rocket has the slowest speed where d is distance in feet and t is time in seconds?

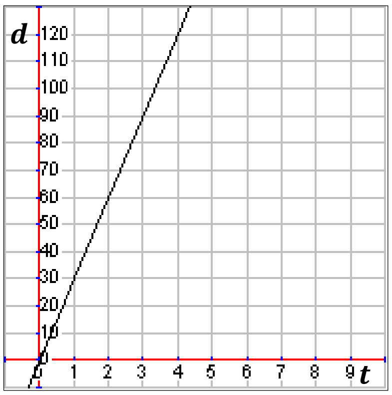
Rocket A
260 feet in 10 seconds

Rocket B
$d = 25t$

Rocket C

t	2	4	6	8	10
d	40	80	120	160	200

Rocket D



18. Is the following equation linear or non-linear? How do you know?

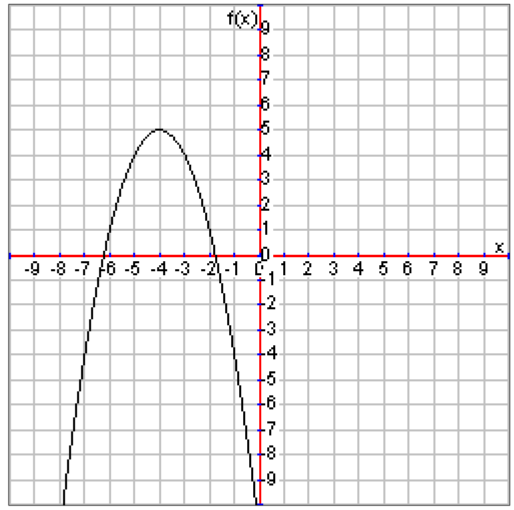
$$y = x^2 + 17$$

19. A runner can sprint 2 meters per second and has a 5 meter head start in a race. Write a linear function that models this situation using d as distance in meters and t as time in seconds.

20. Determine the rate of change and initial value of the linear function described by the following table.

x	2	4	6	8	10
y	7	12	17	22	27

21. A man pays $25 for every 3 months of membership at the local country club plus a $50 initial sign-up fee. If the man paid a total of $175, how many months was he a member of the country club?

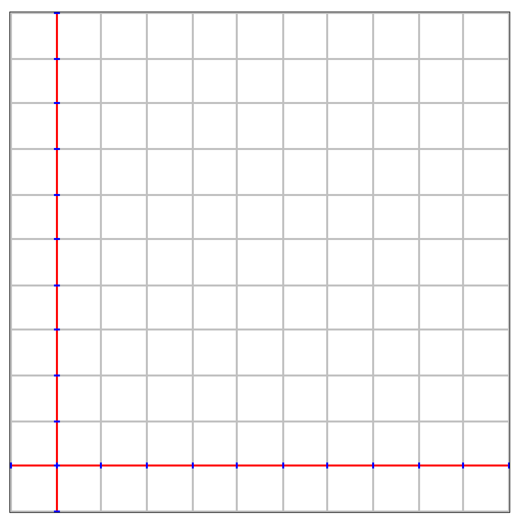


Use for #22 and #23

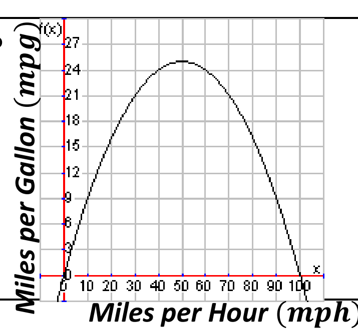
22. Where is the function above increasing and where is it decreasing?

23. Does the function above have a max, a min, or neither? If it has a max or min, tell what it is.

24. Draw a graph comparing distance from home to time that matches the story. *A boy started at home and slowly walked toward his friend's house. After a while, he ran as fast as he could. Once at his friend's house, he stayed there for the rest of the day.*



25. What is the best gas mileage this car can get?



Unit 5: Solving Equations

Pre-Test Unit 5: Solving Equations

No calculator necessary. Please do not use a calculator.

Solve the following equations for the given variable. There may be a single solution, infinite solutions, or no solutions. *(5 pts; 2 pts for simplification, 2 pts for inverse operation, 1 pt for answer)*

1. $7g + 3g - 10 + 3 = 43$

2. $4(x + 2) - \frac{1}{2}x = 22$

3. $6y + 9 - 7 = 4y + 12$

4. $6(b + 2) = 6b + 13$

5. $6y + 9 - \frac{4}{3}y = 4y + \frac{2}{3}y + 12$

6. $6(b + 2) = 6b + 12$

Write and solve an equation for the following situations. *(5 pts; 2 pts for correct equation, 2 pts for correct simplification and inverse operations, 1 pt for answer)*

7. A man buys four books from the store and a Preferred Reader discount card for \$20. Later that day, he goes back and buys five more books. He also got a \$5 discount using his new card. If he spent a total of \$87 at the bookstore, how much did each book cost assuming every book cost the same amount?

8. A girl bought two packs of gum from the store. Later she thought she would need some more gum and went back to the store to buy three more packs of gum. If she paid a total of \$5.70 for gum, how much was each pack of gum?

9. A store discounted the price of Doritos $\$0.35$ and then a man bought 5 bags. If he paid a total of $\$12.70$ for the bags of chips, how much was each bag originally?

10. A man bought 4 cups of coffee and left a $7 tip. A woman bought 8 cups of coffee and only left a $2 tip. If they paid the same amount, how much was each cup of coffee?

Answer the following questions about creating equations. *(5 pts; partial credit at teacher discretion)*

11. Create and solve a multi-step equation with exactly one solution.

12. How would you know that your equation has exactly one solution without actually solving it?

13. Create and solve a multi-step equation with no solutions.

14. How would you know that your equation has no solutions without actually solving it?

15. Create and solve a multi-step equation with infinite solutions.

16. How would you know that your equation has infinite solutions without actually solving it?

Solve the following equations. *(5 pts; 2 pts for correct inverse operation, 3 pts for answer or answers)*

17. $x^2 = 64$

18. $x^2 = -16$

19. $x^3 = 64$

20. $x^3 = -27$

5.1 Solving Equations by Combining Like Terms

When we are solving equations, we are attempting to isolate the variable in order to determine what specific value that variable has in the given equation. We do this using inverse operations. Sometime we refer to this as "undoing" the operations that have happened to the variable. Let's review the basics of solving one- and two-step equations.

Solving Basic Equations

Let's start with an example problem. The variable x is being added by 3. To "undo" the addition by 3, we subtract 3 from both sides.

$$\begin{aligned} x + 3 &= 8 \\ -3 \quad & -3 \\ x &= 5 \end{aligned}$$

We do the same thing with two-step equations except that there are two steps that need to be "undone" using inverse operations. Also, remember that we "undo" those operations in reverse order. For example, in the following problem x is being multiplied by 2 and then subtracted by 4. Working in reverse order, we need to add 4 to both sides of the equation and then divide by 2.

$$\begin{aligned} 2x - 4 &= 10 \\ +4 \quad & +4 \\ \frac{2x}{2} &= \frac{14}{2} \\ x &= 7 \end{aligned}$$

Combining Like Terms

Now that we remember how to solve these equations, we can solve more complex problems. Before we begin inverse operations to solve a problem, we want to make the equation as simple as possible. This involves performing any operations that we know how to do. For example, consider the following equation:

$$3x + 4x - 9 + 4 = 9$$

Do we know how to add $3x$ and $4x$? Then we do so and get $7x$. Do we know how to perform the operation $-9 + 4$? Then we do so and get -5. Put that together and our new equation looks like this which we'll leave as an exercise to solve and verify the solution of $x = 2$ since we have down to a two-step equation:

$$7x - 5 = 9$$

Sometimes this step is called combining like terms because we added/subtracted like terms. Like terms are those pieces of the equation are the same type of number. In our example, the $3x$ and $4x$ were both variables while the -9 and $+4$ were both regular numbers. Therefore we combined them. If it's helpful, we can mark like terms in some way to show that they are alike. Consider this where the circles show one set of like terms and the triangles show another:

$$3x + 4x \boxed{-\ 9} \boxed{+\ 4} = 9$$

Lesson 5.1

Solve each equation by combining like terms when necessary.

1. $2x + 3 = 23$

2. $-7y + 2 = 16$

3. $\frac{q}{3} + 7 = 12$

4. $\frac{z+3}{4} = 5$

5. $2g + 2g - 4 + 3 = 43$

6. $4 + 3h - h = 2 + 10$

7. $2t + 8 + \frac{1}{2}t - t = 11$

8. $4a + 6a + 3 - 8 = 15$

9. $3w + 15 - 5 + 2w = 5$

10. $\frac{5}{2}r + 2r + 7 - 2 = 5$

11. $2y + 2y + 6 + 10 = 18$

12. $5x + 7 - 2x + 1 = 12$

13. $3 - 2x + 4x + 6 = 9$

14. $10 - 5 + 3x + 2x = 0$

Write an equation for each situation and then solve by combining like terms when necessary.

15. Great Uncle Wilbert splits his inheritance equally between his five nieces and nephews. Unfortunately each of them must pay a $7500 inheritance fee to the state government. If each niece or nephew got $237,500, how much money was Great Uncle Wilbert's inheritance worth?

16. The Department of Designing the Death Star had a lot of money in a bank account and then received a large donation of $13,000 from George Lucas. They decided to split their money equally between the three research projects: Tie Fighters, Mega Lasers, and Air Conditioning/Power Grid. If each research project got $25,000, how much money did the Department of Designing the Death Star have in the bank account originally?

17. Logan collected pledges for the charity walk-a-thon. He will receive total contributions of $68 plus $20 for every mile that he walks. How many miles will he need to walk to raise $348?

18. Jasmine bought 6 CDs, all at the same price. The tax on her purchase was $5.04, and the total was $85.74. What was the price of each CD?

19. A farmer buys 6 sheep to start his wool farm. He then decides to buy insurance for $100 just in case something baaaa…d happens. The farmer realizes that his six sheep just aren't enough and decides to buy 10 more sheep. He also thought the sheep would sleep better at night if he bought them a small space heater for $25. If the farmer paid a total of $925, how much did each sheep cost?

20. Nikki buys 7 packs of SillyBanz from the store. After school the next day, she decides to buy 3 more packs to give to her friend Olivia. Then she realized that if she didn't buy something for Kerrie too, Kerrie would be mad. So Nikki then went back to the store again and bought 2 more packs of SillyBanz to give to Kerrie. If Nikki spent a total of $14.40, how much was each pack of SillyBanz?

21. During the spring car wash, the Activities Club washed 14 fewer cars than during the summer car wash. They washed a total of 96 cars during both car washes. How many cars did they wash during the summer car wash?

22. The Marsh family took a vacation to Disney World that covered a total distance of 1356 miles. (That includes the trip there and the trip back.) The trip back was 284 miles shorter than the trip there. How long was the trip to Disney World (meaning the trip there)?

5.2 Solving Equations with the Distributive Property

When we are solving equations, we are attempting to isolate the variable in order to determine what specific value that variable has in the given equation. We do this using inverse operations. Sometime we refer to this as "undoing" the operations that have happened to the variable. Let's review the basics of solving one- and two-step equations.

Solving with the Distributive Property

Another way to simplify a problem before applying inverse operations is to use the distributive property. It is true that we can solve without using the distributive property, but often it is easier to use it.

Remember that the distributive property says $a(b+c)=ab+ac$ which means whenever you multiply a number, a in this case, by a set of parentheses, you multiply by everything in the parentheses. So to multiply the parentheses by a we actually multiply a by b and then a by c before adding those. Note that subtraction would work the same way so we get $a(b-c)=ab-ac$.

Let's look at this example which we should verify has a solution of $x=6$:

$$3(x-2)=12$$

$$3*x-3*2=12$$

$$3x-6=12$$

Add six to both sides of the equation and then divide by three to get the final answer.

There may also be times where we have to use the distributive property and then combine like terms. In that case, we follow the order of operations and do our multiplication before combining like terms with addition. Consider the following example which we should verify has a solution of $x=3$:

$$3(x+2)-7+2x=14$$

$$3x+6-7+2x=14$$

$$\begin{aligned} 5x-1 &= 14 \\ +1 \quad &\quad +1 \end{aligned}$$

$$5x=15$$

$$\frac{5x}{5}=\frac{15}{5}$$

$$x=3$$

Lesson 5.2

Solve each equation by using the distributive property and combining like terms.

1. $2(x+7)+x=20$

2. $2(x-1)+3x=3$

3. $3(m+1)-2m=0$

4. $z+4(2z+3)=15$

5. $-\frac{1}{2}(b+2)+3b=-1$

6. $4(n+2)-2n=0$

7. $4+2(1+x)=12$

8. $-(x+3)+\frac{3}{4}x+5=0$

9. $2(2x+3)-2=5$

10. $2(3x-1)+2(4x+5)=8$

Write an equation for each situation and then solve by using the distributive property and combining like terms.

11. A gym charges a \$50 activation fee and \$17 per month for a membership. If you spend \$356, for how many months do you have a gym membership?

12. Suppose you go to a concert and purchase 3 identical T-shirts and a hat. The hat cost \$21 and you spend \$60 in all. How much does each T-shirt cost?

13. A store had homemade sweaters on sale for \$20 off the original price. Aunt Ethel jumped at the bargain and bought a sweater for all 15 members of her family. If Aunt Ethel paid \$375 for all the sweaters, what was the original price of each sweater?

14. After an oil pipeline burst one morning, gas prices went up by $2.20 per gallon. If that afternoon you bought 10 gallons of gas for $53.90, what was the price per gallon before the oil pipeline burst that morning?

15. For Christmas, Maryland purchased subscriptions to Xbox Live for her four children. Each subscription costs $5 per month plus a $15 sign-up fee. If she received a bill for $120, for how many months did she purchase subscriptions for her children?

16. When Apple sells their iPads, they increase the price $50 from what it costs them to actually make the iPads. One Apple store sold 10 iPads one day which cost a total of $5000. How much does an iPad cost to actually make?

5.3 Solving Equations with Variables on Both Sides

Another concept that we will need to solve equations is knowing how to deal with equations where there are variables on both sides of the equation. Since we only know how to solve equations where the variable is on one side, we need to get the equations into that form. Let's start with the following example:

$$4(x-2)+2x=3(x-1)-11$$

Note that there are variables on both sides of the equation, but we also need to do some simplification. It is almost always easiest to simplify first, so let's do so.

$$4x-8+2x=3x-3-11$$

$$6x-8=3x-14$$

Now there are still variables on both sides. We have the $6x$ on the left side and $3x$ on the other side. It would be nice if the $3x$ were not on the right. So how do we get rid of it? We can subtract $3x$ from both sides of the equation because we know that $3x-3x=0$ which will eliminate the variable on the right side of the equation. Observe:

$$\begin{array}{rl} 6x-8 &= 3x-14 \\ -3x \quad\;\; & \;\; -3x \\ 3x-8 &= -14 \end{array}$$

Now we have it down to a two-step equation which we know how to solve. We'll leave it as an exercise to verify the solution is $x=-2$.

Let's look at one more example of getting the variable on one side of the equation where the solution should be $x=7$:

$$\begin{array}{rl} -2x-6 &= -4x+8 \\ +4x \quad\;\; & \;\; +4x \\ 2x-6 &= 8 \end{array}$$

Add $4x$ to both sides to eliminate the $-4x$ on the right side.

Lesson 5.3

Solve each equation by using the distributive property, combining like terms, and eliminating the variable on one side of the equation.

1. $2y + 3 + 4 = 5y + 10$
2. $2p + 4p - 3 = 2p + 1$
3. $8k + 5 + 2k = 23 + k$
4. $4r + \frac{9}{4}r + 14 = 5r - \frac{3}{4}r + 1 - 3$
5. $2x + 3 = 2x - (3 + 2x) + 6$
6. $4(x - 1) + 2x = 2(x + 2)$
7. $-5(f + 2) = 3f + 2$
8. $\frac{3}{2}c - 3c + 4 = \frac{5}{2}c + 7 - 3$
9. $10(a + 1) = 2(a + 2) - 2$
10. $5x - 3x + 7 = 3x - 1$
11. $5d - 25 + 2d = 2d$
12. $4(2t + 1) + t = 3(t + 2)$
13. $\frac{1}{2}q + 2(q + 5) = -4(q + 1) + 1$
14. $4(1 - 2u) = 2(u + 2)$
15. $5z - z + 3 = z + 3 + 1$
16. $6x - 3x + 26 = 5(x + 8)$
17. $6(x + 1) = 4\left(1 + \frac{1}{4}x\right) + 6 + 3x$
18. $9m - m + 3 = -2(m + 1)$
19. $-(y - 4) + 3y = 4(y + 1)$
20. $-2(j + 5) + 6 = 4(j + 2)$

Write an equation for each situation and then solve by using the distributive property, combining like terms, and eliminating the variable on one side of the equation.

21. Tao is making a 7 feet high door. If the height is 1 foot more than twice its width, what is its width?

22. Terikka bought three bags of popcorn at the concession and a drink for $1.50. If she paid $3.75 total, how much was each bag of popcorn?

23. Naphtali's cell phone company charges $0.25 per text plus a $10 flat fee. Asher's cell phone company charges $0.10 per text plus a $25 flat fee. At how many texts are Naphtali and Asher paying exactly the same amount?

24. Stanley bought five packs of Yu-Gi-Oh cards, $7 worth of bubble gum, and then eight more packs of Yu-Gi-Oh cards. Simon bought four packs of Yu-Gi-Oh cards, $10 worth of Cheetos, $12 worth of Mt. Dew, and then six more packs of Yu-Gi-Oh cards. If they paid the same amount, how much was each pack of Yu-Gi-Oh cards?

25. Toby sells his framed paintings for $20 each. Ishmael sells his paintings for $14 each and charges a flat fee of $18 for framing. How many paintings need to be purchased for Toby and Ishmael to charge the same amount?

26. The original price of Doritos is the same at both Wal-Mart and County Market. Jon found out that Wal-Mart had Doritos on sale at $0.50 off per bag and bought four bags. Later that day, he found out that County Market had Doritos on sale at $1 off per bag and bought six bags. If he paid the same amount at both stores, what was the original price of Doritos?

5.1 – 5.3 Algebraic Word Problems

There will probably never be a time in life when someone runs up to you and says, "Quick! What is the answer to $2x + 7 = -13$!" In most cases, algebraic equations can be used to solve problems that come up through real life situations.

Setting up Multi-Step Equations

Bob went to a store to buy Xbox games. In this particular store, all the games are the same price. He went on Monday and bought 2 games and paid $3 extra for the game insurance (in case the games were scratched). Bob enjoyed those games so much that he went back the next Monday and bought 3 more games and got an $8 discount because he was a repeat customer. If Bob paid a total $35 over the two Mondays, how much were each of the games?

This is a typical problem that we can use algebra to solve, but we need to set up the equation first. In order to do that, we must define our variable in the problem. What is it that we are trying to find? The cost of each game. Therefore, we will let g stand for the cost of a single Xbox game.

Now that we know what variable we will use, let's begin to set up our equation. The first important part of the problem is that Bob "bought 2 games." How can we represent that algebraically? Well, we know that we have a total cost, so we need our equation to be in terms of money. This means that $2g$ will work because that means we take two times the cost of a single game, which gives us the cost of both games.

Next Bob "paid $3 extra." This means we will need to increase the total amount paid by three. So we will have $2g + 3$.

Then Bob buys three more games. What will the equation look like now? Yes, we'll need to add another $3g$ to the equation we are building, so we have $2g + 3 + 3g$ so far.

Finally, Bob gets an $8 discount, which means that money is subtracted from his total cost. Therefore we have $2g + 3 + 3g - 8$ which gives us his total cost of $35. That means our final equation is as follows, which we know how to solve:

$$2g + 3 + 3g - 8 = 35$$

$$5g - 5 = 35$$

$$5g - 5 + 5 = 35 + 5$$

$$\frac{5g}{5} = \frac{40}{5}$$

$$g = 8$$

After we combine like terms and use inverse operations to solve, we get that each game costs $8.

Setting up Distributive Property Equations

Let's look at one more example problem. Billy loves macaroni and cheese. In fact, he loves it so much that he has decided to go to the local super market to buy 20 boxes (which should last him at least 4 days). When he gets to the super market, to his great joy he finds that the macaroni and cheese boxes are having a special price reduction of $0.50 off each box. If Billy paid a total of $9.60 for his 20 boxes of mac n' cheese, what was the original cost of each box?

Again, we need to set up our equation first by deciding on our variable. For this problem, we'll let m be the cost of a single box of mac n' cheese.

From there we know that we need to take 20 times the cost of each box, but each box was first reduced in price by $0.50. We'll need to do the subtraction before the multiplication. That means we will get $20(m - 0.50)$ as our beginning part of the equation. We need to be careful because the parentheses are important. The price was reduced first, so the subtraction needs to happen first.

Now that we know the total cost, we should be able to finish setting up the equation and solve as follows:

$$20(m - 0.50) = 9.60$$

$$20m - 10 = 9.60$$

$$20m - 10 + 10 = 9.60 + 10$$

$$20m = 19.60$$

$$\frac{20m}{20} = \frac{19.60}{20}$$

$$m = 0.98$$

This means that the cost of a single box of mac n' cheese was $0.98 originally. What a deal!

5.4 Solving Equations with Infinite or No Solutions

So far we have looked at equations where there is exactly one solution. It is possible to have more than solution in other types of equations that are not linear, but it is also possible to have no solutions or infinite solutions. No solution would mean that there is no answer to the equation. It is impossible for the equation to be true no matter what value we assign to the variable. Infinite solutions would mean that any value for the variable would make the equation true.

No Solution Equations

Let's look at the following equation:

$$2x + 3 = 2x + 7$$

Note that we have variables on both sides of the equation. So we'll subtract $2x$ from both sides to eliminate the $2x$ on the right side of the equation. However, something odd happens.

$$\begin{aligned} 2x + 3 &= 2x + 7 \\ -2x \quad\; & \quad\, -2x \\ 3 &= 7 \end{aligned}$$

That can't be right! We know that three doesn't equal seven. It is a false statement to say $3 = 7$, so we know that there can be no solution. Does that make sense though? Well if we took twice a number and added three, would it ever be the same as twice a number and adding seven?

Let's look at another example equation:

$$3(x + 4) = 3x + 11$$

Note that we need to simplify and that there are variables on both sides of the equation. So we'll first multiply through the parentheses with the distributive property and then subtract $3x$ from both sides to eliminate the $3x$ on the right side of the equation.

$$\begin{aligned} 3(x + 4) &= 3x + 11 \\ 3x + 12 &= 3x + 11 \\ -3x \quad\;\; & \quad\, -3x \\ 12 &= 11 \end{aligned}$$

We again get a false statement and therefore we know there are no solutions. Sometimes we use the symbol Ø to represent no solutions. That symbol means "empty set" which means that the set of all answers is empty. In other words, there is no answer. So if we want to use Ø to represent no solution, we may.

Infinite Solutions Equations

Let's look at the following equation:

$$2x + 3 = 2x + 3$$

Note that we have variables on both sides of the equation. So we'll subtract $2x$ from both sides to eliminate the $2x$ on the right side of the equation. However, something different happens this time.

$$\begin{aligned} 2x + 3 &= 2x + 3 \\ -2x \quad\;\; & \quad -2x \\ 3 &= 3 \end{aligned}$$

When does three equal three? All the time! This means that it doesn't matter what value we substitute for x, the equation will always be true. Go ahead and try plugging in a couple of your favorite numbers to verify this is true.

Also note that twice a number plus three is equal to itself in our original equation. When is something equal to itself? Always! So there are infinite solutions. Sometimes we use the symbol ∞, which means infinity, to represent infinite solutions.

Let's look at one more example with simplification necessary.

$$\begin{aligned} -2(x + 3) &= -2x - 6 \\ -2x - 6 &= -2x - 6 \\ +2x \quad\;\; & \quad +2x \\ -6 &= -6 \end{aligned}$$

We again get a statement that is always true and therefore we know there are infinite solutions.

Creating Multi-Step One Solution Equations

Now that we understand how to solve the different types of equations, we should be able to create them. To create a one solution equation, we can honestly create an equation using any number we want as long as we don't have the same amount of variables on both sides of the equation. For example, this equation would have a single solution because the variables will not "disappear" from both sides of the equation as we simplify:

$$x + 2x + 3 + 4 = 5x + 6x + 7 + 8$$

What is the solution for that equation?

$$\begin{aligned} 3x + 7 &= 11x + 15 \\ -3x \quad\;\; & \quad -3x \\ 7 &= 8x + 15 \\ -15 \quad & \quad\quad -15 \\ -8 &= 8x \\ \div 8 \;\; & \;\; \div 8 \\ -1 &= x \end{aligned}$$

Creating Multi-Step No Solution Equations

To create a no solution equation, we can need to create a mathematical statement that is always false. To do this, we need the variables on both sides of the equation to cancel each other out and have the remaining values to not be equal. Take this simple equation as an example.

$$\begin{array}{rcl} x+1 &=& x+2 \\ -x & & -x \\ 1 &=& 2 \end{array}$$

Since one does not equal two, we know we have an equation with no solution. However, we want multi-step equations, so we'll need to make it a bit more complex. Let's look at this example:

$$x+2x+1=3x+2+3$$

$$\begin{array}{rcl} 3x+1 &=& 3x+5 \\ -3x & & -3x \\ 1 &=& 5 \end{array}$$

Notice that we combined like terms first and then eliminated the variable from one side. When that happened, the variable on the other side was eliminated as well, giving us a false result. There the key to creating equations with no solutions is to have the coefficients (number in front of the variable) match and the constants (regular numbers after that) not match.

Creating Multi-Step Infinite Solutions Equations

If we needed to create a false math statement for no solutions, what type of math statement do we need to create one with infinite solutions? Yes, we need one that will be always true. Consider the following example:

$$x+2x+3+3=3(x+2)$$

$$\begin{array}{rcl} 3x+6 &=& 3x+6 \\ -3x & & -3x \\ 6 &=& 6 \end{array}$$

Again, the coefficients matched after we combined like terms and used the distributive property, but in this case the constants also matched. This gives us the true statement that six does equal six. Therefore there are infinite solutions. Let's look at one more example.

$$4(x+1)=4x+4$$

$$4x+4=4x+4$$

We should able to stop here as we notice that the two sides are exactly the same. Four times a number plus four is always equal to four times that number plus four. Therefore there are infinite solutions.

Lesson 5.4

Solve the following equations. Some equations will have a single answer, others will have no solution, and still others will have infinite solutions.

1. $2x + 2x + 2 = 4x + 2$
2. $3(x - 1) = 2x + 9$
3. $2x + 8 = 2(x + 4)$
4. $2x - x + 7 = x + 3 + 4$
5. $-2(x + 1) = -2x + 5$
6. $4x + 2x + 2 = 3x - 7$
7. $2(x + 2) + 3x = 2(x + 1) + 1$
8. $4(x - 1) = \frac{1}{2}(x - 8)$
9. $x + 2x + 7 = 3x - 7$
10. $3x - x + 4 = 4(2x - 1)$
11. $4(2x + 1) = 5x + 3x + 9$
12. $10 + x = 5(\frac{1}{5}x + 2)$
13. $8(x + 2) = 2x + 16$
14. $3 + \frac{3}{2}x + 4 = 4x - \frac{5}{2}x$
15. $\frac{3}{2}(2x + 6) = 3x + 9$
16. $\frac{1}{2}(2 - 4x) + 2x = 13$
17. $12 + 2x - x = 9x + 6$
18. $4x + 1 = 2(2x + 3)$
19. $4(x + 3) - 4 = 8\left(\frac{1}{2}x + 1\right)$
20. $x + 5x + 4 = 3(2x - 1)$
21. $5(x + 2) - 3x = 2(x + 5)$
22. $3x + 1 = 3(x - 1) + 4$
23. $4x + 2x - 5 = 7x - 1$
24. $-2(x + 1) = 2(x - 1)$
25. $2(x + 5) = 2x + 5$
26. $2(3x + 3) = 3(2x + 2)$
27. $2x + 1 - 4 = -2x - 3$
28. $4(x + 1) = 4(2 - x)$
29. $3x + 7x + 1 = 2(5x + 1)$
30. $6(x + 1) + 5 = 13 - 2 + 6x$

Create multi-step equations with the given number of solutions.

31. A single solution

32. Infinite solutions

33. No solution

34. Infinite solutions

35. No solution

36. A single solution

37. No solution

38. A single solution

39. Infinite solutions

40. A single solution

41. Infinite solutions

42. No solution

5.5 Solving Exponent Equations

It may have become evident that squaring a number and taking the square roots are opposites. In fact, we call them inverse operations. In the same way, cubing a number and taking the cube root are inverse operations. We can use that fact to solve equations that involve exponents.

Solving $x^2 = p$ where p is rational

Let's start by looking at equations where *p* is not only rational, but also an integer. Remember that our goal in solving equations is to isolate the variable so that we will know what value the variable is equal to. So we are trying to "undo" everything that happens to the variable. Consider the following example. In order to get the variable by itself, we have to undo the square. We use the square root to do so and notice that whatever we do to one side of the equation, we must do to the other to maintain equality.

$$x^2 = 16$$

$$\sqrt{x^2} = \sqrt{16}$$

$$x = \pm 4$$

Notice that we had to give the answer of plus or minus four. Plug in both values for x to verify this is true. When $x = 4$ we see that $(4)^2 = 16$, and when $x = -4$ we see that $(-4)^2 = 16$.

It is also possible to solve an equation where the answer is a fraction. For example, consider the following problem:

$$x^2 = \frac{4}{9}$$

$$\sqrt{x^2} = \sqrt{\frac{4}{9}}$$

We know we need to take the square root of both sides of the equation, but how do we take the square root of a fraction? In particular, what fraction times itself is $\frac{4}{9}$? Note that $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ and therefore $\sqrt{\frac{4}{9}} = \frac{2}{3}$. This means that to take the square root of a fraction, we can take the square root of the numerator and the square root of the denominator. This is because when we multiply fractions the numerator gets multiplied by the numerator and the denominator by the denominator. The numerators and denominators stay separate, so we can take the square root separately. So let's complete our problem, don't forget the plus or minus in the answer.

$$x^2 = \frac{4}{9} \qquad \sqrt{x^2} = \sqrt{\frac{4}{9}} \qquad x = \pm\frac{2}{3}$$

Realize that we could have a problem where the answer will be irrational. In that case, we would either leave the answer as a plus or minus square root or approximate the solution.

$$x^2 = 50$$

$$\sqrt{x^2} = \sqrt{50}$$

$$x = \pm\sqrt{50} \approx \pm 7.1$$

Solving $x^3 = p$ where p is rational

If we used the square root to solve problems where the variable had been squared, what will need when the variable has been cubed?

$$x^3 = 27$$

If you thought we would need to take the cube root, you are correct. Since the cube root is the inverse operation to cubing, we will use that to isolate the variable as follows:

$$x^3 = 27$$

$$\sqrt[3]{x^3} = \sqrt[3]{27}$$

$$x = 3$$

Remember that we can take cube roots of negative numbers, unlike square roots. That means that the following problem is possible.

$$x^3 = -8$$

$$\sqrt[3]{x^3} = \sqrt[3]{-8}$$

$$x = -2$$

Notice that we could also take the cube root of a fraction in the same way we did with square roots.

$$x^3 = \frac{27}{64}$$

$$\sqrt[3]{x^3} = \sqrt[3]{\frac{27}{64}}$$

$$x = \frac{3}{4}$$

We will limit problems of this nature to ones where we can actually find the cube root.

Lesson 5.5

Solve.

1. $x^2 = 100$
2. $x^2 = 196$
3. $x^2 = 25$
4. $x^2 = 1$
5. $x^2 = 81$
6. $x^3 = 1$
7. $x^3 = 64$
8. $x^3 = -27$
9. $x^3 = -64$
10. $x^3 = -1$
11. $x^2 = \frac{25}{36}$
12. $x^2 = \frac{49}{16}$
13. $x^2 = \frac{64}{81}$
14. $x^3 = -\frac{27}{64}$
15. $x^3 = \frac{1}{8}$
16. $x^2 = 64$
17. $x^2 = 49$
18. $x^2 = 144$
19. $x^3 = -8$
20. $x^3 = 1000$
21. $x^3 = -125$
22. $x^2 = \frac{100}{121}$
23. $x^2 = \frac{4}{36}$
24. $x^3 = \frac{1}{125}$
25. $x^3 = 0.125$
26. $x^2 + 25 = 50$
27. $x^2 - 25 = 0$
28. $x^2 - 16 = 10$
29. $x^2 + 13 = 36$
30. $x^2 = 200$
31. $x^3 + 5 = 13$
32. $x^3 + 1 = 28$
33. $x^3 - 2 = 62$
34. $x^3 - 10 = 115$
35. $x^3 = \frac{8}{27}$

Review Unit 5: Equations

No calculator necessary. Please do not use a calculator.

Unit 5 Goals

- Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. *(8.EE.7a)*
- Solve linear equations with rational number coefficients, including equations that require using the distributive property and combining like terms. *(8.EE.7b)*
- Use square root and cube root symbols to represent solutions to equations of the form $\mathrm{x}^2 = \mathrm{p}$ and $\mathrm{x}^3 = \mathrm{p}$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. *(8.EE.2)*

Solve the following equations for the given variable, state that there are infinite solutions, or state that there are no solutions.

1. $2x + \frac{5}{2}x - 8 + 2 = 3$

2. $3(m + 4) - 2m = 9$

3. $3(2y + 5) - 6 = 4y + 10$

4. $2(b + 2) = \frac{2}{3}(b + 6)$

5. $5g + 6 - 2 = 4g + 4 + g$

6. $4y + 9 - y = 3(y + 3)$

7. $5x - (x + 2) = 4x + 8$

8. $2 + 4j - 5 = 10j$

Write and solve an equation for the following situations.

9. Soda is on sale for \$0.20 off each 12-pack. A man bought 4 12-packs and paid a total of \$4.20. How much did each 12-pack originally cost?

10. A local baseball team bought 75 shirts from a t-shirt company. The team was charged \$6.25 for total shipping but received a \$20 team discount. If the total bill was \$398.75, how much was each t-shirt?

11. A boy buys four video games from the store and a Frequent Gamer discount card for \$10. Later that day, he goes back and buys two more video games. He also got a \$2 discount using his new card. If he spent a total of \$218 at the game store, how much did each video game cost assuming every video game cost the same amount?

12. Verizon Wireless offers an internet plan for \$20 a month plus a \$20 setup fee. Consolidated Communications offers an internet plan for \$25 a month plus a \$10 setup fee. After how many months will the plans be equal?

Answer the following questions about creating equations.

13. Create and solve a multi-step equation with infinite solutions.

14. How would you know that your equation has exactly one solution without actually solving it?

15. Create and solve a multi-step equation with no solutions.

16. How would you know that your equation has no solutions without actually solving it?

17. Create and solve a multi-step equation with exactly one solution.

18. How would you know that your equation has infinite solutions without actually solving it?

Solve the following equations.

19. $x^3 = 8$

20. $x^2 = 25$

21. $x^2 = -36$

22. $x^3 = -64$

23. $x^3 = -125$

24. $x^2 = 4$

Unit 6: Systems of Equations

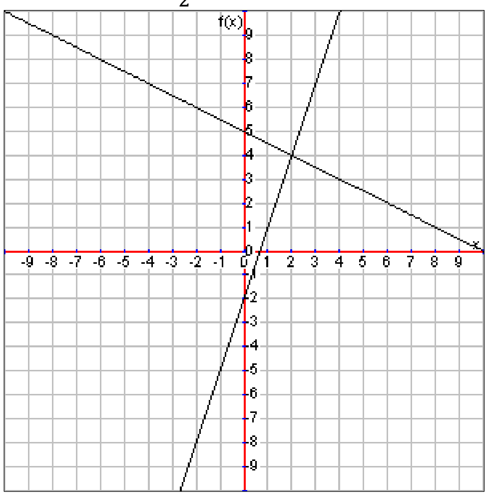
Pre-Test Unit 6: Systems

No calculator necessary. Please do not use a calculator.

Estimate the solution to the system of equations using the graph provided. Give your answer in the form of a point. *(10 pts; 3 pts for* x*-value within ½ unit, 3 pts for* y*-value within ½ unit, 4 pts for listing as point in order)*

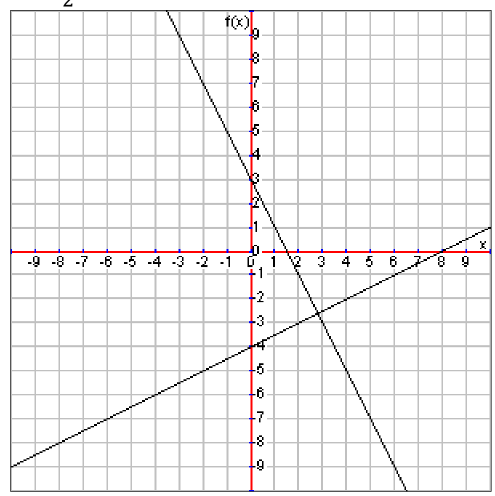
1. $y = 3x - 2$

   $y = -\frac{1}{2}x + 5$



2. $y = -2x + 3$

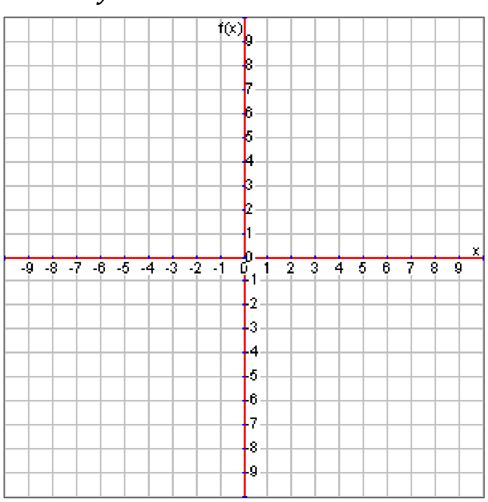
   $y = \frac{1}{2}x - 4$



Estimate the solution to the system of equations by graphing each equation on the graph provided. Give your answer in the form of a point. *(10 pts; 2 pts for each correctly graphed equation, 2 pts for each correct coordinate of the apparent solution to within ½ unit, 2 pts for listing as point in order)*

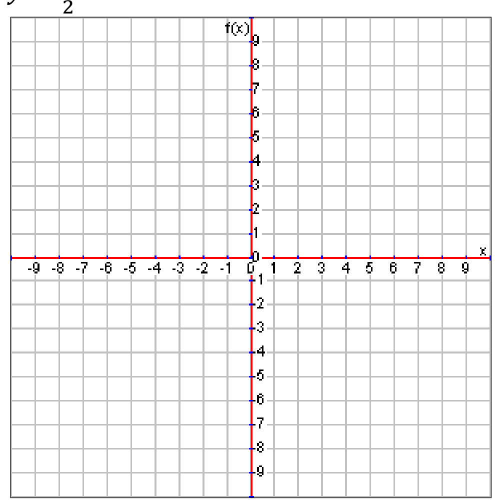
3. $x = 3$

   $y = -3x + 1$



4. $y = 3x + 3$

   $y = \frac{1}{2}x - 2$



Solve the following systems of equations using any method. There could be one solution, infinite solutions, or no solution. *(10 pts; 3 pts for correct application of solution method, 2 pts for correct value of first variable, 3 pts for correct substitution of that value and inverse operations, 2 pts for correct value of second variable; no credit without work or explanation when solving by inspection)*

5. $y = 1$
   $4x - 3y = -7$

6. $\frac{1}{2}x + 2y = 3$
   $2x - y = 3$

7. $2x + 3y = 1$
   $2x + 3y = -7$

8. $4x + 2y = 8$
   $2x + y = 4$

Write and solve equations for the following situations. *(10 pts; 1 pt for each correct equation, 2 pts for correct use of solution method, 2 pts for correct value of first variable, 2 pts for correct substitution and inverse operation work for second variable, 2 pts for correct value of second variable)* ***YOU MAY USE A CALCULATOR ON THESE!***

9. Candy worth \$1.05 a pound was mixed with candy worth \$1.35 a pound to produce a mixture worth \$1.17 a pound. How many pounds of each kind of candy were used to make 30 pounds of the mixture?

10. The perimeter of a rectangle is 56 cm. The length of the rectangle is 2 cm more than the width. Find the dimensions of the rectangle.

6.1 Graphing with Slope-Intercept Form

Before we begin looking at systems of equations, let's take a moment to review how to graph linear equations using slope-intercept form. This will help us because one way we can solve systems of equations is to graph the equations and see where the lines cross.

Slope-Intercept Form

Any linear equation can be written in the form $y = mx + b$ where m is the slope and b is the y-intercept. Sometimes the equation we need to graph will already be in slope-intercept form, but if it's not, we'll need to rearrange the equation to get it into slope-intercept form. Take a look at the following equations:

Example 1

$$y = 2x - 1$$

This equation is already in slope-intercept form. Nothing needs to be done.

Example 2

$$2x + y = 7$$

This equation is not in slope-intercept form. We need to subtract $2x$ from both sides to get the y by itself.

$$2x - 2x + y = 7 - 2x$$

$$y = -2x + 7$$

Example 3

$$3x - 2y = 4$$

This example is also not in slope-intercept form. We'll first subtract $3x$, but then notice that we'll be left with a $-2y$. Be careful because that negative sign is important. Next divide by -2 to get y by itself.

$$3x - 3x - 2y = 4 - 3x$$

$$-2y = -3x + 4$$

$$\frac{-2y}{-2} = \frac{-3x + 4}{-2}$$

$$y = \frac{3}{2}x - 2$$

Example 4

$$-4x + 2y = 8$$

This is not in slope-intercept form. We'll first need to get rid of the $-4x$ by adding $4x$ and then we'll have to get rid of the times by 2 by dividing by 2. That will get y by itself.

$$-4x + 4x + 2y = 8 + 4x$$

$$2y = 4x + 8$$

$$\frac{2y}{2} = \frac{4x + 8}{2}$$

$$y = 2x + 4$$

So, step one in graphing is to get the equation in slope-intercept form.

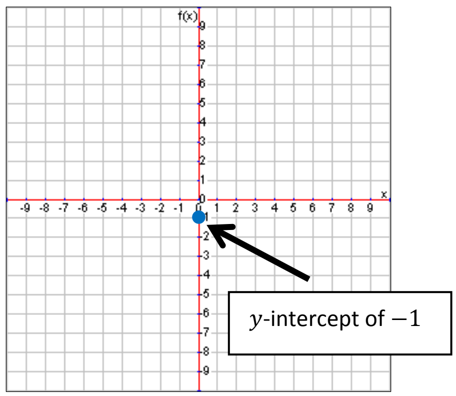
The y-Intercept and the Slope

Once you have an equation in slope-intercept form, start by graphing the y-intercept on the coordinate plane. From the y-intercept, move the rise and run of the slope to plot another point. Finally, draw the line that connects the two points. Let's use our previous equations to graph step-by-step.

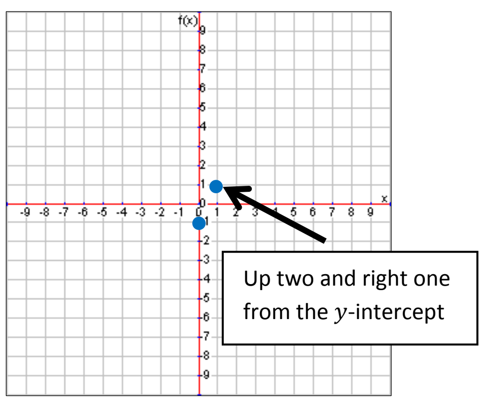
Example 1

$$y = 2x - 1$$

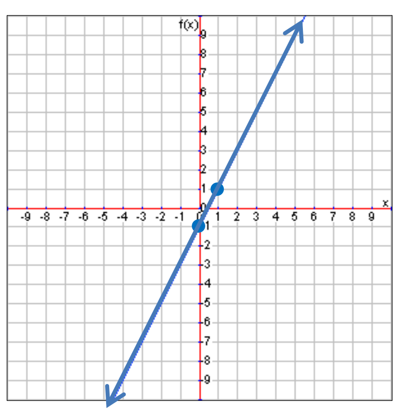
The y-intercept is -1, so we plot a point at -1 on the y-axis to start.



Next we know the slope is 2 which means a rise of 2 and a run of 1.
So we'll move up two and right one to plot the next point.

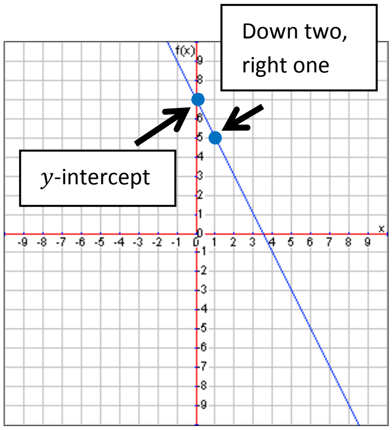


Finally, connect the dots with a line.



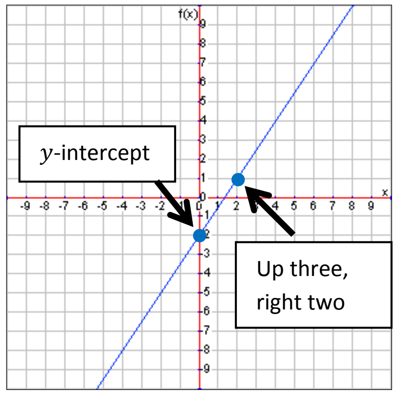
Example 2

$y = -2x + 7$



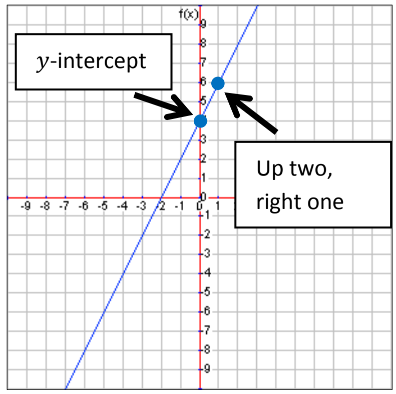
Example 3

$y = \frac{3}{2}x - 2$



Example 4

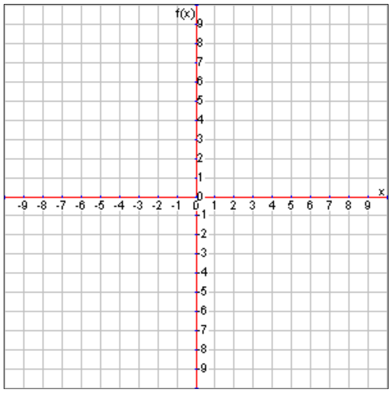
$y = 2x + 4$



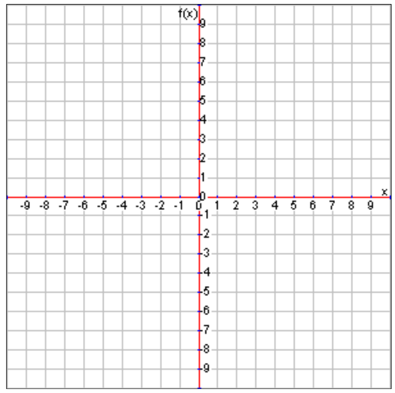
Lesson 6.1

Graph the following linear equations using slope-intercept form.

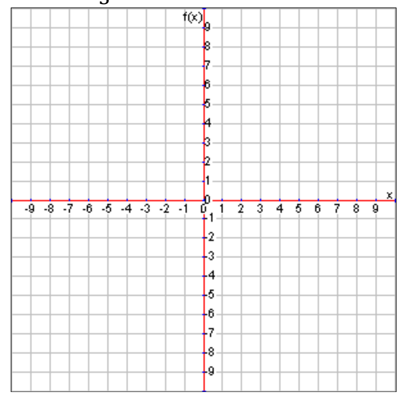
1. $y = 2x + 1$



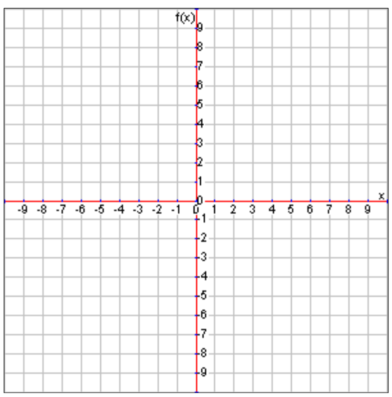
2. $y = 3x - 4$



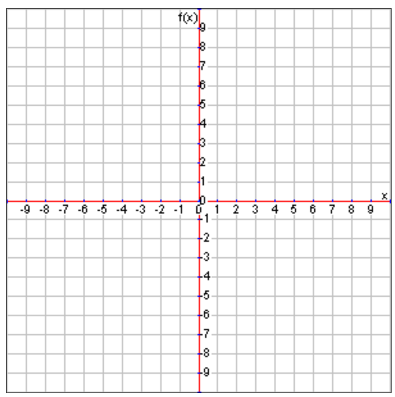
3. $y = \frac{2}{3}x + 5$



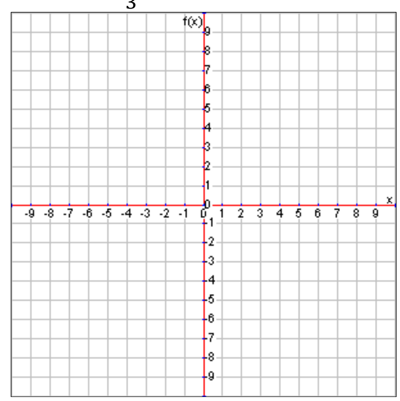
4. $y = 7$



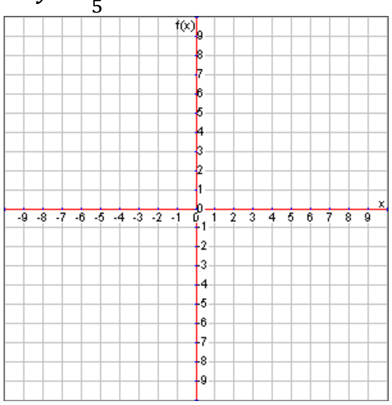
5. $y = -3x - 2$



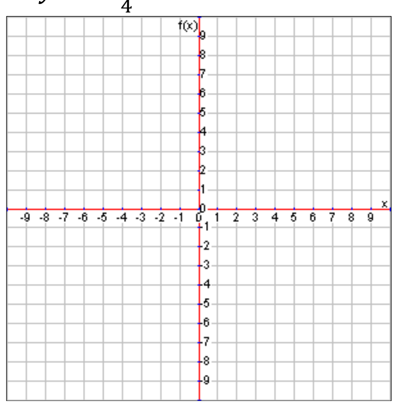
6. $y = -\frac{1}{3}x + 5$



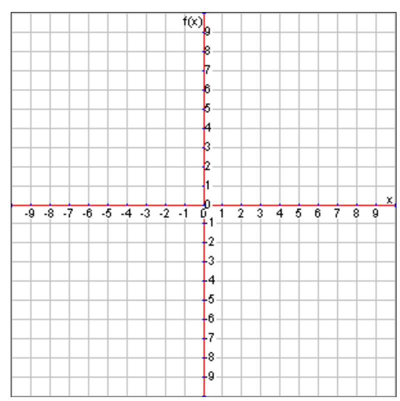
7. $y = \frac{2}{5}x - 2$



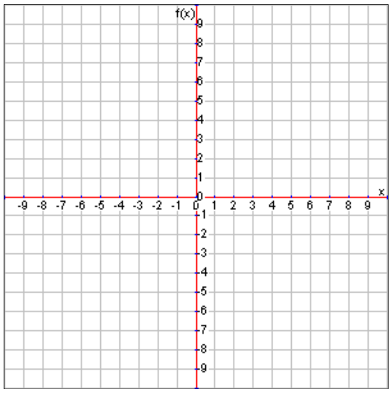
8. $y = -\frac{3}{4}x - 1$



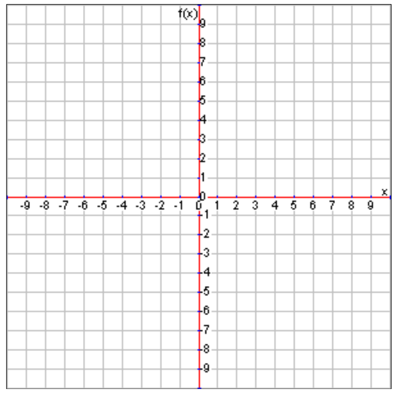
9. $y = -4$



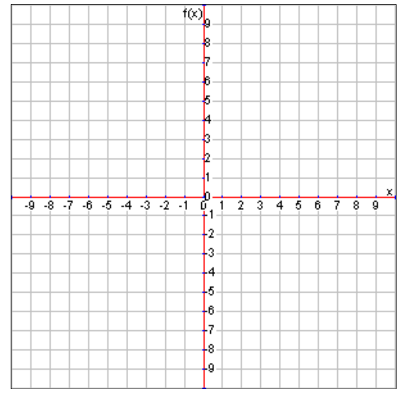
10. $2x + y = 2$



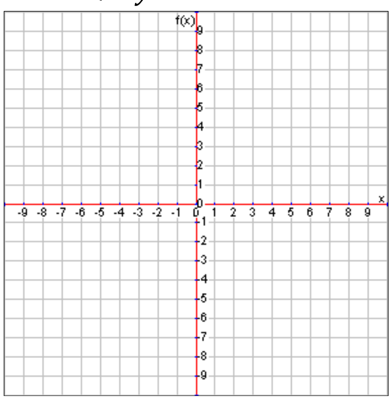
11. $-3x + y = 4$



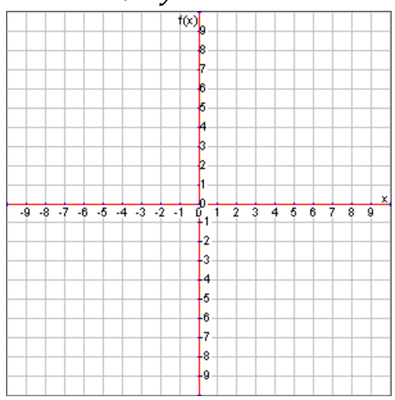
12. $4x + y = -5$



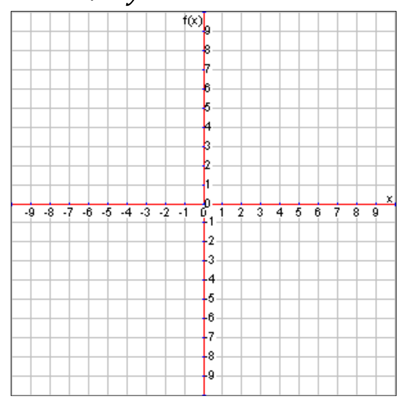
13. $4x + 2y = 6$



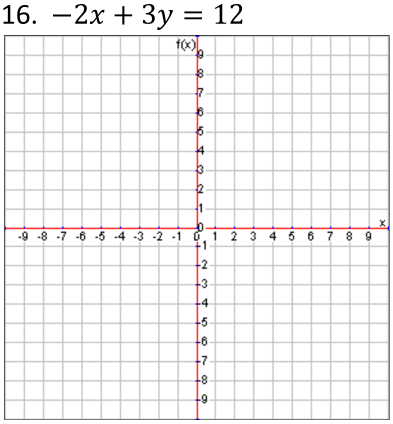
14. $-6x + 3y = -9$



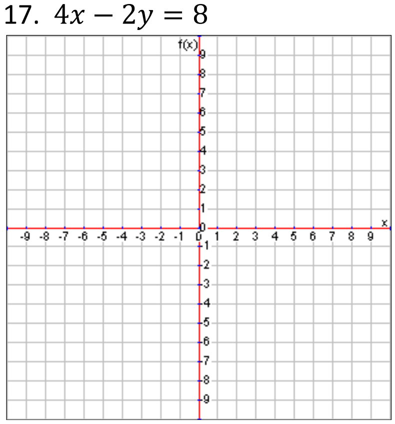
15. $x + 3y = 6$



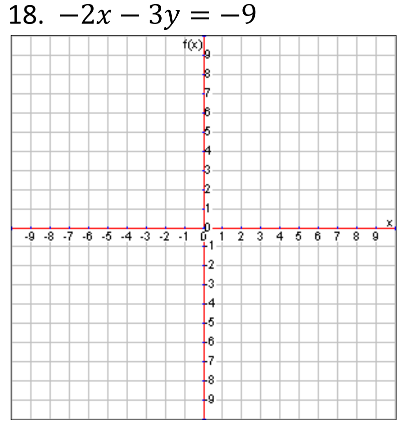
16. $-2x + 3y = 12$



17. $4x - 2y = 8$



18. $-2x - 3y = -9$



6.2 Solving Systems Graphically

Now that we know how to solve complicated equations, we move on to solving what are called systems of equations. A system of equations is when we have multiple equations with multiple variables and we are looking for values that the variables represent so that all of the equations are true at the same time.

We will mainly be dealing with two variables and two equations, but you can solve most systems of equations as long as you have the same number of equations as variables. As a quick example, consider the following system:

$$x + y = 5$$

$$x - y = 1$$

It doesn't take too much work to verify the solution of this system is $x = 3$ and $y = 2$. Notice that those values for x and y make both equations true at the same time.

$$3 + 2 = 5$$

$$3 - 2 = 1$$

The question remains, how do we get that solution?

Solving with Graphs

If we have our equations set up using the x and y variables, we can graph both equations. Let's see how this helps us. To start with, let's graph the first equation $x + y = 5$. Remember that we can do this in a couple of ways. We could simply make an x/y chart and plot the points. Alternately, we could get the equation in slope-intercept form and then graph.

Let's start with an x/y chart. Remember that in an x/y chart we pick x values and substitute those into the equation to find y values. Confirm on your own that this x/y chart is correct for $x + y = 5$:

x	-2	-1	0	1	2
y	7	6	5	4	3

Now we can plot those points on a coordinate plane and connect them to get our graph.

If we don't like the x/y chart method, we can turn the equation into slope-intercept form by isolating the y variable on the left side like so:

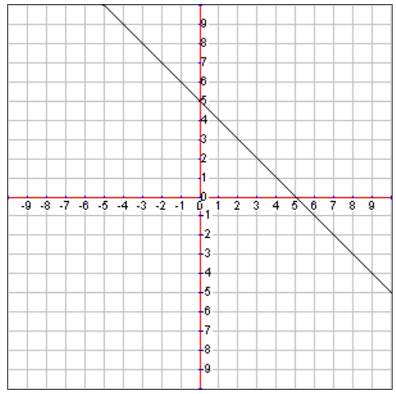
$$x + y = 5$$

$$x - x + y = 5 - x \longleftarrow \boxed{\text{Subtract } x \text{ from both sides}}$$

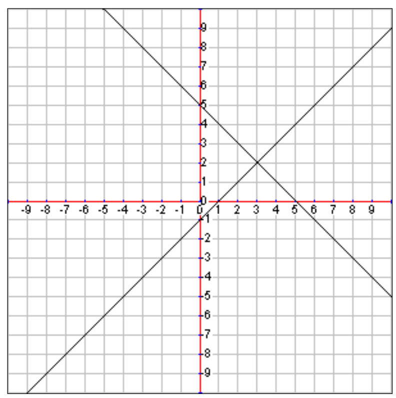
$$y = 5 + (-x) \longleftarrow \boxed{\text{Subtract means add a negative}}$$

$$y = -x + 5 \longleftarrow \boxed{\text{Commutative property}}$$

Either way, we'll get a graph that looks like this:



Now we graph the second equation, $x - y = 1$, in the same way. It turns into $y = x - 1$ and gives us an overall graph like the following:



What do you notice about those two lines? They intersect. At what point do they intersect? The intersection is at the point (3,2) which means that $x = 3$ and $y = 2$. What does this tell us about solving systems of equations using graphs?

Yes, the point of intersection is the solution to the system because that point is the only point on both lines (assuming we're dealing with only linear equations for now). In fact, we sometimes write the solution to a system of equations as a point. So the solution to this system is (3,2).

Let's try another example. What is the solution to the following system of equations?

$$4x + 2y = 6$$

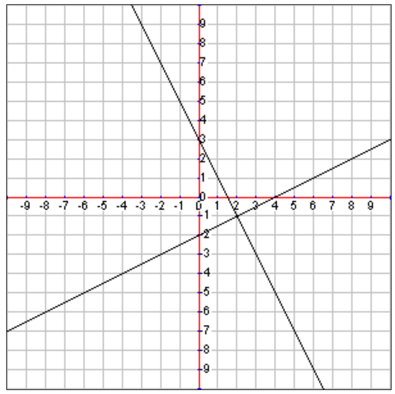
$$-x + 2y = -4$$

We'll leave it as an exercise to verify that the following equations are the same system just written in slope-intercept form:

$$y = -2x + 3$$

$$y = \frac{1}{2}x - 2$$

Now graph those equations to see where they intersect.



It looks like the graphs intersect at the point (2,-1) which we can verify by substituting into the original equations as follows:

$$4x + 2y = 6 \rightarrow 4(2) + 2(-1) = 6$$

$$-x + 2y = -4 \rightarrow -(2) + 2(-1) = -4$$

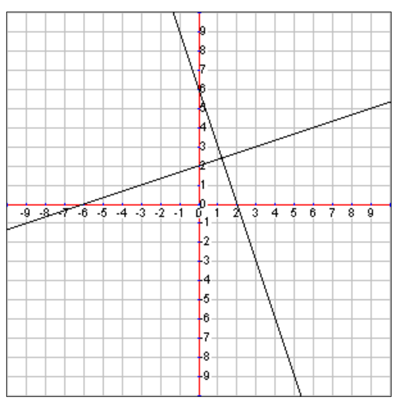
That means we have the correct solution.

Estimating Using a Graph

So far our solutions have been integer values, but that won't always be the case. We can still use the graphing method to get a decent estimate even if it's not a very nice solution. For example, consider the following equations and graphs.

$$y = \frac{1}{3}x + 2$$

$$y = -3x + 6$$

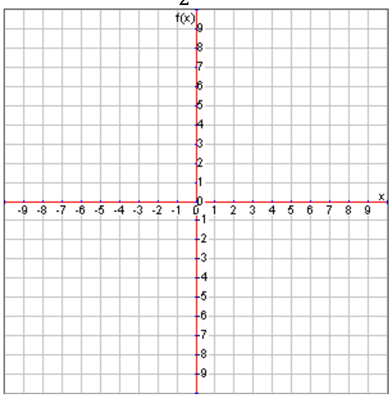


Note that the x coordinate where the lines intersect is a little more than 1 and the y coordinate of intersection is little more than 2. We might estimate this solution as $(1\frac{1}{4}, 2\frac{1}{3})$ or the decimal equivalent. The actual solution is $(1.2, 2.4)$ for this system, but we'll discover how to find the exact solution later.

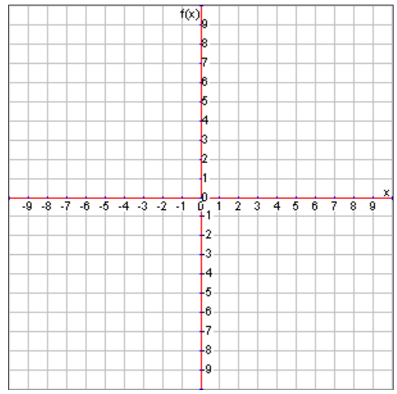
Lesson 6.2

Graph the following systems of equations and estimate the solution from the graph.

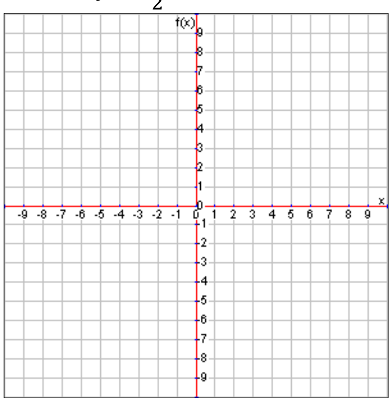
1. $y = x + 1$
   $y = -\frac{3}{2}x + 6$



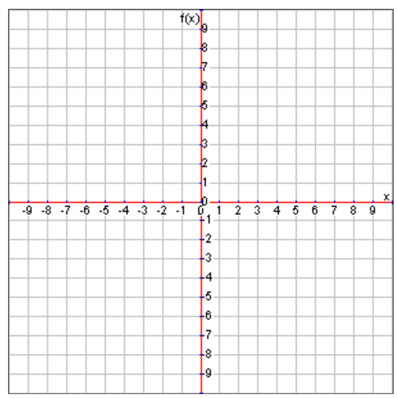
2. $y = 2x + 8$
   $y = x + 6$



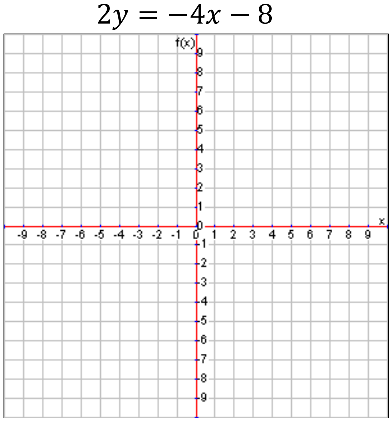
3. $y = -2x + 3$
   $y = \frac{1}{2}x - 4$



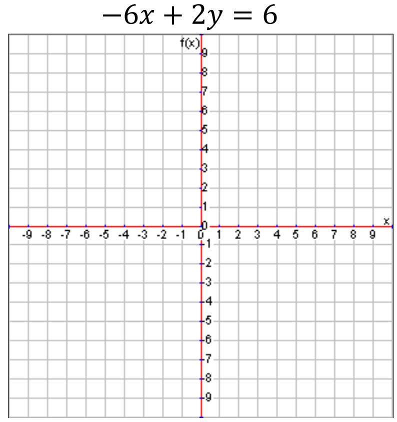
4. $y = 2x + 3$
   $y = 4$



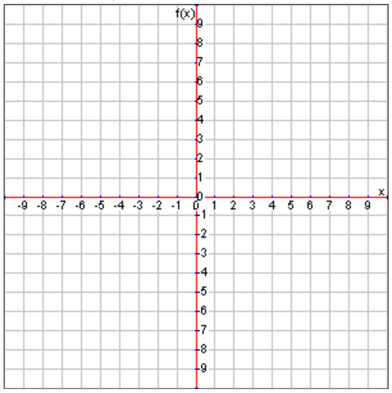
5. $3y = x + 9$
   $2y = -4x - 8$



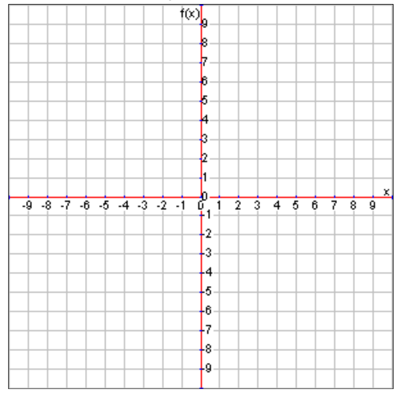
6. $4x + 2y = 6$
   $-6x + 2y = 6$



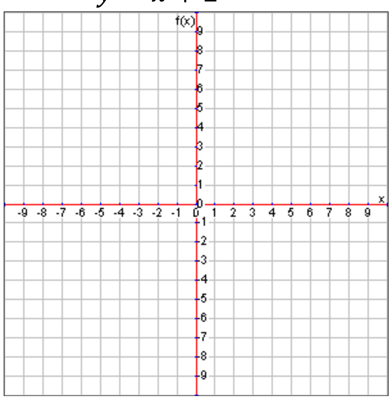
7. $3y = 6$
$2y = -x + 12$



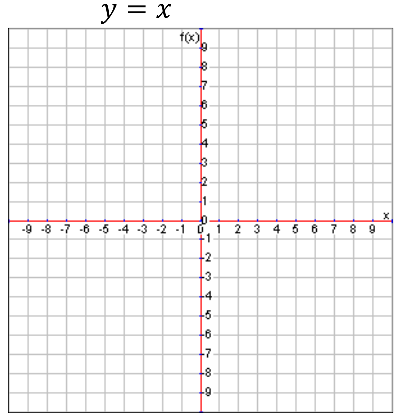
8. $-x + 3y = -18$
$x + 2y = -4$



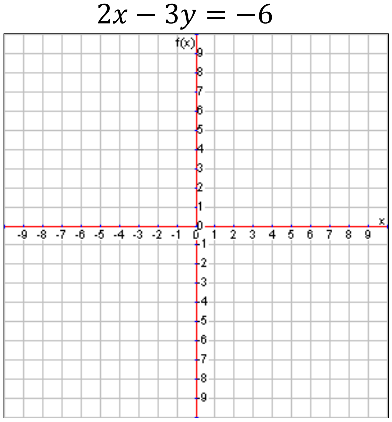
9. $y = -x$
$y = x + 2$



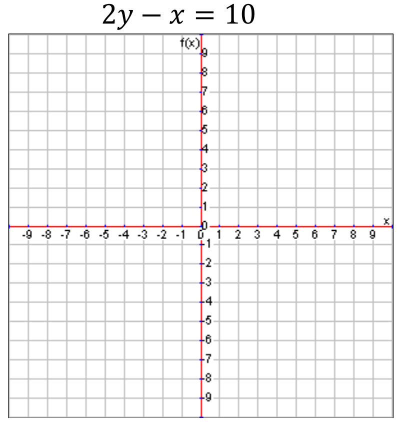
10. $x + y = 6$
$y = x$



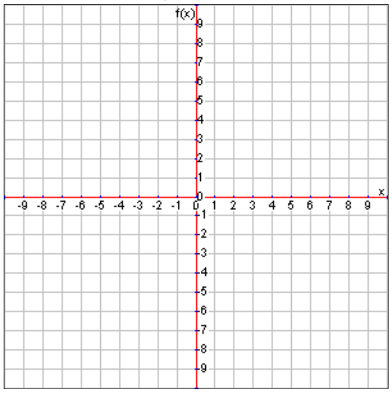
11. $4x + 3y = 24$
$2x - 3y = -6$



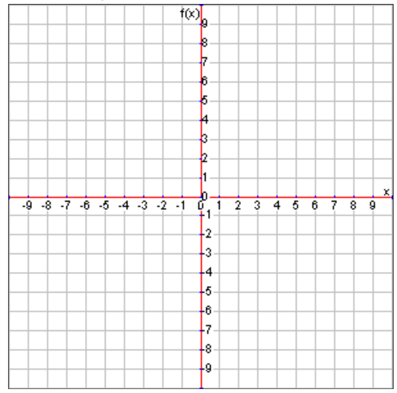
12. $x + y = 2$
$2y - x = 10$



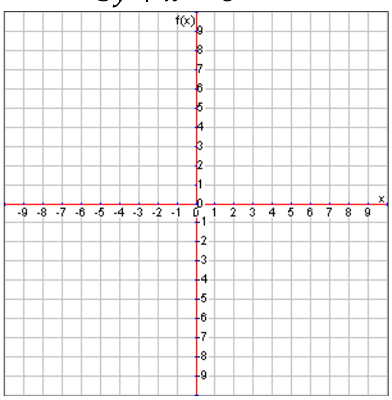
13. $x - 2y = 2$
$3x + y = 6$



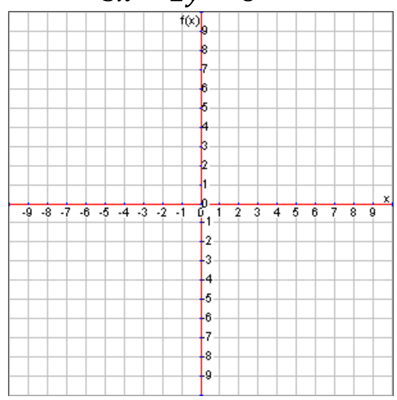
14. $y = -4$
$y = -3x - 4$



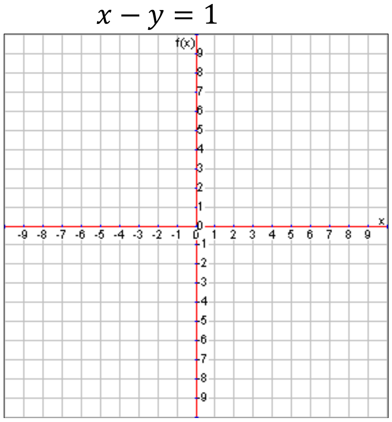
15. $y = x + 3$
$3y + x = 6$



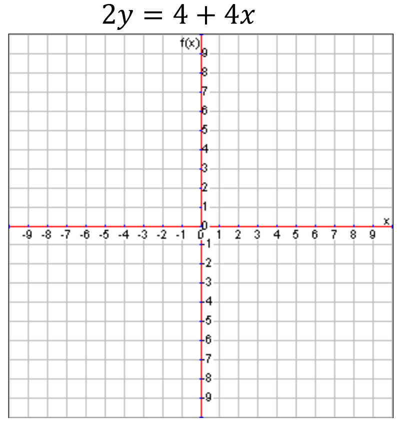
16. $x + 5y = -5$
$3x - 2y = 8$



17. $-x + y = -1$
$x - y = 1$



18. $4x = 2y - 10$
$2y = 4 + 4x$



6.3 Solving Systems with Substitution

While graphing is useful for an estimate, the main way that we can solve a system to get an exact answer is algebraically. There are a few useful methods to do this, and we will begin with the substitution method. The general idea with this method is to isolate a single variable in one equation and substitute that into the other equation.

Isolating a Variable

Consider the following system of equations.

$$3x + y = 1$$

$$3x + 2y = 4$$

It is always best to check if one variable has a coefficient of one and isolate that variable. Remember that a coefficient is a number multiplied by a variable. That means that a coefficient of one will mean that the variable doesn't have a number in front of it because the one is understood to be there, and we don't write it. In this case, notice that the y in the first equation has a coefficient of one. It would probably be easiest to isolate that variable. Let's do so.

$$3x + y - 3x = 1 - 3x$$

$$y = 1 - 3x$$

Substitution

Now that we know what y is equal to in the first equation, we can substitute that expression for y in the second equation. Be careful to not plug back into the first equation or else we'll end up with infinite solutions every time. Since we want a solution that is true in both equations, we must use both equations.

$$3x + 2y = 4$$

$$3x + 2(1 - 3x) = 4$$

$$3x + 2 - 6x = 4$$

$$-3x + 2 = 4$$

Now that we have it down to a simple two-step equation, we can solve like normal and get the following:

$$-3x + 2 - 2 = 4 - 2$$

$$\frac{-3x}{-3} = \frac{2}{-3}$$

$$x = -\frac{2}{3}$$

Finding the Second Variable Value

Now that we know what x equals, we can substitute that back into either of the original equations to find what the y coordinate is at the point of intersection of the two lines. It is also a good idea to plug in this x value into both equations to make sure they give the same y value. We'll start with the first equation.

$$3x + y = 1$$

$$3\left(-\frac{2}{3}\right) + y = 1$$

$$-2 + y = 1$$

$$-2 + 2 + y = 1 + 2$$

$$y = 3$$

Double Check

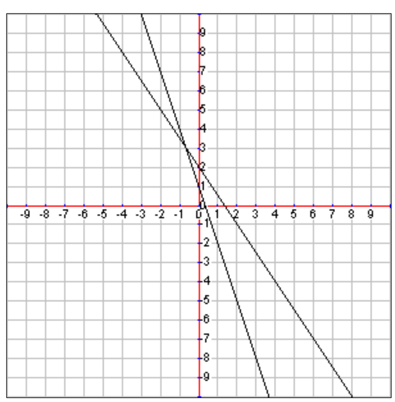
This means that the solution should be the point $(-\frac{2}{3}, 3)$, but we found that y value using the first equation. We need to make sure this point is on the second line as well, so let's substitute the values into that equation.

$$3x + 2y = 4$$

$$3\left(-\frac{2}{3}\right) + 2(3) = 4$$

$$-2 + 6 = 4$$

That statement is true and therefore the point is on the second line as well. So our solution is the point $(-\frac{2}{3}, 3)$ for this system. Just for some extra confidence, examine the following graph of the system of equations and notice that the point we found is indeed the point of intersection.



Coefficients Other Than One

It may be the case that we have all coefficients with values other than one. We can still use substitution, but we'll have to be a bit more careful isolating one variable at the beginning. Let's consider the following system of equations.

$$2x + 4y = 8$$

$$3x + 2y = 7$$

In this case, it might be easier to solve the first equation for x because the coefficient for y and the 8 will easily divide by 2. So let's isolate the x in the first equation as follows:

$$2x + 4y - 4y = 8 - 4y$$

$$\frac{2x}{2} = \frac{8 - 4y}{2}$$

$$x = 4 - 2y$$

Now substitute that x value into the second equation as follows:

$$3x + 2y = 7$$

$$3(4 - 2y) + 2y = 7$$

$$12 - 6y + 2y = 7$$

$$12 - 4y = 7$$

$$12 - 12 - 4y = 7 - 12$$

$$-4y = -5$$

$$\frac{-4y}{-4} = \frac{-5}{-4}$$

$$y = \frac{5}{4}$$

Now that we have the y coordinate, we can plug in to find the x value.

$$2x + 4y = 8$$

$$2x + 4\left(\frac{5}{4}\right) = 8$$

$$2x + 5 = 8$$

$$2x + 5 - 5 = 8 - 5$$

$$2x = 3$$

$$\frac{2x}{2} = \frac{3}{2}$$

$$x = \frac{3}{2}$$

So our solution is $\left(\frac{3}{2}, \frac{5}{4}\right)$. We'll leave it as an exercise to double check using the second equation.

Infinite and No Solutions

It is still possible to get infinite solutions or no solution for a system of equations. After the substitution step, if we get down to a number equals a number statement that is always true, there are infinite solutions. If we get down to a number equals a number statement that is false, there are no solutions. This is really the application of what we learned earlier in this unit about solving equations with one variable and getting infinite or no solutions.

Lesson 6.3

Solve the following systems using the substitution method.

1. $2x + 8y = 12$
   $x - 2y = 0$

2. $x + y = 7$
   $2x + y = 5$

3. $y = 5$
   $2x - y = 9$

4. $y = -\frac{1}{2}x + 1$
   $2x + 3y = 6$

5. $2x + y = -16$
   $x - 2y = -28$

6. $4y = 8$
   $2x + 5y = 11$

7. $x + y = 2$
   $-2x + 4y = -19$

8. $x + 2y = 4$
   $3x - 4y = -3$

9. $2x + y = 4$
   $2y = -4x + 8$

10. $x + y = 2$
    $x + y = 5$

11. $y = 3x$
    $3x + 3y = 4$

12. $y = 2x + 3$
    $y = 4x - 1$

13. $x - 3y = 0$
    $\frac{1}{3}x + y = 2$

14. $2x - \frac{1}{3}y = -9$
    $-3x + y = 15$

15. $x = 2$
$2x + y = 4$

16. $4x = 3y + 3$
$x = 2$

17. $\frac{3}{2}x = 2y$
$y = x - 1$

18. $x - 2y = -1$
$3y = x + 4$

19. $x + 2y = 0$
$3x + 4y = 4$

20. $2y = -6$
$x + 2y = -1$

21. $x - 4y = 1$
$2x - 8y = 2$

22. $x - 2y = 3$
$4x - 8y = 12$

23. $x = 0$
$3x - 6y = 12$

24. $x = 2y - 3$
$x = 2y + 4$

25. $2x - 3y = -24$
$x + \frac{1}{4}y = -5$

26. $\frac{2}{3}x - 2y = 12$
$x = -2y - 2$

27. $x + y = 6$
$2y = -2x + 2$

28. $x + 2y = 7$
$2x - 8y = 8$

29. $2x = 6y - 14$
$3y - x = 7$

30. $y = -x + 3$
$2y + 2x = 4$

Write and solve a system of equations using any method (graphing, elimination, or substitution) for each of the following situations.

31. Leonard sells small watermelons for $\$7$ each and large watermelons for $\$10$ each. One day the number of small watermelons he sold was fifteen more than the number of large watermelons, and he made a total of $\$394$. How many small and how many large watermelons did he sell?

32. The perimeter of a rectangle is 28 cm. The length of the rectangle is 2 cm more than twice the width. Find the dimensions of the rectangle.

33. The sum of Julian's and Kira's age is 58. Kira is fourteen less than twice as old as Julian. What are their ages?

34. A 3% solution of sulfuric acid was mixed with an 18% solution of sulfuric acid to produce an 8% solution. How much 3% solution and how much 18% solution were used to produce 15 L of 8% solution?

35. Supplementary angles are two angles whose measures have the sum of 180 degrees. Angles X and Y are supplementary, and the measure of angle X is 24 degrees greater than the measure of angle Y. Find the measures of angles X and Y.

36. At the end of the 2000 baseball season, the New York Yankees and the Cincinnati Reds had won a total of 31 World Series. The Yankees had won 5.2 times as many World Series as the Reds. How many World Series did each team win?

37. Peanuts worth $2.25 a pound were mixed with cashews worth $3.25 a pound to produce a mixture worth $2.65 a pound. How many pounds of each kind of nuts were used to produce 35 pounds of the mixture?

38. Ernesto spent a total of $64 for a pair of jeans and a shirt. The jeans cost $6 more than the shirt. What was the cost of the jeans?

39. The perimeter of a rectangular garden is 68 feet. The length of the garden is 4 more than twice the width. What are the dimensions of the garden?

40. The Future Teachers of America Club at Paint Branch High School is making a healthy trail mix to sell to students during lunch. The mix will have three times the number of pounds of raisins as sunflower seeds. Sunflower seeds cost $4.00 per pound, and raisins cost $1.50 per pound. If the group has $34.00 to spend on the raisins and sunflower seeds, how many pounds of each should they buy?

6.4 Solving Systems with Elimination

Sometimes it is easier to eliminate a variable entirely from a system of equations rather than use the substitution method. We do this by adding opposite coefficients together to get zero of one variable.

Subtracting to Eliminate

We first need to decide which variable is easiest to eliminate. Consider the following system of equations.

$$3x + y = 1$$

$$3x + 2y = 4$$

Notice that in this case the coefficients for x are the same. This means that they will be easily eliminated. If we subtract $3x$ from both sides of the first equation, we will eliminate the variable from the left side but will still have x are the right side. However, if we instead subtracted $(3x + 2y)$ from the left side we could subtract 4 from the right side because we know that $3x + 2y$ is exactly equal to 4 thanks to the second equation. (Remember that if we're going to solve a system of equations, we'll have to use both equations somehow, which is what we just did.)

This is sort of like repossession in a way. If you don't have the money to pay the bank, they can repossess your property up to an equivalent value of what you owe. In the same way, if we don't want to take away $(3x + 2y)$ from the right side, we can take away something equivalent which is 4 in this case. Let's take a look.

$$3x + y = \quad 1$$

$$-(3x + 2y) \quad - 4$$

$$0x - 1y = -3$$

Notice that it almost looks like we subtracted the second equation from the first. What we actually did was subtract expressions that are equal from both sides to keep the first equation balanced. Now we can solve since we have zero x's left.

$$-1y = -3$$

$$\frac{-1y}{-1} = \frac{-3}{-1}$$

$$y = 3$$

Now that we know what y equals, we can substitute that back into either equation to find the x value of the solution point.

$$3x + y = 1$$

$$3x + (3) = 1$$

$$3x + 3 - 3 = 1 - 3$$

$$3x = -2$$

$$\frac{3x}{3} = \frac{-2}{3}$$

$$x = -\frac{2}{3}$$

So we get the solution $\left(-\frac{2}{3},\ 3\right)$ which can verify is in the second equation by substituting both values in to make sure it is a true mathematical statement.

Adding to Eliminate

Adding to eliminate a variable will work the same way. In this case we should find one variable with the opposite coefficient of the same variable in the other equation. For example, consider this system of equations:

$$3x + 2y = 4$$

$$x - 2y = 4$$

Notice that the first equation has 2 as the coefficient for y and the second equation has a -2 as the coefficient. That means we should be able to add $(x - 2y)$ to both sides of the first equation. However, remember that we don't want to end up with more of the y variable on the right side, so we will add something equivalent to it. In this case that will be 4.

$$3x + 2y = \quad 4$$

$$+(x - 2y) \quad + 4$$

$$4x = 8$$

We get $0y$ which means that the y variable is eliminated

$$\frac{4x}{4} = \frac{8}{4}$$

$$x = 2$$

We'll leave it as an exercise to show that from here we can get $y = -1$ which means that our solution to this system of equations is the point $(2, -1)$.

When the Coefficients Don't Match

The elimination method works fine when the coefficients match or are opposites, but what about when it is just a messy system of equations like this?

$$2x + 3y = -1$$

$$4x + 5y = -1$$

Solving this system by the substitution method would mean dealing with fractions and the coefficients don't match so it looks like the elimination method won't work either. However, is there a way we can get the coefficients to match?

Notice that in the first equation we have a $2x$ and in the second we have a $4x$. Wouldn't it be nice if the first equation had a $4x$ instead of the $2x$? Is there any way we can make that happen? If we multiply both sides of the first equation by 2, we will maintain equality and have $4x$ to match the second equation. Let's do so.

$$2(2x + 3y) = 2(-1)$$

$$4x + 6y = -2$$

Now that we have matching coefficients we can use the elimination method to continue to solve by subtracting $(4x + 6y)$ from the left side of our new equation and subtracting -2 from the right side since that is equal to $4x + 6y$.

$$4x + 6y = \ \ -2$$

$$-(4x + 5y) \ \ -(-1)$$

$$0x + 1y = -1$$

$$y = -1$$

From here we can again substitute $y = -1$ into either original equation to find that $x = 1$ which gives us the solution of $(1, -1)$.

Infinite and No Solutions

It is still possible to get infinite solutions or no solution for a system of equations. If both variables get eliminated and we get down to a number equals a number statement that is always true, there are infinite solutions. If we get down to a number equals a number statement that is false, there are no solutions.

Lesson 6.4

Solve the following systems using the elimination method.

1. $x + y = 1$
   $x - y = 5$

2. $2x + 3y = 7$
   $-2x + y = 5$

3. $3x + y = 6$
   $3x - 2y = 9$

4. $\frac{1}{2}x + 3y = 1$
   $3x + 3y = 6$

5. $x + y = -3$
   $x - y = 1$

6. $4x + y = -9$
   $4x + 2y = -10$

7. $\frac{1}{5}x + 2y = -10$
   $2x + 2y = -10$

8. $-2x + y = 10$
   $4x + y = -8$

9. $-4x = 4$
   $4x - 3y = -10$

10. $x = 1$
    $6x - 5y = 11$

11. $x - 2y = 5$
    $3x - 2y = 9$

12. $3x + y = 5$
    $2x + y = 10$

13. $x = 5$
    $2x - 3y = 16$

14. $3x + \frac{3}{2}y = 6$
    $3x - 2y = -1$

15. $4x - 3y = 12$
$\frac{2}{3}x + 2y = 12$

16. $-5x + 3y = 6$
$x - y = 4$

17. $3y = 6$
$4x - y = -2$

18. $3x + y = 2$
$6x + 3y = 5$

19. $x + y = 4$
$2x + 2y = 8$

20. $x + y = 2$
$2x + 2y = 8$

21. $x + 3y = 12$
$2x - 3y = 12$

22. $2x + 3y = 10$
$5x + 7y = 24$

23. $5x + 4y = -3$
$10x - 2y = -3$

24. $5x - 4y = -8$
$3x + 8y = 3$

25. $4x - 7y = 10$
$3x + 2y = -7$

26. $\frac{1}{2}x - 3y = -4$
$4y = 8$

27. $3x - 4y = -10$
$5x + 8y = -2$

28. $4x + 3y = 19$
$3x - 4y = 8$

29. $4x + \frac{3}{2}y = 17$
$6x + 5y = 20$

30. $3x + 4y = -25$
$2x = -6$

Write and solve a system of equations using any method (graphing, elimination, or substitution) for each of the following situations.

31. The sum of two numbers is 82 and their difference is 26. Find each of the numbers.

32. Kathryn buys 8 cups of coffee and 2 bagels one day and pays $\$31$. Harry buys 3 cups of coffee and 3 bagels the same day and pays $\$17.25$. How much is each cup of coffee and each bagel?

33. Farmer Deanna looks out her window and counts a total of 64 legs on a total of 20 animals. If she has only sheep and chickens, how many of each does she have? *(Hint: Sheep have 4 legs each and chickens 2 legs each.)*

34. Tyler and Pearl went on a 20-kilometer bike ride that lasted 3 hours. Because there were so many steep hills on the bike ride, they had to walk for most of the trip. Their walking speed was 4 kilometers per hour. Their riding speed was 12 kilometers per hour. How much time did they spend walking?

35. A used book store also started selling used CDs and videos. In the first week, the store sold 40 used CDs and videos at $\$4.00$ per CD and $\$6.00$ per video. The sales for both CDs and videos totaled $\$180.00$. How many CDs and videos did the store sell in the first week?

36. A metal alloy is 25% copper. Another metal alloy is 50% copper. How much of each alloy should be used to make 1000 grams of a metal alloy that is 45% copper?

37. Dried apricots worth $3.25 a pound were mixed with dried prunes worth $4.75 a pound to produce a mixture of dried fruit worth $3.79 a pound. How much of each kind of fruit was used to produce 25 pounds of mixture?

38. One number added to twice another number is 23. Four times the first number added to twice the other number is 38. What are the numbers?

39. The owners of the River View Restaurant have hired enough servers to handle 17 tables of customers, and the fire marshal has approved the restaurant for a limit of 56 customers. How many two-seat and how many four-seat tables should the owners purchase?

40. The Rodriguez family and the Wong family went to a brunch buffet. The restaurant charges one price for adults and another price for children. The Rodriguez family has two adults and three children, and their bill was $40.50. The Wong family has three adults and one child, and their bill was $38.00. Determine the price of the buffet for an adult and the price for a child.

6.2 – 6.4 Solving Systems Word Problems

A word problem with systems of equations will mean that we need to write our own equations for the problems before beginning the solving process. We'll work through a few common examples of system word problems writing the equations together, but leaving the actual solving as an exercise. A note of caution: some problems may be able to be solved using the guess and check strategy, but no credit will be given without work. Since we have been given some specific methods to solve systems, we will be expected to use one of those methods.

Money Systems

Frodo bought 7 packs of Yu-Gi-Oh cards and 3 packs of gum and paid $29.19.
Bilbo bought 4 packs of Yu-Gi-Oh cards and 1 pack of gum and paid $15.98.
How much is each pack of Yu-Gi-Oh cards and each pack of gum?

The most important thing to do is to define our variables. In this case we want the cost of each thing, so we'll let y be the cost of a pack of Yu-Gi-Oh cards and let g be the cost of a pack of gum. Since we get the total cost by multiplying the number of things we buy by their price, we can write the following equation for Frodo.

$$7y + 3g = 29.19$$

In other words, 7 times the cost of each Yu-Gi-Oh pack plus 3 times the cost of pack of gum gives us the total costs of $29.19. In the same way we can write an equation for Bilbo.

$$4y + g = 15.98$$

In this case, substitution or elimination may work best since we would need an extremely zoomed in graph to be accurate to the nearest hundredth. Either way, we get the solution of $y = \$3.75$ and $g = \$0.98$. Notice that it doesn't necessarily make sense to write this solution as a point because we're not using x and y.

Age Systems

Jean-Luc's age plus William's age is equal to 95. Jean-Luc is 11 years older than William. How old are Jean-Luc and William?

In this case, we'll define our variables to let j equal Jean-Luc's age and w equal William's age. Now we can write an equation to show their total age.

$$j + w = 98$$

Since the problem states Jean-Luc is, we know we'll start the second equation with $j =$. Now since he is 11 years older than William, that would be William's age plus 11. Therefore we get our second equation like so:

$$j = w + 11$$

In this case, substitution may work best since we already have j isolated in the second equation. Substituting that back into the first equation will show us that $w = 42$ which means that $j = 53$. So Jean-Luc is 53 years old and William is 42 years old.

"How Many" Systems

An amusement park charges $15 for a child and $25 for an adult. If one day the park made $37,250 and had 2,000 people attend, how many children and how many adults attended the park that day?

We're looking for the number of adults and children so let's have a be the number of adults and c be the number of children. With that in mind, let's start with the total attendance since that is the easier equation. The number of the children plus the number of adults is the total number of people attending.

$$c + a = 2000$$

The second equation will have to do with the money aspect of the park. Again, we find the total amount of money made by multiplying the price of admission times the number of people of that age group for both age groups. Then we'll add them up. That gives us this equation:

$$15c + 25a = 37250$$

In this case, either substitution or elimination will work nicely. For substitution solve the first equation for either variable. For elimination multiply the first equation by either coefficient from the second equation and then subtract. Either way, we find the solution is $c = 1275$ and $a = 725$. This means that there were 1275 children and 725 adults attending the amusement park that day.

Finding Two Numbers Systems

The sum of two numbers is 214. The first number is 18 more than three times the second number. What are the two numbers.

We need to carefully define our variables first. We'll let f be the first number and let s be the second number. We know that sum means add, so our first equation is:

$$f + s = 214$$

The second equation will start with $f =$ since it says "the first number is" and then we'll have three times the second number plus 18 (because it's "more than").

$$f = 3s + 18$$

Substitution may be the suggested method here since we have f isolated already. We should find the solution of $s = 49$ and $f = 165$.

Rectangle Systems

A rectangle has a perimeter of 42 cm and the length is 5 less than the width. What are the length and width of the rectangle?

We'll let l be the length the rectangle and w be the width of the rectangle. We can handle the perimeter in two ways. We know the formula for the perimeter of a rectangle is $p = 2l + 2w$, so we can use that. However, it may be easier to realize that we can divide that equation by 2 on both sides to find the fact that the sum of the length and width is half the perimeter. Let's use that fact for our first equation.

$$l + w = 21$$

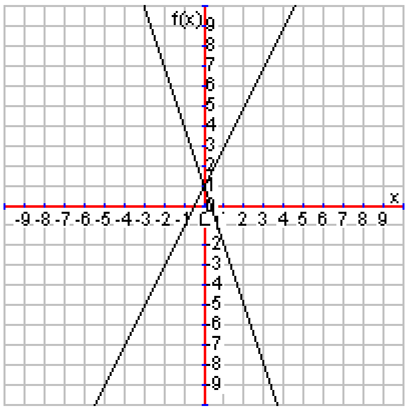
Since the length is 5 less than the width, we get the following for our second equation:

$$l = w - 5$$

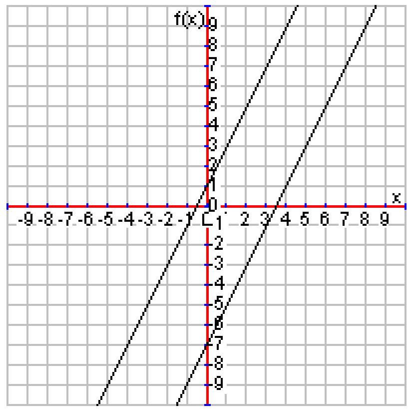
Again, it appears that substitution will work nicely since we have l isolated already. We will get the solution of $w = 13\ cm$ and $l = 8\ cm$. Don't forget the units!

6.5 Solving Systems by Inspection

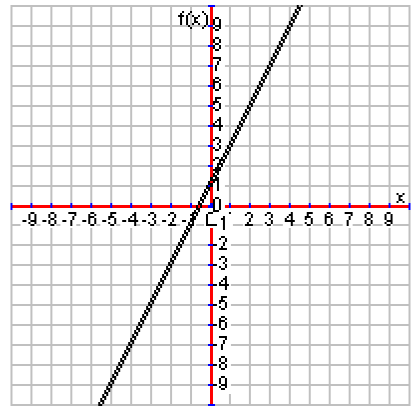
What makes a system of linear equations have a single solution, no solutions, or infinite solutions? One of the first representations we looked at for systems was the graphical representation. What is true about the following systems of linear equations that have either infinite or no solution?



One solution
$y = 2x + 1$
$y = -3x + 1$



No solutions
$y = 2x + 1$
$y = 2x - 7$



Infinite solutions
$y = 2x + 1$
$y = 2x + 1$

Notice that the systems with no solutions and infinite solutions both have the same slope. In other words, the lines are parallel. If those parallel lines have different y-intercepts, then there are no solutions to the system. If those parallel lines are in fact the same exact line (same slope and same y-intercept), then there are infinite solutions. The lines are sitting right on top of each other. Therefore if we could quickly determine whether two lines have the same slope, we could know if it will have infinite or no solutions.

Standard Form

If the two equations are given in slope-intercept form, then we can readily see the slope and y-intercept. Same slope and different intercept would mean no solution. Same slope and same intercept would mean infinite solutions. However, not all equations are given in slope-intercept form. Another common form of a linear equation is called **standard form**, which is: $Ax + By = C$.

Consider the following system of equations given in standard form. We can't readily see the slope or y-intercept since they are both in standard form.

$$2x + y = 5$$

$$4x + 2y = 10$$

So how can we find the slope? We could solve each equation for y, but this method is called inspection. We're looking for a quicker way. Let's get the second equation in slope-intercept form and see if we can find any patterns of where the slope comes from.

$$4x + 2y = 10$$

$$4x + 2y - 4x = 10 - 4x$$

$$2y = -4x + 10$$

$$\frac{2y}{2} = \frac{-4x + 10}{2}$$

$$y = -2x + 5$$

Notice that we got the slope from dividing the coefficients of the variables. Specifically, if we started with the standard form equation $Ax + By = C$, we took $-A$ divided by B. In other words, we can simply look at the ratio of the coefficients in each equation. If they are the same, then the lines will have the same slope meaning it will definitely either have no solutions or infinite solutions. Look at the original system again:

$$2x + y = 5$$

$$4x + 2y = 10$$

Notice that the ratio of the coefficients, $\frac{-A}{B}$, for both equations is equal: $\frac{-2}{1} = \frac{-4}{2} = -2$. That means there are either no solutions or infinite solutions. The y-intercept will tell us which one, but remember that if they two equations are the exact same, there will be infinite solutions. Otherwise it will be no solutions.

If we divided the second equation by 2 on both sides we would get the first equation. Since the two equations would be the same, any point on the line represented by the first equation would be on the line of the second equation. That means we know there are infinite solutions and didn't have to do any work at all.

Now consider the following system. How many solutions are there?

$$3x - 2y = 5$$

$$2x - 3y = 5$$

Check the ratios of the coefficients. Notice that $\frac{-3}{-2} \neq \frac{-2}{-3}$ which means that the lines are not parallel. That tells us there is one solution, and we should use graphing, substitution, or elimination to find the solution.

Solution Steps

In essence, we follow these steps if the equations are not in slope-intercept form:

1) Make sure both equations are in standard form and check if the ratio of the coefficients are equal
    a. If the ratios are not equal, there is a single solution, and you need to solve.
    b. If the ratios are equal, then check if you can make the equations exactly the same.
        i. If the equations can be made the same, there are infinite solutions.
        ii. If the equations cannot be made the same, there are no solutions.

Lesson 6.5

Decide if the following systems of equations have a single solution, no solutions, or infinite solutions. If it has a solution, solve the system.

1. $x + y = 1$
   $x + y = 5$

2. $2x + 3y = 7$
   $4x + 5y = 13$

3. $\frac{1}{2}x + 3y = 1$
   $x + 6y = 2$

4. $x + \frac{1}{3}y = -10$
   $3x + y = 30$

5. $2y = 6$
   $3(x + y) = 12$

6. $x + y = 2$
   $3x + 3y = 6$

7. $x + 5y = 9$
   $x + 5y = 6$

8. $2y = 5$
   $4y = 15$

9. $x + \frac{3}{5}y = 2$
   $y = -2x + 3$

10. $3x + y = 10$
    $y - 10 = -3x$

11. $3x + y = 5$
    $y = -3x + 5$

12. $6x + 4y = 10$
    $3y - 10 = -7x$

13. $2x + y = 4$
    $y - 5 = -2x$

14. $5x - 4y = 3$
    $5x = 4y - 3$

15. $7x + 5y = 3$
    $5y - 3 = -7x$

16. $\frac{2}{3}x - y = 0$
    $2x = 3y$

17. $4x = 4$
    $2x + 2y = 4$

18. $x = 2$
    $2(x + y) = 4$

19. $x + 4y = 2$
    $2(x + 4y) = 10$

20. $10x = 10 - 2y$
    $5x + y = 5$

Write a system of equations for each situation and solve using inspection.

21. The sum of two numbers is 100. Twice the first number plus twice the second number is 200. What are the numbers?

22. The perimeter of a rectangle is 40 in. Twice the length of the rectangle is 20 minus twice the width. What are the length and width?

23. Coffee worth $\$2.95$ a pound was mixed with coffee worth $\$3.50$ a pound to produce a blend worth $\$3.30$ a pound. How much of each kind of coffee was used to produce 44 pounds of blended coffee?

24. Jeri has a total of 40 pets with a total of 160 legs. If she owns only cats and dogs, how many of each does she have?

25. Pam's age plus Tom's age is 65. Twice Pam's age is equal to 130 minus twice Tom's age. How old are they?

26. The sum of two numbers is 50. Three times the first number minus three times the second number is 30. What are the numbers?

27. The perimeter of a rectangle is 30 cm. Four times the length of the rectangle is equal to 120 minus four times the width. What are the length and width?

28. A customer bought six cups of coffee and four bagels and paid $\$10$. Another customer bought three cups of coffee and two bagels and paid $\$15$. How much are each cup of coffee and each bagel?

29. A family went to Six Flags and bought two adult tickets and five child tickets and paid $\$160$. A second family bought two adult tickets and eight child tickets and paid $\$220$. How much is each adult ticket and each child ticket?

30. Jorge bought two T-shirts and four hoodies for the CMS Student Council for $\$80$. Xavier bought one T-shirt and two hoodies for $\$40$. How much is each T-shirt and each hoodie?

Review Unit 6: Systems of Equations

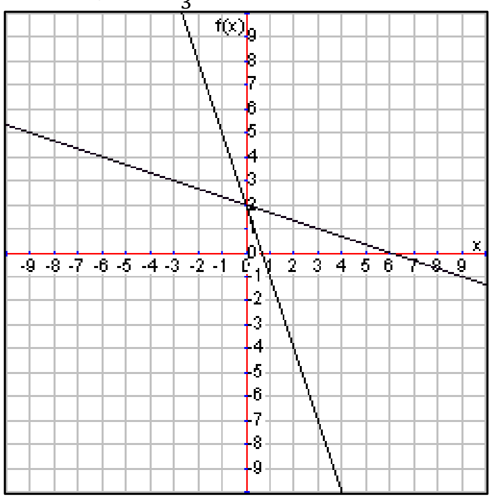
No calculator necessary. Please do not use a calculator.

Unit 6 Goals

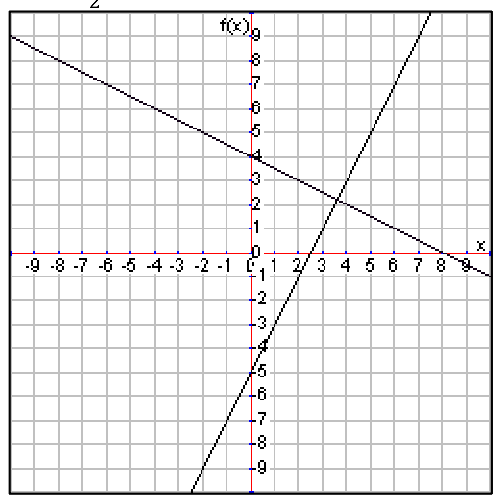
- *Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously. (8.EE.8a)*
- *Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. (8.EE.8b)*
- *Solve real-world and mathematical problems leading to two linear equations in two variables (8.EE.8c)*

Estimate the solution to the system of equations using the graph provided. Give your answer in the form of a point.

1. $y = -3x + 2$
   $y = -\frac{1}{3}x + 2$

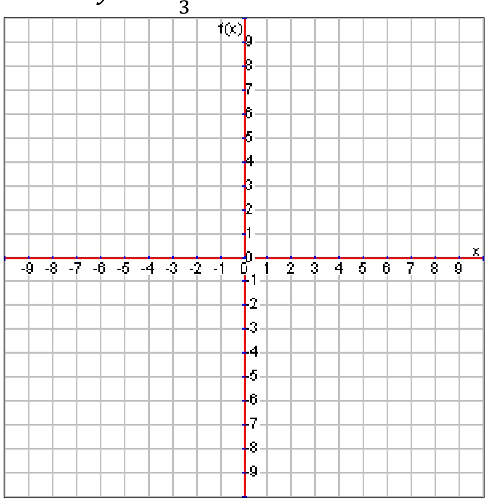


2. $y = 2x - 5$
   $y = -\frac{1}{2}x + 4$

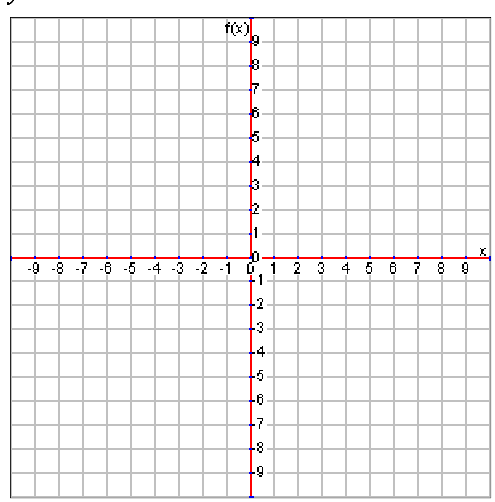


Estimate the solution to the system of equations by graphing each equation on the graph provided. Give your answer in the form of a point.

3. $y = 3x - 7$
   $y = -\frac{1}{3}x + 3$

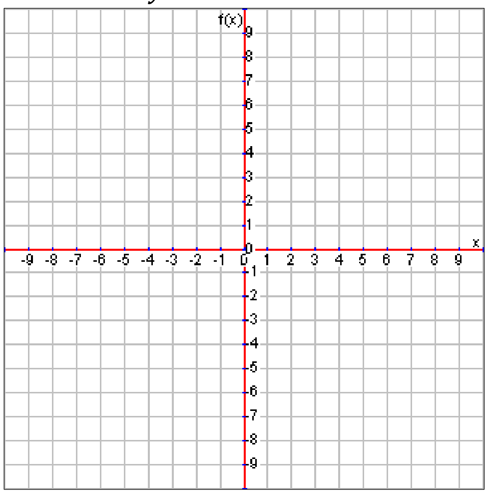


4. $y = 2x + 7$
   $y = 3$

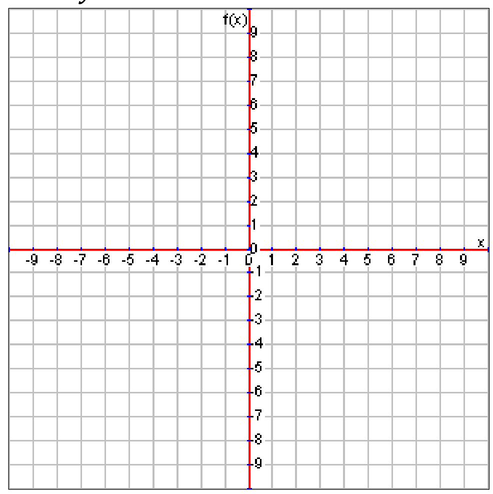


Estimate the solution to the system of equations by graphing each equation on the graph provided. Give your answer in the form of a point.

5. $-2x + 2y = 4$
   $x + y = -8$



6. $x + \frac{1}{3}y = 2$
   $-x + y = -10$



Solve the following systems of equations using any method. There could be one solution, infinite solutions, or no solution.

7. $3x + 3y = 9$
   $x + y = 3$

8. $x - 2y = 8$
   $-\frac{1}{4}x + 2y = -11$

9. $y = 5$
   $x + y = 4$

10. $6x + 2y = 8$
    $3x + y = 4$

11. $x + y = 1$
    $2x + 2y = 4$

12. $y = 3x - 5$
    $6x - 2y = 10$

Write and solve equations for the following situations. YOU MAY USE A CALCULATOR ON THESE!

13. Kera sells glasses of Koolaid for $\$1$ each and lemon shakeups for $\$3$ each. One day she sold 10 more lemon shakeups than glasses of Koolaid, and she made a total of $\$190$ selling. How many glasses of Koolaid and lemon shakeups did she sell?

14. The perimeter of a rectangle is 14 cm. The length of the rectangle is 4 cm more than twice the width. Find the dimensions of the rectangle.

15. A 12% brine solution was mixed with a 16% brine solution to produce a 15% brine solution. How much of the 12% brine solution and how much of the 16% brine solution were used to produce 40 L of the 15% solution?

16. One customer purchased 2 lattes and 1 hot chocolate for $\$9$. The next customer purchased 2 lattes and 3 hot chocolates for $\$13$. How much did each latte and each hot chocolate cost?

3rd Quarter Exam Review

No calculator necessary. Please do not use a calculator.

1. Solve $3x + 8 = 14$

2. Solve $5g + 2g - 10 + 3 = 14$.

3. Solve $2(h + 3) = 30$.

4. Solve $\frac{1}{3}(9h - 3) = 11$.

5. Solve $3(z + 4) = 3(z + 4)$.

6. Solve $4(a + 1) = 5a + 5 - a$.

7. Solve $2(b + 3) = 5b - 3b + 4$.

8. Solve $x^2 = -36$

9. Solve $x^2 = 25$

10. Solve $x^3 = -125$

11. Solve $x^3 = 27$

12. A woman bought five buckets of chicken from the store. Later she bought five more buckets of chicken, but this time she had a coupon for \$5 off. She spent a total of \$95. If you wanted to know how much was each bucket of chicken was, what would be the best variable definitions?

13. A woman bought five buckets of chicken from the store. Later she bought five more buckets of chicken, but this time she had a coupon for \$5 off. She spent a total of \$95. If you wanted to know how much was each bucket of chicken was, write an equation to represent this situation.

14. A woman bought five buckets of chicken from the store. Later she bought five more buckets of chicken, but this time she had a coupon for \$5 off. If she spent a total of \$95 that day, how much was each bucket of chicken?

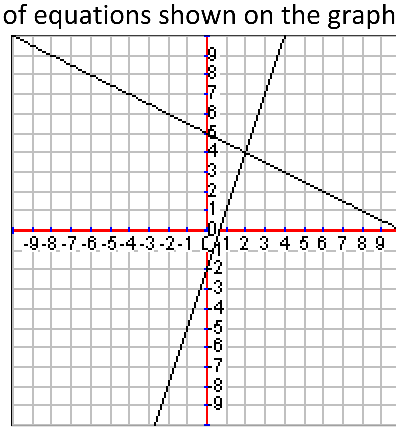
15. Does the following equation have infinite solutions, no solution or one solution?

$$5(x + 3) = 5x + 15$$

16. Does the following equation have infinite solutions, no solution or one solution?

$$4(x - 3) = 7x - 3 - 3x$$

17. Which is the best estimate of the solution to the system of equations shown on the graph?



18. Solve the following system of equations.

$$y = x + 2$$
$$y = 3x - 4$$

19. Solve the following system of equations.

$$x = -2$$
$$y = 3x - 4$$

20. Solve the following system of equations.

$$\frac{1}{3}x + y = 3$$
$$x + y = 7$$

21. Solve the following system of equations.

$$2x + 4y = 7$$
$$4x + 8y = 14$$

22. Solve the following system of equations.

$$2x - y = 8$$
$$2x - y = 10$$

23. Solve the following system of equations.

$$x - 2y = 8$$
$$2x - y = 10$$

24. Bob sold small and large beach umbrellas. Small ones cost $\$10$ each and large ones cost $\$25$ each. One afternoon Bob sold 10 umbrellas and made $\$190$. How many of each size umbrella did he sell?

25. Scotty mixed 50% acid with 100% acid and got made 10 mL of 75% acid. How much of each type of acid did Scotty use?

Unit 7: Real Numbers

Pre-Test Unit 7: Real Numbers

No calculator necessary. Please do not use a calculator.

Convert the following fraction to a decimal or decimal to a fraction. *(5 pts; 3 pts for correct set-up/work, 2 pts for correct answer)*

1. $0.\overline{45}$

2. $\frac{3}{11}$

Identify if the given number is rational or irrational and explain how you know. *(5 pts; 2 pts for correct answer, 3 pts for explanation)*

3. $\sqrt{25}$

4. $\sqrt{2}$

5. π

6. 0.45

Evaluate the following roots. *(5 pts; no partial credit)*

7. $\sqrt{49}$

8. $\sqrt[3]{-27}$

Approximate the square roots to one decimal place. *(5 pts; 2 pts for whole number accuracy, 1 pt if within 0.1)*

9. $\sqrt{20}$

10. $\sqrt{8}$

Compare the following irrational numbers using < or >. *(5 pts; no partial credit)*

11. $\sqrt{21}\ \square\ \sqrt{19}$

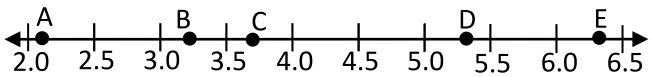
12. $-\sqrt{17}\ \square\ -\sqrt{15}$

List the following numbers in order from least to greatest. *(5 pts; no partial credit)*

13. $\sqrt{7},\ \pi,\ 3,\ 4.1\overline{5},\ \frac{11}{2}$

14. $\sqrt{27},\ \sqrt{23},\ 5,\ \frac{17}{2}$

Match the given number with the letter that approximates that number's position on the number line. *(5 pts; no partial credit)*



15. $\sqrt{14}$ 16. $\sqrt{5}$ 17. $\sqrt{40}$

Estimate the value of the expressions to the nearest whole number. *(5 pts; 3 pts for correct $\sqrt{}$ approximations)*

18. $2\sqrt{35}$ 19. $5 + \sqrt{50}$

Answer the following question. *(5 pts, partial credit at teacher discretion)*

20. A calculator displays the following number in its display: 0.37 which does not fill the display screen. Is this enough to determine whether this number is rational or irrational? In at least one complete sentence, explain why.

7.1 Identifying Irrational Numbers

In previous years you studied rational numbers. Recall that rational numbers are any number that can be expressed as a fraction where the numerator and denominator are both integers. Sometimes you will see this fraction written as $\frac{p}{q}$ where p and q are both integers.

You might also recall that this means that repeating decimals, meaning decimals that follow a repeating numeric pattern as some point are both rational numbers. For example, $0.08\overline{3}$ and $0.\overline{142857}$ are rational numbers because they are a repeating pattern. In fact, $0.08\overline{3} = \frac{1}{12}$ and $0.\overline{142857} = \frac{1}{7}$. Remember that even terminating decimals, meaning decimals that stop, are really repeating decimals and therefore rational. For example, $0.75 = 0.75000000000000 \ldots$ which shows that the zero is repeating meaning 0.75 is rational. In fact, it is equal to $\frac{3}{4}$.

All of this helps us to define irrational numbers. Recall that the prefix *ir-* means "not" so that we can define irrational numbers as numbers that are not rational. In other words, an irrational number cannot be written as a fraction. An irrational number written as a decimal would go forever and have no repeating pattern.

The most common example of this is the number π which you may know is approximately 3.14 or $\frac{22}{7}$. However, both of those values are only rational estimates of π. Other than a few special numbers like π or *e* (which you'll learn about in later math courses), irrational numbers come up most often when dealing with square roots.

Recall that a square root is the inverse operation of squaring a number. In other words, we are asking ourselves, "What number multiplied by itself will equal the given number?" The symbol for square root is $\sqrt{}$ and you should remember some basics such as $\sqrt{25} = 5$ or $\sqrt{0.49} = 0.7$ when we take the principal (or positive) square root.

When square roots don't have exact solutions such as the examples above, they are irrational. So all of the following are irrational numbers because they don't have an exact solution: $\sqrt{60}, \sqrt{11}, \sqrt{2}, \sqrt{77}, \sqrt{21}$. In particular, note that $\sqrt{2}$ is irrational. This is something you should have memorized.

Using Technology

Most calculators have a square root button, but that will not necessarily tell you for sure whether a number is rational or not as you may not be able to plug in the whole number you want to find the square root of due to the display screen size. Also, the calculator may give you an answer such as 0.1344567927 which looks like it could be irrational since we can't see a pattern, but the number $0.\overline{1344567927}$ is rational and we don't know for sure if the pattern repeats because the calculator did not return enough numbers. Rely on your brain and not the calculator to determine if a number is rational or irrational.

Lesson 7.1

Identify which of the following numbers are rational or irrational.

1. $\sqrt{25}$ 2. $\sqrt{24}$ 3. $-\sqrt{36}$ 4. $-\sqrt{64}$ 5. $-\sqrt{27}$ 6. $\frac{3}{8}$

7. 0.45 8. $0.\overline{2}$ 9. $\sqrt{49}$ 10. $\sqrt{18}$ 11. $-\sqrt{10}$ 12. $\frac{11}{21}$

13. $\frac{2}{13}$ 14. $0.\overline{42}$ 15. 0.39 16. $-\sqrt{100}$ 17. $-\sqrt{16}$ 18. $-\sqrt{43}$

19. If the number 0.77 is displayed on a calculator that can only display ten digits, do we know whether it is rational or irrational? In one complete sentence explain why.

20. If the number 0.123456789 is displayed on a calculator that can only display ten digits, do we know whether it is rational or irrational? In one complete sentence explain why.

21. If the number 0.987098709 is displayed on a calculator that can only display ten digits, do we know whether it is rational or irrational? In one complete sentence explain why.

22. If the number 0.425364758 is displayed on a calculator that can only display ten digits, do we know whether it is rational or irrational? In one complete sentence explain why.

7.2 Converting Fractions and Decimals

Sometimes it is easier to deal with a number as a fraction rather than a decimal. For example, when solving the equation $\frac{1}{3}x = 7$, it is much easier to have the fraction rather than the repeating decimal $.\overline{3}$ because we can't work with an infinite amount of digits by hand or with a calculator.

Even terminating decimals are sometimes easier to work with as a fraction. For example, it would be easier to convert 0.03125 to $\frac{1}{32}$ when performing the computation 64×0.03125.

In other cases, we need to convert a fraction to a decimal. For example, a stock price may drop $\$3\frac{1}{4}$ which we would prefer to see as $\$3.25$ since our money uses the decimal representation.

Converting a Fraction to a Decimal

To convert a fraction to a decimal, we simply do what the fraction tells us to do. Remember that the fraction bar means divide. So the fraction $\frac{1}{5}$ means 1 out of 5 or could represent 1 object (a candy bar perhaps) split between 5 people. Reading the fraction computationally, it says $1 \div 5$ which equals 0.2 as a decimal.

Remember that the numerator is the ***dividend*** (goes inside the division box) and the denominator is the ***divisor*** (goes outside the division box). Many people make the common mistake of dividing the bigger number by the smaller number because it's easier, but that would mean that $\frac{1}{5} = 5$, which is not true.

$$\begin{array}{r} 0.2 \\ 5\overline{)1.0} \\ \underline{-\,0} \\ 10 \\ \underline{-\,10} \\ 0 \end{array}$$

Our example happened to be a terminating decimal, but remember that you could end up with a repeating decimal as well such as $\frac{1}{3} = 0.\overline{3}$, $\frac{1}{11} = 0.\overline{09}$, or $\frac{1}{12} = 0.08\overline{3}$.

Converting a Decimal to a Fraction

To convert a terminating decimal to a fraction, follow these three steps:

1.) **Read it.** Say the decimal out loud properly. This means that 0.34 is thirty-four hundredths, not "point 34."

2.) **Write it.** Note that by saying the decimal out loud we know what the furthest right place value is. In the example problem, it is hundredths. Now write thirty-four hundredths as a fraction like this: $\frac{34}{100}$.

3.) **Reduce it**. If possible, reduce the fraction to lowest terms. For our example $\frac{34}{100} \div \frac{2}{2} = \frac{17}{50}$, and since it will not reduce further, we know that $0.34 = \frac{17}{50}$.

Converting a Repeating Decimal to a Fraction

At the 8th grade level, we will be converting decimals with either one or two repeating digits into fractions. Let's work through turning $0.8\overline{36}$ into a decimal to see how we do this conversion.

First, set $x = 0.8\overline{36}$ and then notice that we have two digits repeating, the three and the six. Multiply both sides of that equation by 100 since there are two repeating decimals. (If there was only one repeating decimal, we would multiply both sides by 10. If there were three repeating decimals, we'd multiply by 1000, and so on.)

This gives us the equation $100x = 83.6\overline{36}$ after the multiplication by 100 on both sides. Now we can subtract values from each other, simplify, and solve for x.

$$100x - x = 83.6\overline{36} - 0.8\overline{36}$$

$$99x = 82.8$$

$$\frac{99x}{99} = \frac{82.8}{99}$$

$$x = \frac{82.8}{99}$$

Now we have a fraction value of the original decimal, but it's a decimal mixed with a fraction. Therefore we need to find an equivalent fraction without decimals in the numerator and then reduce to lowest terms. To do so, we'll use the same trick of multiplying by a power of 10, namely $\frac{10}{10}$.

$$x = \frac{82.8}{99} * \frac{10}{10} = \frac{828}{990}$$

$$x = \frac{828}{990} \div \frac{9}{9} = \frac{92}{110} \div \frac{2}{2} = \frac{46}{55}$$

After reducing to lowest terms we see that $0.8\overline{36} = \frac{46}{55}$. Let's do one more example converting the repeating decimal $0.1\overline{6}$ to a fraction in lowest terms and find that $0.1\overline{6} = \frac{1}{6}$.

$$x = 0.1\overline{6}$$

$$10x = 1.6\overline{6}$$

$$10x - x = 1.6\overline{6} - 0.1\overline{6}$$

$$9x = 1.5$$

$$\frac{9x}{9} = \frac{1.5}{9}$$

$$x = \frac{1.5}{9} * \frac{10}{10} = \frac{15}{90} \div \frac{15}{15} = \frac{1}{6}$$

Lesson 7.2

Convert the following fractions to repeating decimals.

1. $\frac{7}{15}$ 2. $\frac{2}{3}$ 3. $\frac{7}{9}$ 4. $\frac{10}{33}$ 5. $\frac{1}{9}$ 6. $\frac{2}{11}$

7. $\frac{11}{12}$ 8. $\frac{1}{3}$ 9. $\frac{5}{6}$ 10. $\frac{5}{11}$ 11. $\frac{1}{6}$ 12. $\frac{7}{18}$

Convert the following repeating decimals to fractions.

13. $0.\overline{2}$ 14. $0.1\overline{5}$ 15. $0.3\overline{6}$ 16. $0.4\overline{8}$ 17. $1.2\overline{3}$ 18. $1.\overline{5}$

19. $0.\overline{81}$ 20. $0.\overline{35}$ 21. $0.2\overline{15}$ 22. $0.1\overline{23}$ 23. $1.\overline{16}$ 24. $3.\overline{25}$

7.3 Evaluation of Roots

Previously we used the square root to help us approximate irrational numbers. Now we will expand beyond just square roots and talk about cube roots as well. For both we will be finding the roots where the answer will be an integer and therefore rational. In other words, we won't have to worry about approximately because these will work out nicely.

Evaluating Square Roots of Perfect Squares

Remember that perfect squares are numbers that have integer square roots. Perfect squares are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256,… and so forth. Since the numbers are perfect squares, we get integer answers when we take the square root. So $\sqrt{1} = 1$ and $\sqrt{36} = 6$ and on and on.

However, strictly speaking each square root actually has two solutions. The reason that $\sqrt{36} = 6$ is because $6 \times 6 = 36$, but notice that $-6 \times -6 = 36$ is also true. That means that $\sqrt{36} = -6$ is a true statement as well. So for each square root there are actually two solutions, one positive and one negative.

To prevent confusion about which number we want, the positive or the negative, the mathematical community decided that when we see the square root symbol, we will always give the principal square root which is the positive answer. This means that whenever you see something like $\sqrt{49}$ you should know that we only want the positive root, which is 7. If we want the negative root, it will be written like this: $-\sqrt{49} = -7$. Let's look at a few examples just to make sure we understand.

$\sqrt{144} = 12$ $\qquad$ $-\sqrt{100} = -10$ $\qquad$ $\sqrt{64} = 8$ $\qquad$ $-\sqrt{225} = -15$

It is also possible to ask for both roots. For example, the directions for homework may say, "Find both square roots of the given number." Then you would list them both as follows: $\sqrt{36} = 6$ and -6. The alternate way to list both square roots is to use the plus or minus sign, $\pm$, to represent both. So you could say $\sqrt{36} = \pm 6$ because the square of 36 is positive 6 and negative 6.

We could also take square roots of certain decimals nicely. For example, $\sqrt{0.36} = 0.6$ or $\sqrt{0.09} = 0.3$. However, we will limit ourselves to integers for now.

Lastly, remember that we cannot take the square root of negative numbers. So $\sqrt{-64}$ has no solution because nothing times itself is -64. Any number times itself will always be positive.

Evaluating Cube Roots of Perfect Cubes

Just like there are square roots, there are also cube roots. The cube root of a given number is like asking what number cubed (meaning to the third power) will give you that original number. So the cube root of 8 is 2 because $2^3 = 8$. We represent the cube root with a symbol exactly like the square root symbol except there is a "3" in the "v" of the symbol.

Look at the following examples.

$$\sqrt[3]{8} = 2 \qquad \sqrt[3]{27} = 3 \qquad \sqrt[3]{64} = 4$$

The numbers 1, 8, 27, 64, 125, 216, 343… and so forth are called perfect cubes because they have an integer cube root.

Notice that a cube root does not have two answers. There are not positive and negative cube roots. Each cube root only has one real number solution. For example we know $\sqrt[3]{8} \neq -2$ because $(-2)^3 = -8$ instead of eight.

However, this means we can take the cube root of negative numbers. So $\sqrt[3]{-8} = -2$ is a true statement. Let's look at a couple more examples.

$$\sqrt[3]{-1} = -1 \qquad \sqrt[3]{-27} = -3 \qquad \sqrt[3]{-64} = -4$$

Approximating Irrational Numbers

It is not always practical to work with irrational numbers. For example, you would not go to the store and order $\sqrt{15}$ packs of bubble gum. Instead it would be better to realize that $\sqrt{15} \approx 4$ and order four packs of bubble gum. How do we make those approximations?

One of the easiest ways to do this is to think of the perfect squares. Recall that the perfect squares are the numbers 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, and so forth. They are the numbers that have a whole number square root.

If you want to approximate $\sqrt{15}$, notice that $\sqrt{15} \approx \sqrt{16} = 4$. We simply see which perfect square the number inside the square root is closest to and use that to make an estimate. This works well for square roots that are relatively close to a perfect square.

However, some square roots we may want to approximate with more precision. By hand this will mean making an educated guess, checking that guess, and refining that guess. For example, let's approximate $\sqrt{22}$ to one decimal place of accuracy.

First note that $\sqrt{22}$ is between $\sqrt{16}$ and $\sqrt{25}$ which means that our solution is between 4 and 5. The solution is closer to 5 because $\sqrt{22}$ is closer to $\sqrt{25}$. Therefore we might guess that $\sqrt{22} \approx 4.7$ for an initial guess. Now let's check by examining $4.7 \times 4.7 = 22.09$. That's pretty close, and since 4.7^2 is just over 22, we might check 4.6 to see if it's a better solution. We find that $4.6 \times 4.6 = 21.16$ which is much farther away from 22 than our first guess was. This means that $\sqrt{22} \approx 4.7$ is the most approximate solution to one decimal place. Note that if we wanted our solution as an improper fraction, we could easily convert 4.7 to $\frac{47}{10}$.

Let's say that you want to make a square blanket for a baby and want it to be as large as possible. You have to buy cloth by the square foot and you can only afford 30 square feet since it costs $2 per square foot and you have $60 to spend. What is the approximate side length of the cloth square that you will need to have the fabric store cut for you to the nearest one decimal place?

$\sqrt{30}$ is between $\sqrt{25} = 5$ and $\sqrt{36} = 6$ and just barely closer to $\sqrt{25}$. So our first guess might be 5.4. Checking we get that $5.4 \times 5.4 = 29.16$, which is a little below the 30 we are looking for. So now we try $5.5 \times 5.5 = 30.25$, which is just above 30, but is a closer estimate. Therefore we will say $\sqrt{30} \approx 5.5$ and have the fabric store cut a cloth square that is 5.5 feet by 5.5 feet.

Using Technology

Calculators give you an approximation of irrational numbers whenever you find a square root of a non-perfect square. For example, plugging in $\sqrt{22}$ to a calculator gives us $\sqrt{22} \approx 4.6904157598$. This is an approximation because we know that the actual solution as a decimal goes on forever, but the calculator has to stop at some point and display the answer. If you were asked to *round* the solution of $\sqrt{22}$ to one decimal place, then you could simply plug in $\sqrt{22}$ to your calculator and then round it to $\sqrt{22} \approx 4.7$.

Most calculators also have a square root button that looks like $\boxed{\sqrt{\ }}$ or $\boxed{\sqrt{x}}$. They also have another root button that is used for cube roots or even higher roots that looks like $\boxed{\sqrt[x]{y}}$. To use this button type the number you want the root of, then hit the button, then which root you want. So if you wanted to find $\sqrt[3]{-8}$ you would type in -8, hit the $\boxed{\sqrt[x]{y}}$ button, then type 3 before hitting equals. While the calculator can perform these operations, it will be unnecessary since we confine ourselves to small perfect squares and cubes. In other words, you once again should be doing these type of problems by hand.

Lesson 7.3

Find both square roots of the given numbers.

1. 49 2. 64 3. 25 4. 16 5. 1 6. 121

7. 9 8. 196 9. 625 10. 4 11. 36 12. 81

Evaluate the following roots giving the principal root.

13. $\sqrt{81}$ 14. $-\sqrt{100}$ 15. $\sqrt{36}$ 16. $\sqrt{-4}$

17. $\sqrt{144}$ 18. $-\sqrt{225}$ 19. $-\sqrt{169}$ 20. $\sqrt{400}$

21. $-\sqrt{900}$ 22. $\sqrt[3]{-27}$ 23. $\sqrt[3]{125}$ 24. $\sqrt[3]{1}$

25. $\sqrt[3]{-1}$ 26. $\sqrt[3]{-64}$ 27. $\sqrt[3]{216}$ 28. $\sqrt[3]{8}$

29. $\sqrt[3]{-1000}$ 30. $\sqrt[3]{27}$ 31. $-\sqrt[3]{27}$ 32. $-\sqrt[3]{1}$

Approximate the following irrational numbers to the nearest whole number.

33. $\sqrt{28}$ 34. $\sqrt{14}$ 35. $-\sqrt{39}$ 36. $-\sqrt{56}$ 37. $-\sqrt{77}$ 38. $\sqrt{18}$

39. $\sqrt{2}$ 40. $\sqrt{41}$ 41. $\sqrt{21}$ 42. $-\sqrt{65}$ 43. $-\sqrt{12}$ 44. $-\sqrt{120}$

45. $\sqrt{8}$ 46. $\sqrt{13}$ 47. $\sqrt{32}$ 48. $\sqrt{47}$ 49. $-\sqrt{99}$ 50. $-\sqrt{5}$

Approximate the following irrational numbers to one decimal place.

51. $\sqrt{30}$ 52. $\sqrt{10}$ 53. $-\sqrt{40}$ 54. $-\sqrt{17}$ 55. $\sqrt{101}$ 56. $\sqrt{7}$

57. $\sqrt{3}$ 58. $\sqrt{90}$ 59. $\sqrt{35}$ 60. $-\sqrt{11}$ 61. $-\sqrt{22}$ 62. $\sqrt{61}$

63. $\sqrt{50}$ 64. $\sqrt{6}$ 65. $\sqrt{67}$ 66. $\sqrt{140}$ 67. $-\sqrt{55}$ 68. $-\sqrt{45}$

7.4 Comparing and Ordering Irrational Numbers on a Number Line

To compare irrational numbers that are square roots, we can simply examine the number that we are taking the square root of. For example, we know that $\sqrt{15} < \sqrt{17}$ because 15 is less than 17.

However, when we compare irrational numbers such as $\sqrt{10}$ and π, it is simplest to compare rational approximations of each written as a decimal. We know that $\sqrt{10} \approx 3.16$ and that $\pi \approx 3.14$. Therefore we can say that $\sqrt{10} > \pi$. Notice that it was useful to approximate the irrational numbers to two decimal places in this case even though it wasn't entirely necessary.

The same is true for comparing irrational and rational numbers. By finding a rational approximation of the irrational numbers, we can compare values such as π and $\frac{22}{7}$. For these numbers we may have to go to a three decimal places for our approximation and the use of a calculator would make sense. Rounded to three decimal places, we find that $\pi \approx 3.142$ and $\frac{22}{7} \approx 3.143$ which means that $\pi < \frac{22}{7}$.

Once we know how to compare two numbers, we can then order a set of numbers through comparison of two numbers at a time. For example, we could list from least to greatest $\sqrt{10}$, π, 3.14, and $\frac{22}{7}$. We know the following:

$$\sqrt{10} \approx 3.162$$

$$\pi \approx 3.142$$

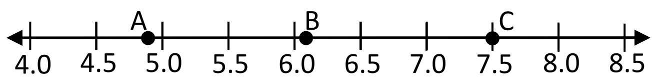
$$\frac{22}{7} \approx 3.143$$

We see that $\sqrt{10}$ is greater than all the other numbers given. We also note that 3.14 is the smallest because it is equal to 3.140 to three decimal places. Therefore we can list them in order like so: $3.14, \pi, \frac{22}{7}, \sqrt{10}$. Notice that the closer the numbers are to each other, the more decimal places of accuracy we need in our rational approximation.

Locating Irrational Numbers on a Number Line

Again, rational approximations of irrational numbers will be our friend. On a number line, we generally list rational number markers. On the simplest number lines, we count by integers. On a standard English ruler, we count by fractions, usually $\frac{1}{16}$ inch or $\frac{1}{8}$ inch. On a standard metric ruler, we count by millimeters which are each .1 centimeters. No matter how the number line is set up, we will still need the rational approximations of the irrational numbers.

For example, let's try to place the following irrational numbers on the number line: $\sqrt{37}$, $\sqrt{42}$, and $\sqrt{24}$. First we will make a quick, one decimal place approximation of each. $\sqrt{37} \approx 6.1$ since 37 is just over 36, $\sqrt{56} \approx 7.5$ since 56 is about half-way between the perfect squares of 49 and 64, and $\sqrt{24} \approx 4.9$ since 24 is just under 25. Now examine where the dots are located on the following number line.



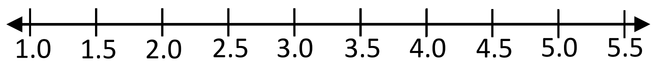
Note that point A must be $\sqrt{24}$ since it is just under 5, point B must be $\sqrt{37}$ since it is just over 6, and point C must be $\sqrt{56}$ since it is right at 7.5 on the number line.

In the same way that you can identify which point on a number line goes with which irrational number, you can also place points on a number line to represent the irrational number.

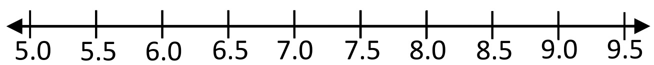
Lesson 7.4

Place a point on the number line given for each of the following irrational numbers.

1. Point A: $\sqrt{2}$ 2. Point B: $\sqrt{17}$ 3. Point C: $\sqrt{11}$ 4. Point D: $\sqrt{8}$ 5. Point E: $\sqrt{5}$

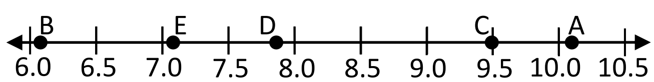


6. Point V: $\sqrt{26}$ 7. Point W: $\sqrt{88}$ 8. Point X: $\sqrt{77}$ 9. Point Y: $\sqrt{37}$ 10. Point Z: $\sqrt{30}$

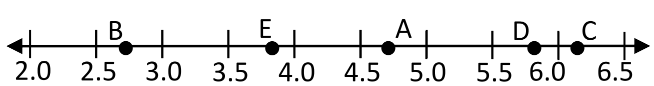


Name the point on the number line associated with each irrational number.

11. $\sqrt{50}$ 12. $\sqrt{103}$ 13. $\sqrt{62}$ 14. $\sqrt{90}$ 15. $\sqrt{37}$



16. $\sqrt{7}$ 17. $\sqrt{22}$ 18. $\sqrt{34}$ 19. $\sqrt{38}$ 20. $\sqrt{15}$



Compare the following numbers using < or >.

21. $\sqrt{32}\ \square\ 5.1$ 22. $\sqrt{38}\ \square\ \sqrt{42}$ 23. $\sqrt{17}\ \square\ \frac{9}{2}$ 24. $\sqrt{49}\ \square\ 7.1$

25. $\sqrt{99}\ \square\ \frac{28}{3}$ 26. $\sqrt{17}\ \square\ 4.5$ 27. $\frac{43}{5}\ \square\ \sqrt{65}$ 28. $\sqrt{12}\ \square\ \sqrt{21}$

29. $\sqrt{16}\ \square\ 3.9$ 30. $\sqrt{2}\ \square\ \frac{7}{4}$ 31. $\sqrt{50}\ \square\ \frac{15}{2}$ 32. $\sqrt{9}\ \square\ 3.01$

List the following numbers in order from least to greatest.

33. $\sqrt{16}$, 4.2, $\frac{39}{8}$

34. $\sqrt{24}$, $\sqrt{33}$,5.1

35. $\sqrt{100}$, $\sqrt{110}$, $\frac{32}{7}$

36. 9.4 , $\frac{19}{2}$, $\sqrt{80}$

37. $\sqrt{35}$, $\sqrt{32}$, $\sqrt{37}$, $\frac{22}{3}$

38. $\sqrt{10}$, 3.5, $\sqrt{15}$, $\frac{13}{3}$

39. $\sqrt{65}$, $\sqrt{60}$, 8.5, $\frac{37}{4}$

40. $\sqrt{39}$, $\sqrt{25}$, 5.3, $\sqrt{26}$, $\frac{23}{4}$

41. $\sqrt{12}$, $\sqrt{15}$, 4.3, $\sqrt{9}$, $\frac{14}{5}$

42. $\sqrt{49}$, $\sqrt{63}$, 7.3, $\sqrt{38}$, $\frac{15}{2}$

7.5 Estimating Values of Expressions

Surprise! We're going to use rational approximations of irrational numbers again. We're basically going to be looking at adding, subtracting, and multiplying irrational numbers and how to quickly estimate an answer.

For example, let's try to add the following irrational numbers: $\sqrt{37}+\sqrt{24}$. As a very fast estimate, we know that $\sqrt{37}\approx 6$ since 37 is just over 36 and $\sqrt{24}\approx 5$ since 24 is just under 25. That means we would estimate $\sqrt{37}+\sqrt{24}\approx 11$.

We could further fine tune our estimates by approximating the irrational numbers to one decimal place. Here's an example:

$$\sqrt{37}+\sqrt{56}\approx 6.1+7.5\approx 13.6$$

One last concept we need to be familiar with is multiplication involving irrational numbers. Recall that the expression $5x$ means five times x. In the same way, $5\sqrt{15}$ means five times the square root of fifteen. To estimate that expression we can approximate in the following way:

$5\sqrt{15}\approx 5(4)\approx 20$ or with more precision $5\sqrt{15}\approx 5(3.9)\approx 19.5$

Now combining all of those qualities we can estimate more complex expressions involving all of the operations of addition, subtraction, and multiplication. Just don't forget to follow the order of operations! For example, to the nearest whole number we could estimate the following expression:

$$2\sqrt{13}+5\sqrt{5}-\sqrt{37}\approx 2(3.5)+5(2)-\ 6\approx 11$$

Note that it was useful to approximate $\sqrt{13}$ to 3.5 in the middle of the problem since it led to a nice whole number solution at the end. In general, if we want a whole number answer, it might be a good idea to approximate each irrational as either a whole number or the nearest half value.

Lesson 7.5

Estimate the following expressions to the nearest whole number.

1. $\sqrt{8}+\sqrt{18}$	2. $11-\sqrt{80}$	3. $4\sqrt{48}$	4. $3\sqrt{24}+3$	5. $2\sqrt{35}-3\sqrt{8}$
6. $\sqrt{14}+\sqrt{26}$	7. $\sqrt{120}-7$	8. $2\sqrt{63}$	9. $4\sqrt{15}-5$	10. $2\sqrt{66}-3\sqrt{5}$
11. $\sqrt{9}+\sqrt{10}$	12. $20-\sqrt{102}$	13. $2\sqrt{15}$	14. $3\sqrt{15}+1$	15. $4\sqrt{24}-3\sqrt{3}$
16. $\sqrt{14}+\sqrt{34}$	17. $\sqrt{105}-9$	18. $5\sqrt{26}$	19. $2\sqrt{83}-8$	20. $3\sqrt{17}-2\sqrt{1}$
21. $\sqrt{47}+\sqrt{8}$	22. $8-\sqrt{48}$	23. $7\sqrt{10}$	24. $4\sqrt{5}+9$	25. $3\sqrt{24}-5\sqrt{5}$
26. $\sqrt{65}+\sqrt{63}$	27. $\sqrt{100}-2$	28. $6\sqrt{5}$	29. $2\sqrt{26}-3$	30. $4\sqrt{26}-3\sqrt{4}$

Review Unit 7: Real Numbers

No calculator necessary. Please do not use a calculator.

Unit 7 Goals

- *Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert to a decimal expansion which repeats eventually into a rational number. (8.NS.1)*
- *Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions. (8.NS.2)*

Convert the following fractions into decimals or decimals into lowest terms fractions.

1. $\frac{2}{15}$ 2. $\frac{5}{9}$ 3. $0.2\overline{8}$ 4. $0.\overline{16}$

Identify if the given number is rational or irrational and explain how you know.

5. $\sqrt{21}$ 6. $0.\overline{4}$ 7. $\sqrt{0.04}$ 8. $-\frac{2}{3}$

Evaluate the following roots.

9. $\sqrt{81}$ 10. $-\sqrt{4}$ 11. $\sqrt[3]{-1000}$ 12. $\sqrt[3]{125}$

Approximate the square roots to one decimal place.

13. $\sqrt{84}$ 14. $-\sqrt{27}$ 15. $-\sqrt{50}$ 16. $\sqrt{95}$

Compare the following numbers using < or >.

17. $\sqrt{20}\ \square\ 5.1$ 18. $-\frac{20}{21}\ \square\ -\sqrt{2}$ 19. $\sqrt{15}\ \square\ \sqrt{13}$ 20. $\pi\ \square\ \sqrt{5}$

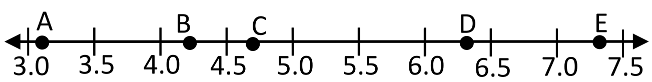
List the following numbers in order from least to greatest.

21. $\sqrt{8}$, π, 9, 3.1, $\frac{15}{7}$

22. $\sqrt{35}$, $\sqrt{40}$, 6.01, $\frac{22}{3}$

23. $\sqrt{3}$, 1.1, 2, $\frac{1}{2}$

Match the given number with the letter that approximates that number's position on the number line.



24. $\sqrt{10}$

25. $\sqrt{40}$

26. $\sqrt{18}$

27. $\sqrt{21}$

Estimate the value of the expressions to the nearest whole number.

28. $3\sqrt{15}$

29. $4 + \sqrt{35}$

Answer the following questions.

30. A calculator displays the following number in its display: 0.4153 which does not fill the display screen. Is this enough to determine whether this number is rational or irrational? In at least one complete sentence, explain why.

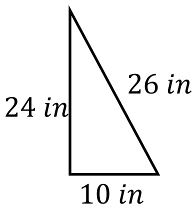
Unit 8: Geometry Applications

Pre-Test Unit 8: Geometry Applications

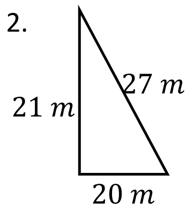
You may use a calculator.

Determine if each of the following is a right triangle or not using the Pythagorean Theorem. *(5 pts; 3 pts for set-up/work or explanation, 2 pts for correct answer)*

1.

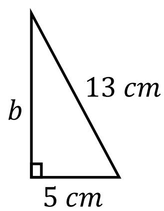


2.

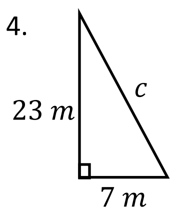


Find the length of the missing side of each right triangle. Round your answer to three decimal places if necessary. *(5 pts; 3 pts for set-up/work or explanation, 2 pts for correct answer)*

3.

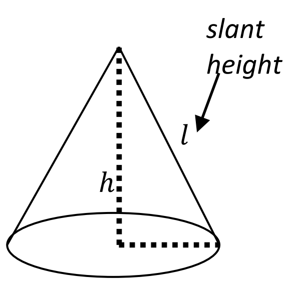


4.

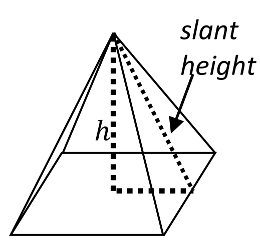


Find the value of the variable. Round your answer to three decimal places if necessary. *(5 pts; 3 pts for set-up/work or explanation, 2 pts for correct answer)*

5. The following cone has a radius of $6\ mm$ and a height of $8\ mm$. What is l, the slant height?

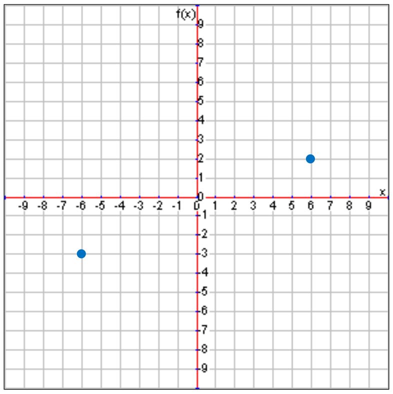


6. The following pyramid has a square base that is $50\ ft$ on each side. The slant height is $50\ ft$. What is h, the height of the pyramid?

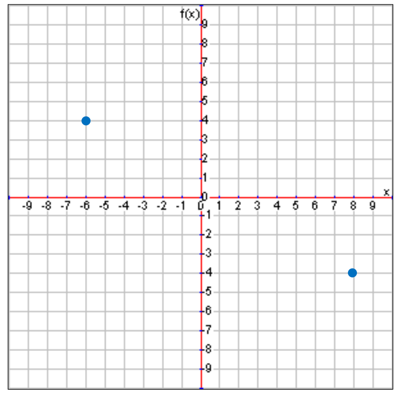


Determine the distance between the given points. Round your answer to three decimal places if necessary.
(5 pts; 3 pts for set-up/work or explanation, 2 pts for correct answer)

7. $(-6,-3)$ and $(6,2)$



8. $(-6,4)$ and $(8,-4)$



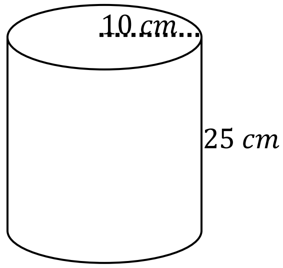
Solve the following problems. *(5 pts; 3 pts for set-up/work, 2 pts for correct final answer)*

9. A hospital helicopter must go pick up a patient that is six miles west and eight miles north of the hospital. How many miles total will the helicopter travel to pick up the patient *and bring him back* to the hospital?

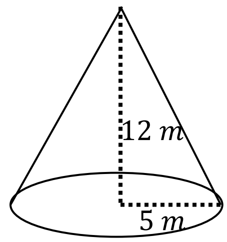
10. A nature area has a rectangle field that is 10 miles by 5 miles and wants to put a fence along the diagonal of the field that will costs $\$1{,}000$ per mile. How much will the fence cost to the nearest dollar?

Find the given dimension of each shape either using $\pi \approx 3.14$ or giving your answer in terms of π. Round your answer to two decimal places if necessary. *(5 pts; 3 pts for set-up/work, 2 pts for correct answer)*

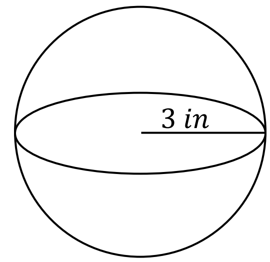
11. Find the volume



12. Find the volume



13. Find the volume



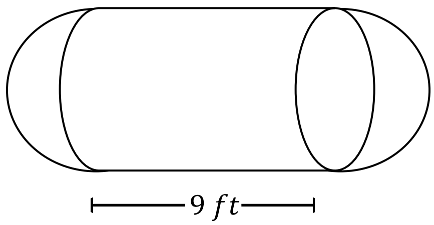
14. Find the radius of a cylindrical fire hose that is $200\ ft$ long and has a volume of $39.25\ ft^3$.

15. Find the height of a waffle cone for ice cream that has a volume of $25.12\ in^3$ and a radius of $2\ in$.

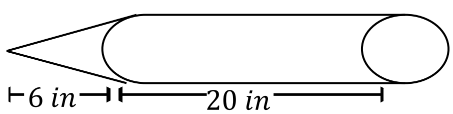
16. Find the radius of a spherical water balloon with a volume of $904.32\ cm^3$.

Find the volume of each shape either using $\pi \approx 3.14$ or giving your answer in terms of π. Round your answer to two decimal places if necessary. *(10 pts; 4 pts for work/volume of each shape, 2 pts for final answer)*

17. A cylindrical propane gas tank with half spheres on either end that is $9\ ft$ long (not including the half spheres) and has a $3\ ft$ radius



18. A caulking gun with a radius of $1\ in$, cone height of $6\ in$, and cylinder height of $20\ in$

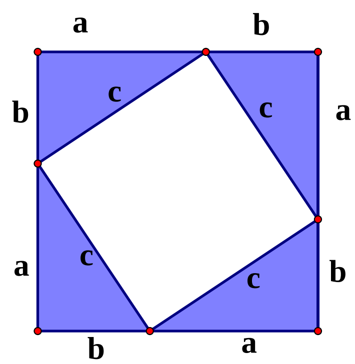


8.1 Pythagorean Theorem and Converse

The Pythagorean Theorem states that if a triangle is a right triangle, then the sum of the squares of the lengths of the legs equals the square of the hypotenuse lengths. That's a complicated way to say that if the legs of the triangle measure a and b and the hypotenuse measures c, then $a^2 + b^2 = c^2$. While you may have heard this in the past, we will now prove it.

Proof of the Pythagorean Theorem

There are many ways to prove the Pythagorean Theorem, but take a look at the following picture. We will refer to this for our proof.



In this picture we have a large square whose side lengths are equal to $a + b$ and an inner square whose side lengths are c. Notice that if we find the area of the large square and subtract the area of the triangles we get the area of the inner square. So let's do that algebraically.

The area of the larger square is:

$$(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = a^2 + ab + ab + b^2$$

$$= a^2 + 2ab + b^2$$

The area of each triangle is $\frac{1}{2}ab$ and since there are four of them, the total area of the triangles is $2ab$.

The area of the inner square is c^2.

This means the large square minus the triangles would look like this:

$$a^2 + 2ab + b^2 - 2ab = c^2$$

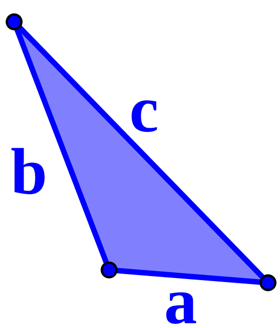
Notice that the $+2ab$ and the $-2ab$ cancel each other out (become zero), so we do get the result we expect which is that $a^2 + b^2 = c^2$.

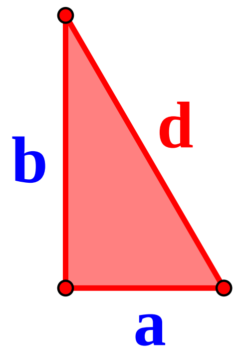
Do a search online to see if you can find another proof for this vital theorem.

Proof of the Pythagorean Theorem Converse

The converse of the Pythagorean Theorem states that if a triangle with side lengths a, b, and c has the property that $a^2 + b^2 = c^2$, then it is a right triangle. We will now prove this.

Assume you have a triangle with side lengths a, b, and c has the property that $a^2 + b^2 = c^2$. Now construct another triangle with side lengths a and b, but make it a right triangle this time with a hypotenuse of length d. The picture would look like this with the original triangle on the left (the one that we don't know whether it is a right triangle or not) and the new triangle on the right (the one we make specifically to be a right triangle).





Since we know the Pythagorean Theorem is true, we know that $a^2 + b^2 = d^2$ which means that $d = \sqrt{a^2 + b^2}$ by taking the square root of both sides.

This means that $d = c$ since $c = \sqrt{a^2 + b^2}$ as well by the original statement that for this triangle $a^2 + b^2 = c^2$. Since all three side lengths are the same, the two triangles are congruent which means that the first triangle must be a right triangle just like the second one we made.

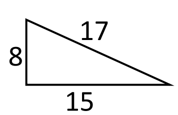
We can't use what has been called the LLP, or the Looks Like it Postulate. Just because the triangle on the left doesn't look like a right triangle, doesn't mean it actually isn't based on the facts we are given about it. The picture is inaccurate in this case.

Implications for the Pythagorean Theorem and its Converse

Now that we know both if a triangle is right then $a^2 + b^2 = c^2$ and if $a^2 + b^2 = c^2$ then the triangle is right, we can solve multiple types of problems. Given any two side lengths of a right triangle we can solve for the third side length using the Pythagorean Theorem. Given three side lengths of a triangle we can test if it's a right triangle using the Pythagorean Theorem converse.

Is it Right?

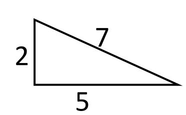
Because of the Pythagorean Converse, we can check whether a triangle is a right triangle or not. Consider the following two triangles. If their side lengths make the Pythagorean Theorem true, they are right.



$8^2 + 15^2 = 17^2$

$64 + 225 = 289$

True, so this is a right triangle.



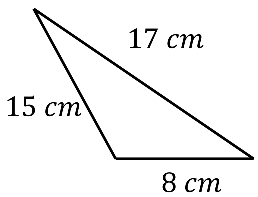
$2^2 + 5^2 = 7^2$

$4 + 25 \neq 49$

False, $4 + 25$ is not 49, so it is not a right triangle.

Lesson 8.1

Determine if the following triangles are right triangles or not using the Pythagorean Theorem.

1.


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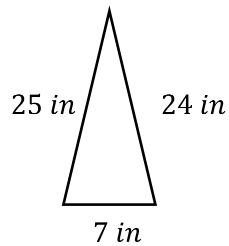

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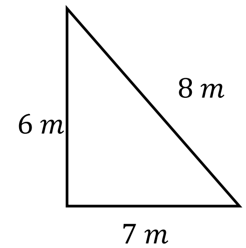

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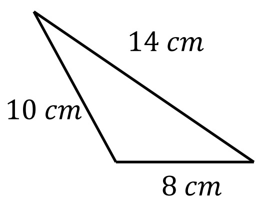

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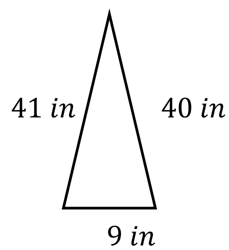
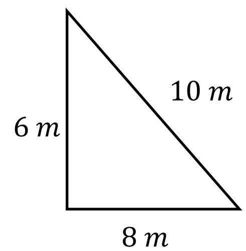


6.


7. $a = 12\ ft$

   $b = 16\ ft$

   $c = 25\ ft$

8. $a = 12\ km$

   $b = 35\ km$

   $c = 37\ km$

9. $a = 10\ mm$

   $b = 24\ mm$

   $c = 27\ mm$

10. $a = 20\ ft$

    $b = 21\ ft$

    $c = 29\ ft$

11. $a = 5\ km$

    $b = 12\ km$

    $c = 17\ km$

12. $a = 5\ mm$

    $b = 12\ mm$

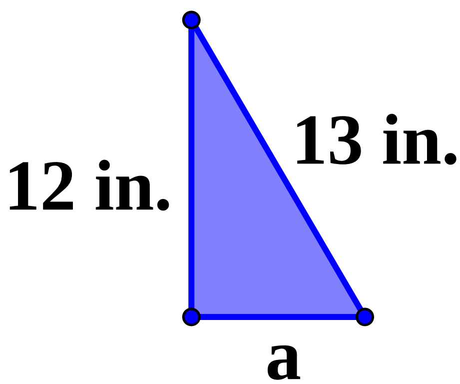
    $c = 13\ mm$

8.2 2D Applications of the Pythag. Theorem

Since we know that in a right triangle the statement $a^2 + b^2 = c^2$ must be true, we can now solve for any missing side length given the other two side lengths. The process of solving for a missing leg (a or b) is only slightly different from solving for a missing hypotenuse (c).

Solving for a Missing Leg

Let's first solve for a missing leg. First note that it makes no difference which leg we label as a and which leg we label as b. This is because the commutative property says that we can add in any order. In other words, whether we have $a^2 + b^2$ or $b^2 + a^2$ doesn't matter, it will always equal c^2. So if we are missing the length of a leg, it might be easiest to always assume it is a that is missing.



Given the fact that this is a right triangle, we can solve for the missing leg length, a. Just substitute everything we know into the Pythagorean Formula. We know that the hypotenuse length, c, is 13 inches and that the other leg length, b, is 12 inches.

$$a^2 + b^2 = c^2$$

$$a^2 + (12)^2 = (13)^2$$

Now go ahead and multiple out those exponents to get the following statement:

$$a^2 + 144 = 169$$

Notice this is a two-step equation where a is being squared and then increased by 144. Applying inverse operations, we know we should subtract 144 from both sides and then take the square root. That looks like this:

$$a^2 + 144 = 169$$

$$-144 \quad -144$$

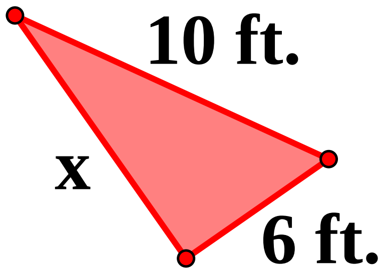
$$a^2 = 25$$

$$\sqrt{a^2} = \sqrt{25}$$

$$a = 5$$

We have just proved that the missing side length must be 5 inches.

Sometimes the missing side length will be labeled with a different variable just to throw us off. Just remember that the legs are always a and b in the Pythagorean Formula and that c, or the hypotenuse, is always the longest side length. For example, in the following picture which are the legs and which is the hypotenuse?



The hypotenuse is always opposite (or across from) the right angle and is the longest side. So the hypotenuse in this picture is 10 ft. That means that the 6 ft and the x must be the two sides. Notice that the legs can also be identified by the fact that they are the sides that make up the right angle. Now substitute into the Pythagorean Formula to solve for x.

$$x^2 + (6)^2 = (10)^2$$

$$x^2 + 36 = 100$$

$$-36 \quad -36$$

$$x^2 = 64$$

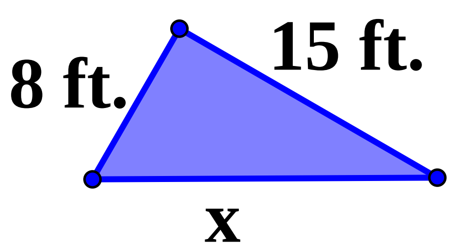
$$\sqrt{x^2} = \sqrt{64}$$

$$x = 8$$

So we know that the missing side length is 8 ft. in this particular triangle.

Solving for a Missing Hypotenuse

Let's now solve for a missing hypotenuse. Remember that the hypotenuse is always the longest side and the side opposite the right angle. Take a look at this example.



Note that 8 ft. and 15 ft. must the lengths of the legs since they make up the right angle. That means that x in this case is the missing hypotenuse. Plugging those values into the Pythagorean Formula yields the following:

$$(8)^2 + (15)^2 = x^2$$

$$64 + 225 = x^2$$

Be careful at this point. Many students mistakenly try to subtract either 64 or 225 from both sides, but that is not accurate. We always combine like terms before using inverse operations, and in this case we still need to combine the 64 + 225 to get 289. So our next steps should look like this:

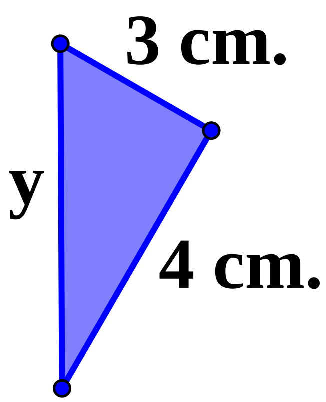
$$289 = x^2$$

$$\sqrt{289} = \sqrt{x^2}$$

$$17 = x$$

This means that the missing hypotenuse length is 17 feet. Note that the only inverse operation we needed to apply in this case was the square root.

Let's look at one more example of solving for a missing hypotenuse. Consider the following picture.



Note that y is the hypotenuse in this case because the sides with lengths 3 and 4 make up the right angle. Plug these values into the Pythagorean Formula.

$$(3)^2 + (4)^2 = y^2$$

$$9 + 16 = y^2$$

$$25 = y^2$$

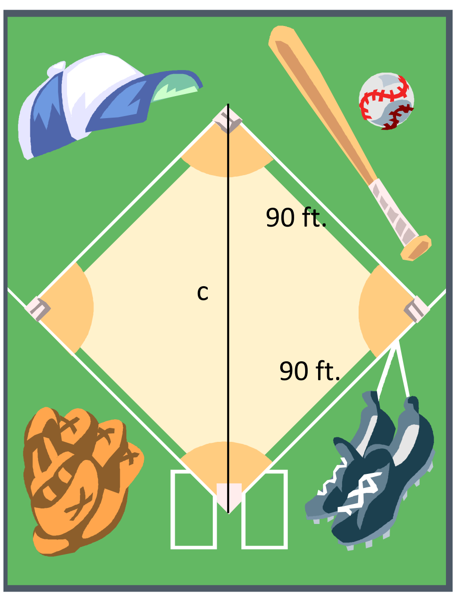
$$\sqrt{25} = \sqrt{y^2}$$

$$5 = y$$

So the hypotenuse has a length of 5 centimeters in this case.

Pythagorean Theorem Word Problems

The use of the Pythagorean Theorem can applied to word problems just as easily. For example, if we know that it is 90 feet from home plate to first base and 90 feet from first base to second base, how far would the catcher have to throw the baseball to get a runner out who is stealing second base? The best tip to give for solving word problems like this is to draw a picture.



In this case, note that the distance from second base to home plate is the hypotenuse of the triangle. That means that the 90 feet distances are the legs. We can now solve as follows.

$$(90)^2 + (90)^2 = c^2$$

$$8100 + 8100 = c^2$$

$$16200 = c^2$$

$$\sqrt{16200} = \sqrt{c^2}$$

$$127.3 \approx c$$

For this problem, there was no exact square root. That means that $\sqrt{16200}$ is irrational and it's probably best to estimate this number. Our answer is approximated to the nearest one decimal place giving us about 127.3 feet. So the catcher would have to throw just over 127 feet to get out the runner trying to steal second base.

Lesson 8.2

Find the length of the missing side of each right triangle. Round your answers to three decimal places if necessary.

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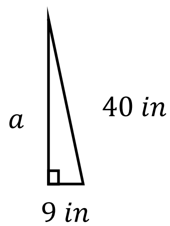

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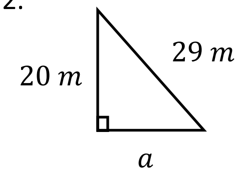

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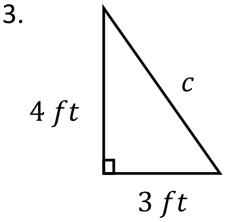

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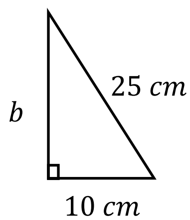

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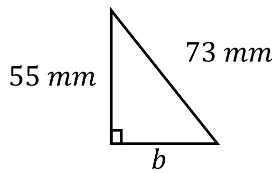

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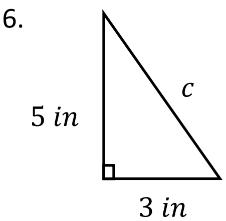

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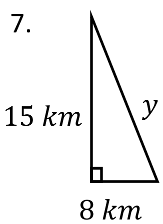

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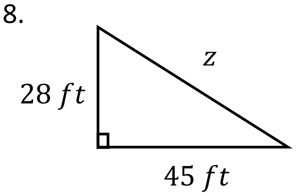

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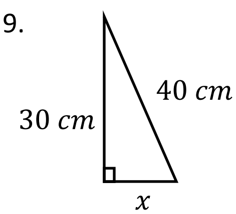

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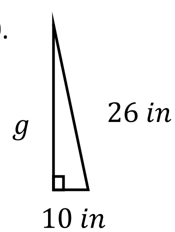

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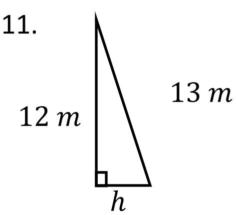

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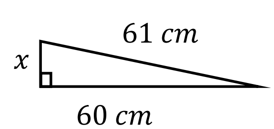

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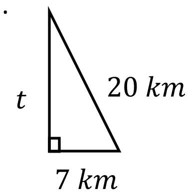

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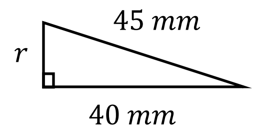

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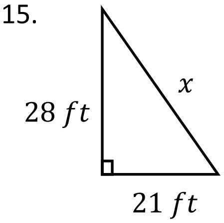


Solve the following problems. Round your answers to the nearest whole number when necessary.

16. You're locked out of your house, and the only open window is on the second floor 25 feet above the ground. There are bushes along the side of the house that force you to put the base of the ladder 7 feet away from the base of the house. How long of a ladder will you need to reach the window?

17. Shae takes off from her house and runs 3 miles north and 4 miles west. Tired, she wants to take the shortest route back. How much farther will she have to run if she heads straight back to her house?

18. Televisions are advertised by the length of their diagonals. If a 42 inch television measures 18 inches high, how wide is the television?

19. A soccer field is 100 yards by 60 yards. How long is the diagonal of the field?

20. Leonard walks 14 meters south and 48 meters east to get to school. If he takes the straight path home after school, how far will he have to walk?

21. You place a 24 foot ladder 10 feet away from the house. The top of the ladder just reaches a window on the second floor. How high off the ground is the window?

22. The dimensions of a basketball court are 74 feet and 42 feet. What is the length of the diagonal of the court?

23. Televisions are advertised by the length of their diagonals. If a TV measures 22 inches high and 45 inches wide, by what size will the TV be advertised.

24. A rectangular garden measures 5 feet wide by 12 feet long. If a hose costs $5 per foot, how much would it cost to place a hose through the diagonal of the garden?

25. A football field is 160 feet wide and 360 feet long. The coach wants to put spray paint along the diagonal of the field. If the spray paint costs approximately $1 per foot of coverage, how much should the coach budget for spray paint?

26. A rectangular park measures 8 miles long by 6 miles wide. The park director wants to put a fence along both sides of the trail that runs diagonally through the park. If the fence costs $150 per mile, how much will it cost to buy the fence?

27. A rectangular pool has a diagonal of 17 yards and a length of 15 yards. If the paint costs $2 per yard of coverage, how much will it cost the owner to paint the width of both ends of the pool?

28. A rectangular dog pen is 3 meters by 4 meters. If a chain costs $1.75 per meter, how much would it cost to put a chain along the diagonal of the pen?

29. Architects built a doorway that was 4 feet wide by 7 feet tall. The diagonal measured 7.3 feet. Are the angles in the doorway right angles?

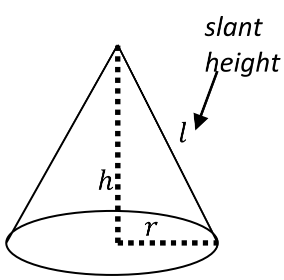
30. A rectangular garden measures 3 meters wide by 4 meters long. The diagonal of the garden measures 5 meters. Are the angles in the garden right angles?

8.3 3D Applications of the Pythag. Theorem

Now that we know how to use the Pythagorean Theorem to find either a missing leg or missing hypotenuse, we can move this concept into three-dimensional concepts. Let's look at some examples.

Regular Cones and Pyramids

In both cones and pyramids we can use the Pythagorean Theorem to find the height or slant height. We can also find the radius in a cone or the side length of the base of a pyramid. First look at this cone.



The height and radius are fairly obvious, but the slant height might be new vocabulary. The slant height, usually referred to as l in problems, is the height from the outer bottom edge of the cone up to the tip. It is not the actual height because it is not perpendicular to the base. Notice that each of these three variables form a right triangle. Therefore if we know two of them we can find the other one.

For example, assume that $r = 5\ in.$ and $h = 12\ in.$ What is l, or the slant height, in inches?

$$a^2 + b^2 = c^2$$

$$r^2 + h^2 = l^2$$

$$5^2 + 12^2 = l^2$$

$$25 + 144 = l^2$$

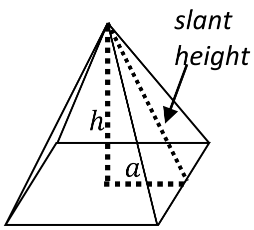
$$169 = l^2$$

$$\sqrt{169} = \sqrt{l^2}$$

$$13\ in. = l$$

Why is this useful? Knowing each dimension allows us to find the volume or surface area of the shape. For example, the formula for the surface of a cone is $SA = \pi r^2 + \pi r l$. Now that we know $l = 13\ in.$ we can find the surface area is $SA = 25\pi + 65\pi = 90\pi\ in^2$.

Let's look at a pyramid missing its height.



Assume that the base of the pyramid (bottom) is a square with side length of $12\ cm.$ and the slant height is $10\ cm$. What is the height of the pyramid? Since we have the side length of the square, we only need half of that to form the short leg of our right triangle, or $6\ cm$. This gives us the following:

$$a^2 + b^2 = c^2$$

$$a^2 + h^2 = l^2$$

$$6^2 + h^2 = 10^2$$

$$36 + h^2 = 100$$

$$-36 \qquad\qquad -36$$

$$h^2 = 64$$

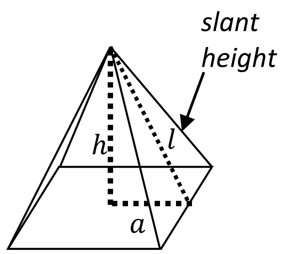
$$\sqrt{h^2} = \sqrt{64}$$

$$h = 8\ cm.$$

Again, we can now answer further questions about this shape like finding the volume. The volume of a regular pyramid uses the formula $V = \frac{1}{3}Bh$ where B is the area of the base shape (the square). This means that the volume is $\frac{1}{3}(144)(8) = 384\ cm^3$.

Lesson 8.3

Use the picture below to find information about the pyramid with a square base in problems 1-14. Round your answers to three decimal places if necessary.



1. The pyramid has a square base that is $70\ ft$ on each side. The slant height is $37\ ft$. What is h, the height of the pyramid?

2. The pyramid has a square base that is $120\ in$ on each side. The slant height is $61\ in$. What is h, the height of the pyramid?

3. The pyramid has a square base that is $50\ m$ on each side. The slant height is $30\ m$. What is h, the height of the pyramid?

4. The pyramid has a square base that is $14\ cm$ on each side. The slant height is $25\ cm$. What is h, the height?

5. The pyramid has a square base that is $14\ cm$ on each side. The height is $24\ cm$. What is l, the slant height?

6. The pyramid has a square base that is $24\ ft$ on each side. The height is $5\ ft$. What is l, the slant height?

7. The pyramid has a square base that is $70\ mm$ on each side. The height is $10\ mm$. What is l, the slant height?

8. The pyramid has a square base that is $26\ ft$ on each side. The height is $82\ ft$. What is l, the slant height?

9. The height of the pyramid is $15\ cm$, and the slant height is $39\ cm$. Find the value of a in the diagram.

10. The height of the pyramid is $80\ in$, and the slant height is $82\ in$. Find the value of a in the diagram.

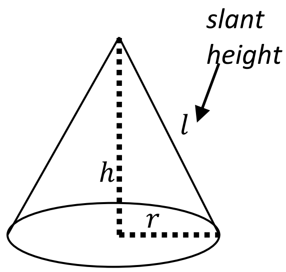
11. The slant height is $17\ ft$ and the height is $8\ ft$. What is s, the side length of the base?

12. The slant height is $10\ cm$ and the height is $8\ cm$. What is s, the side length of the base?

13. The slant height is $26\ mm$ and the height is $10\ mm$. What is s, the side length of the base?

14. The slant height is $50\ ft$ and the height is $32\ ft$. What is s, the side length of the base?

Use the picture below to find information about the pyramid in problems 15-26. Round your answers to three decimal places if necessary.



15. The cone has a radius of $12\ cm$ and a height of $5\ cm$. What is l, the slant height of the cone?

16. The cone has a radius of $15\ mm$ and a height of $8\ mm$. What is l, the slant height of the cone?

17. The cone has a radius of $24\ in$ and a height of $70\ in$. What is l, the slant height of the cone?

18. The cone has a radius of $40\ cm$ and a height of $42\ cm$. What is l, the slant height of the cone?

19. The cone has a radius of $30\ ft$ and a slant height of $34\ ft$. What is h, the height of the cone?

20. The cone has a radius of $33\ m$ and a slant height of $65\ m$. What is h, the height of the cone?

21. The cone has a radius of $16\ in$ and a slant height of $20\ in$. What is h, the height of the cone?

22. The cone has a radius of $30\ cm$ and a slant height of $50\ cm$. What is h, the height of the cone?

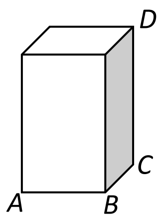
23. The cone has a height of $16\ cm$ and a slant height of $65\ cm$. What is r, the radius of the cone?

24. The cone has a height of $48\ ft$ and a slant height of $50\ ft$. What is r, the radius of the cone?

25. The cone has a height of $4\ in$ and a slant height of $6\ in$. What is r, the radius of the cone?

26. The cone has a height of $14\ cm$ and a slant height of $55\ cm$. What is r, the radius of the cone?

Use the picture below to find lengths of segments in the rectangular prism in problems 27-38. Round your answers to three decimal places if necessary.



27. The length of $\overline{AB}$ is 6 ft and the length of $\overline{BC}$ is 8 ft. Find the length of $\overline{AC}$.

28. The length of $\overline{AB}$ is 40 mm and the length of $\overline{BC}$ is 42 mm. Find the length of $\overline{AC}$.

29. The length of $\overline{AB}$ is 23 cm and the length of $\overline{BC}$ is 70 cm. Find the length of $\overline{AC}$.

30. The length of $\overline{AB}$ is 7 in and the length of $\overline{BC}$ is 7 in. Find the length of $\overline{AC}$.

31. The length of $\overline{AC}$ is 13 mm and the length of $\overline{DC}$ is 84 mm. Find the length of $\overline{AD}$.

32. The length of $\overline{AC}$ is 5 ft and the length of $\overline{DC}$ is 12 ft. Find the length of $\overline{AD}$.

33. The length of $\overline{AC}$ is 11 mm and the length of $\overline{DC}$ is 30 mm. Find the length of $\overline{AD}$.

34. The length of $\overline{AC}$ is 5 in and the length of $\overline{DC}$ is 4 in. Find the length of $\overline{AD}$.

35. The length of $\overline{AB}$ is 4 ft, the length of $\overline{BC}$ is 3 ft and the length of $\overline{DC}$ is 12 ft. Find the length of $\overline{AD}$.

36. The length of $\overline{AB}$ is 12 cm, the length of $\overline{BC}$ is 5 cm and the length of $\overline{DC}$ is 84 cm. Find the length of $\overline{AD}$.

37. The length of $\overline{AB}$ is 2 ft, the length of $\overline{BC}$ is 3 ft and the length of $\overline{DC}$ is 10 ft. Find the length of $\overline{AD}$.

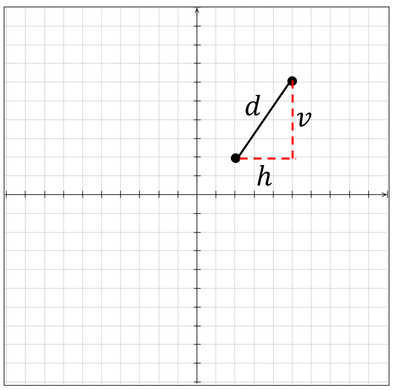
38. The length of $\overline{AB}$ is 6 mm the length of $\overline{BC}$ is 8 mm and the length of $\overline{DC}$ is 50 mm. Find the length of $\overline{AD}$.

8.4 The Distance Between Points

A final application of the Pythagorean Theorem is on the coordinate plane. We can easily find the distance between two points vertically or horizontally on a coordinate plane just by counting, but finding the exact distance diagonally we have not been able to do until now.

The Distance between Any Two Points

On a coordinate plane, we can now find the distance between any two points by drawing in a right triangle and using the Pythagorean Theorem. Consider the following example:



Notice that if we want to find the distance between these two points, $(2,2)$ and $(5,6)$, we need to find the length of d. Also note that h is the horizontal distance between the points and v is the vertical distance between the points. With all those values we now have a right triangle and can use the Pythagorean Theorem as follows:

$$h^2 + v^2 = d^2$$

$$3^2 + 4^2 = d^2$$

$$9 + 16 = d^2$$

$$25 = d^2$$

$$5 = d$$

So we know that the distance between these points is five units. While this is easy to see when drawn out on the coordinate plane, there are times when we are given the two points without a picture. In that case, we have two options. We can either draw the points on the coordinate plane as above, or we can find the horizontal and vertical distance between the points in another way.

To do this without graphing, we realize that the horizontal distance between two points is the difference in their x values. Why is this? Similarly, the vertical distance between two points is the difference in their y values. Again, can you explain why?

So let's look at our two points again, $(2,2)$ and $(5,6)$. The horizontal distance would be the difference between 2 and 5. Since difference means subtract, we can take $5 - 2 = 3$ to find the horizontal distance is 3. Similarly we can subtract the y values to get $6 - 2 = 4$ meaning a vertical distance of 4. We can then plug in 3 and 4 into the Pythagorean Theorem and solve exactly as above.

Enrichment: The Distance Formula

Using the information above, how would we find the distance between two generic points? We typically represent generic points with the notation of (x_1, y_1) and (x_2, y_2). So what would the horizontal and vertical distance between these two points be?

Horizontal distance: $h = x_2 - x_1$

Vertical distance: $v = y_2 - y_1$

Finally, let's substitute these into the Pythagorean Theorem of $h^2 + v^2 = d^2$ as follows and then solve for d since d is the actual distance between the points.

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{d^2}$$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d$$

The final result is what is known as the distance formula. Let's use this formula to find the distance between the points $(-3, 4)$ and $(3, -4)$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(3 - (-3))^2 + ((-4) - 4)^2}$$

$$d = \sqrt{(6)^2 + (-8)^2}$$

$$d = \sqrt{36 + 64}$$

$$d = \sqrt{100}$$

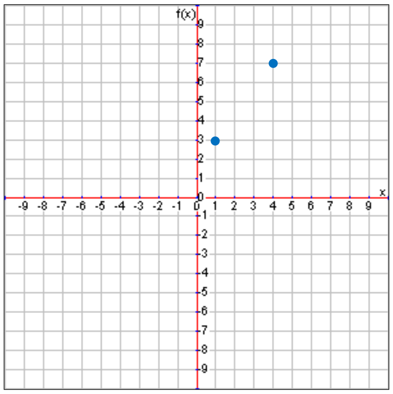
$$d = 10$$

We see that the distance between those two points is ten units. While the distance formula works, it is often easier to simply visualize the horizontal and vertical distance between two points mentally or on a coordinate plane. The distance formula is basically a fancy way to use the Pythagorean Formula and is meant for enrichment only.

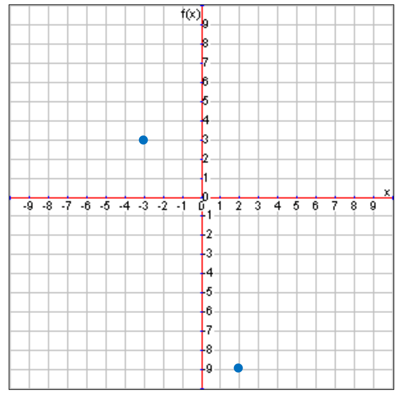
Lesson 8.4

Determine the distance between the given points. Round your answers to three decimal places if necessary.

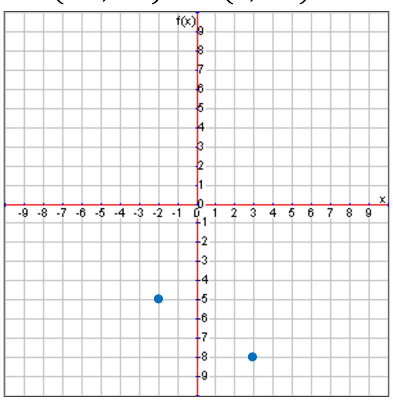
1. $(1,3)$ and $(4,7)$



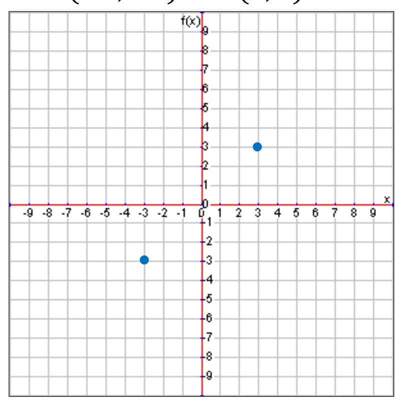
2. $(-3,3)$ and $(2,-9)$



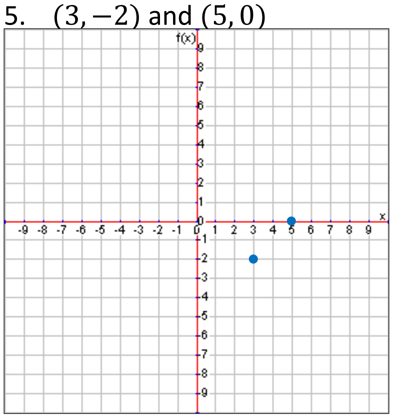
3. $(-2,-5)$ and $(3,-8)$



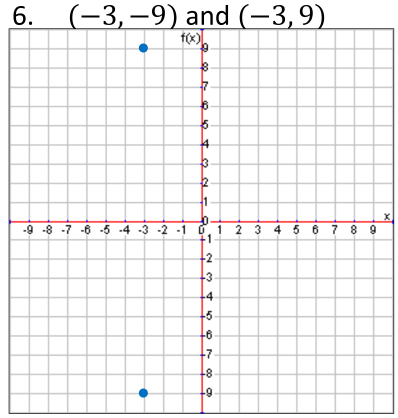
4. $(-3,-3)$ and $(3,3)$



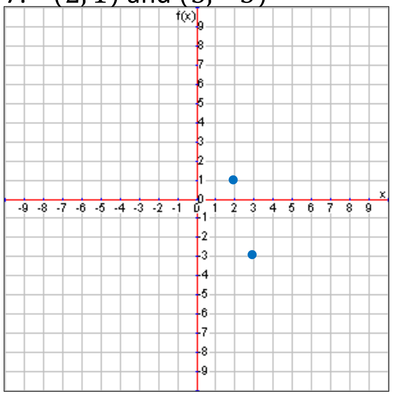
5. $(3,-2)$ and $(5,0)$



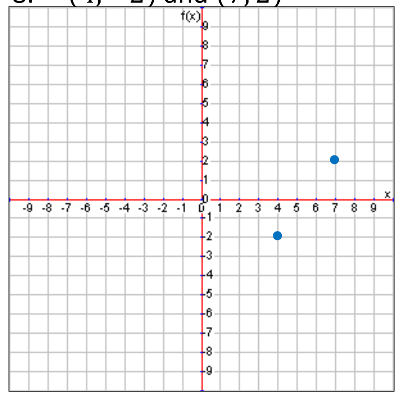
6. $(-3,-9)$ and $(-3,9)$



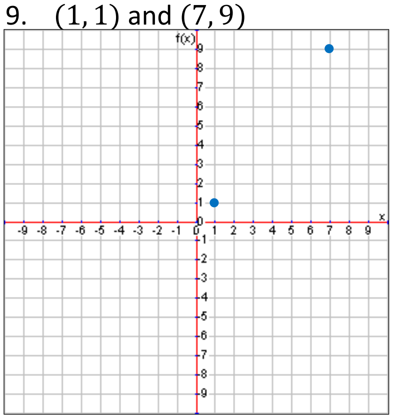
7. $(2, 1)$ and $(3, -3)$



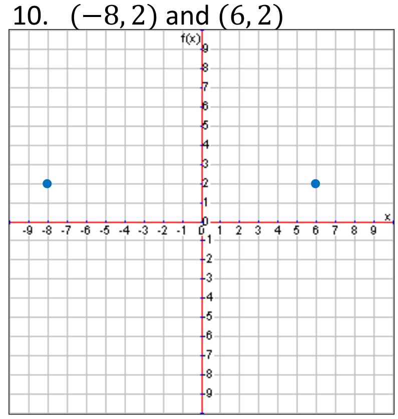
8. $(4, -2)$ and $(7, 2)$



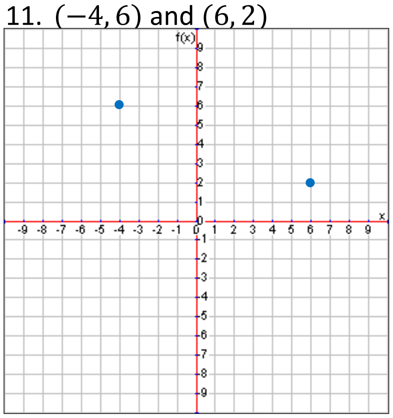
9. $(1, 1)$ and $(7, 9)$



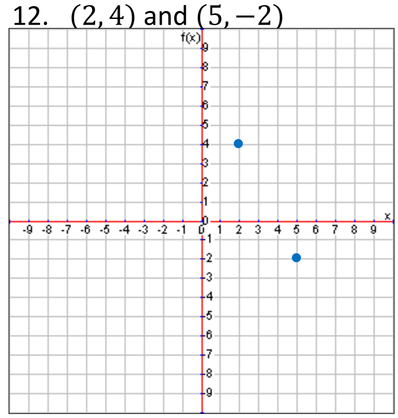
10. $(-8, 2)$ and $(6, 2)$



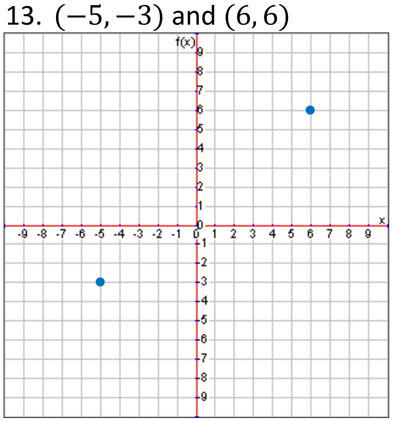
11. $(-4, 6)$ and $(6, 2)$



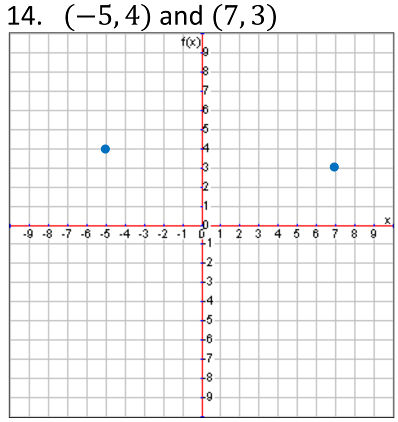
12. $(2, 4)$ and $(5, -2)$



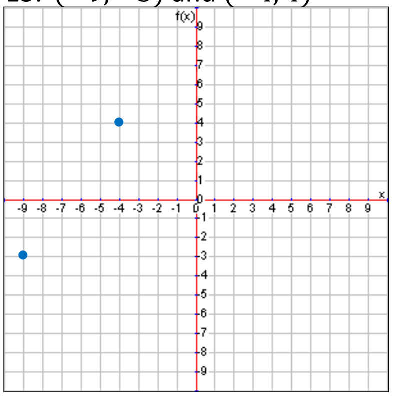
13. $(-5, -3)$ and $(6, 6)$



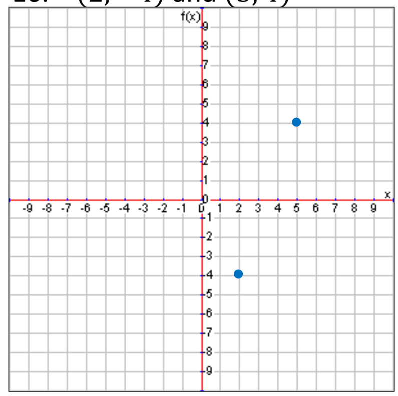
14. $(-5, 4)$ and $(7, 3)$



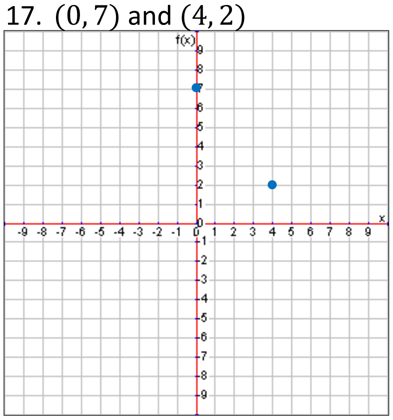
15. $(-9,-3)$ and $(-4,4)$



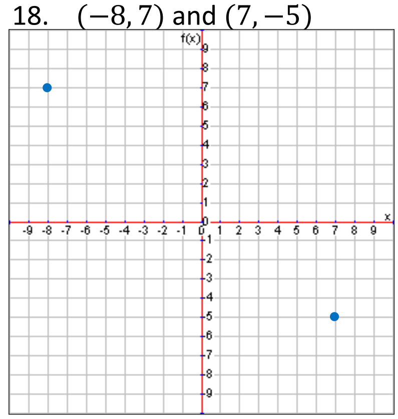
16. $(2,-4)$ and $(5,4)$



17. $(0,7)$ and $(4,2)$



18. $(-8,7)$ and $(7,-5)$

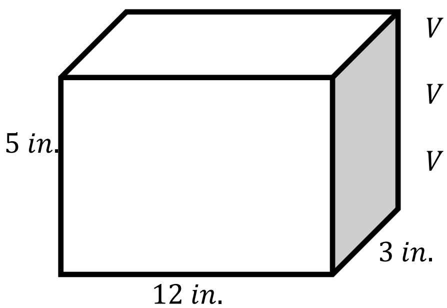


8.5 Volume of Rounded Objects

A basic definition of **volume** is how much space an object takes up. Since this is a three-dimensional measurement, the unit is usually cubed. For example, we might talk about how many cubic feet of water are in a pool (or ft^3) or how many cubic millimeters of ink are in an ink pen (or mm^3). Let's explore some previously learned concepts about volume before diving into cylinders.

Volume Basics

Perhaps the most recognizable formula for volume comes from a rectangular prism. This easily remembered formula is $V = lwh$, or the volume is the length times the width times the height. So in the example below, we see that the volume is $180\ in^3$.



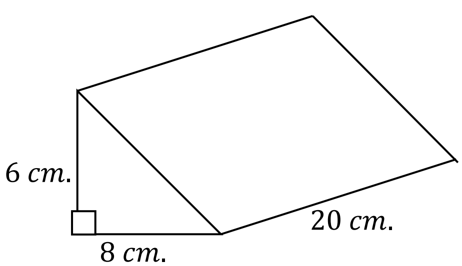
$$V = lwh$$

$$V = 12 * 3 * 5$$

$$V = 180\ in^3$$

This is a nice neat formula, but unfortunately it promotes the idea that all we do to find the volume is multiply all the numbers we see. This simply isn't true, so we need to know where this formula came from.

Hopefully the idea of length times width sounds familiar. We should recognize that as the area formula for a rectangle, but which rectangle specifically on our rectangular prism above? It is the bottom face of the prism, the 12 by 3 rectangle. This bottom face is known as the **base** of the prism. The base shapes of a prism are the shapes that are congruent and parallel, or the top and bottom in our prism above. This means that we could rewrite the volume formula to say $V = Bh$ or volume equals the area of the base times the height of the prism.



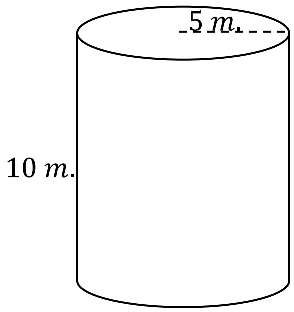
This formula is much more useful because it will work for all prisms. For example, consider the triangular prism below. In this case the base shape is a triangle. Since the **height** of a prism is the perpendicular distance between the bases, we know the height is 20 cm. So we apply our formula of $V = Bh$ by finding the area of the triangle and multiplying by 20, the height.

$$V = Bh = \left(\frac{8 * 6}{2}\right)(20) = (24)(20) = 480\ cm^3$$

In essence, what we should take from this is that the volume is like taking the base shape and stacking up more and more of those shapes until it hits the height of the prism. In other words, $V = Bh$ is a very nice and widely applicable formula.

Volume of a Cylinder

So let's take that concept of the area of the base shape multiplied by the height and transfer it to the cylinder. While the cylinder is not a prism, it is similar. The base is a circle, which we know the area to be πr^2, and the height is the distance between the circles. This means that we can still use the formula $V = Bh$ or in this case, $V = \pi r^2 h$.



$$V = \pi r^2 h$$

$$V = \pi(5)^2(10)$$

$$V = \pi(25)(10)$$

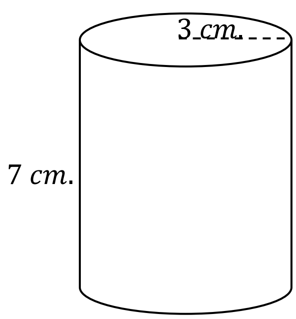
$$V = \pi(250)$$

$$V = 250\pi \approx 250 * 3.14 \approx 785\ m^3$$

This means that this particular cylinder takes up approximately 785 cubic meters of space. Another way to think about it is that about 785 little cubes that are a meter on each side would fit inside this cylinder.

In Terms of Pi

Typically it makes sense to plug in the approximation of 3.14 for pi. This gives us an idea of the actual number or size we're dealing with. However, sometimes a problem will ask for an answer in terms of pi. That means to not actually plug in 3.14 for pi. Just leave pi in the answer. So in the above example, the volume would just be $250\pi\ m^3$. Let's look at another example.



$$V = \pi r^2 h$$

$$V = \pi(3)^2(7)$$

$$V = \pi(9)(7)$$

$$V = \pi(63)$$

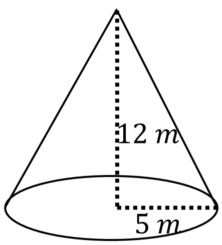
$$V = 63\pi\ cm^3$$

So our final answer in terms of pi is $63\pi\ cm^3$. Of course we can always plug in 3.14 for pi to get an approximate answer which in this case is $\approx 197.82\ cm^3$.

Volume of a Cone

Cones are similar to prisms and therefore we will use a variant of the $V = Bh$ formula. In fact, the cone is nearly the same as a cylinder except that a cone only has one base shape. At the top of the cone is a vertex instead of a second congruent circle. This means that a cone with the same exact circular base and height as a cylinder will hold less. The question is how much less?

We may recall from previous courses that the volume of a pyramid uses the formula $V = \frac{1}{3}Bh$. Since a cone is very similar to a pyramid, it would be reasonable to expect to use the same formula. It turns out this is correct. The actual proof for this takes some calculus, so we'll take it on faith for right now. Could you design an experiment to gather some evidence that this formula will work for a cone?



Let's find the volume of this cone.

$$V = \frac{1}{3}Bh$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi(5)^2(12)$$

$$V = \frac{1}{3}\pi(25)(12)$$

$$V = \frac{1}{3}\pi(300)$$

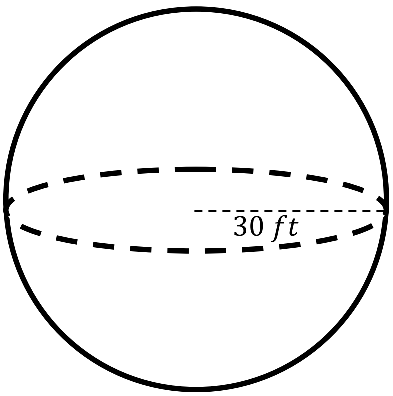
$$V = 100\pi \approx 314\ m^3$$

Again we can leave our answer in terms of pi or use 3.14 to approximate the answer.

Volume of a Sphere

While pyramids and cones share a volume formula, spheres have their own. Spheres use the formula $V = \frac{4}{3}\pi r^3$. Applying this formula is similar to applying the previous volume formulas. In the biz we call this "plug and chug" math.

Let's find the volume of this sphere.



$$V = \frac{4}{3}\pi r^3$$

$$V = \frac{4}{3}\pi(30)^3$$

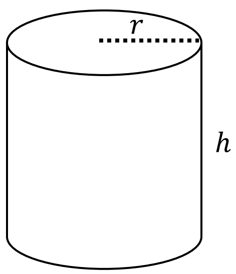
$$V = \frac{4}{3}\pi(27000)$$

$$V = 36000\pi \approx 113040\ ft^3$$

Keep in mind that we are still following order of operations. So we take the radius cubed first. After that we apply the commutative property and multiply the radius cubed by the fraction value of $\frac{4}{3}$. The last thing we do is multiply by pi because sometimes we want to leave it in terms of pi. If we don't want to leave it in terms of pi, we multiply by the approximation of pi which is 3.14. Finally, don't forget the unit for the answer.

Lesson 8.5

Answer the following questions either using $\pi \approx 3.14$ or giving your answer in terms of π. Round your answer to the nearest hundredth where necessary.



1. Find the volume of a cylinder with a radius of 3 in and a height of 10 in.

2. Find the volume of a cylinder with a radius of 10 mm and a height of 2 mm.

3. Find the volume of a cylinder with a radius of 5 cm and a height of 15 cm.

4. Find the volume of a cylinder with a diameter of 22 m and a height of 5 m.

5. Find the volume of a cylinder with a diameter of 4 ft and a height of 1 ft.

6. Find the volume of a cylinder with a radius of 9 in and a height of 9 in.

7. Find the volume of a can of green beans with a radius of 3 cm and a height of 8 cm.

8. Find the volume of a cylindrical can of oatmeal with a radius of 8 cm and a height of 45 cm.

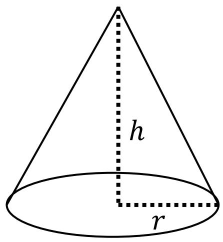
9. Find the volume of a cylindrical water bottle with a diameter of 4 cm and a height of 30 cm.

10. Find the volume of a can of Pepsi with a diameter of 2 in and a height of 3.5 in.

11. Find the volume of a water pipe with a radius of 0.75 ft and a length of 16 ft.

12. Find the volume of a straw used for drinking with a radius of 2 mm and a height of 170 mm.

13. Find the volume of a cone with a radius of 3 in and a height of 10 in.



14. Find the volume of a cone with a radius of 10 mm and a height of 3 mm.

15. Find the volume of a cone with a radius of 5 cm and a height of 15 cm.

16. Find the volume of a cone with a radius of 12 m and a height of 5 m.

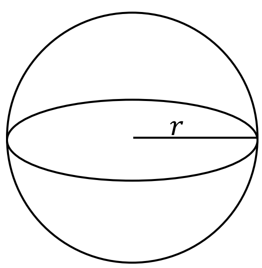
17. Find the volume of a cone with a diameter of 4 ft and a height of 9 ft.

18. Find the volume of a cone with a diameter of 18 in and a height of 9 in.

19. Find the volume of a waffle cone for ice cream with a radius of 4 cm and a height of 12 cm.

20. Find the volume of a cone birthday hat with a radius of 2 in and a height of 9 in.

21. Find the volume of a funnel with a diameter of 10 cm and a height of 9 cm.



22. Find the volume of a sphere with a diameter of 6 in.

23. Find the volume of a sphere with a diameter of 18 mm.

24. Find the volume of a sphere with a radius of 6 cm.

25. Find the volume of a sphere with a radius of 12 m.

26. Find the volume of a sphere with a radius of 2 ft.

27. Find the volume of a sphere with a radius of 5 in.

28. Find the volume of a mini basketball with a radius of 3.5 in.

29. Find the volume of the Earth with a diameter of approximately $12,756 \text{ km}$.

30. Find the volume of the moon with a diameter of approximately 3475 km.

31. Find the volume of a gumball with a radius of 3 mm.

8.6 Solving for a Missing Dimension

Given the volume of a shape, we can solve for a missing dimension such as the height or radius. It should come as no surprise that to isolate the variable in the equation, we will use inverse operations.

Solving for the Height

Let's start with a cylinder with a volume of approximately $314\ in^3$ and a radius of $5\ in$. Since we know the formula for the volume of the cylinder, we can plug in and work backwards.

$$V = \pi r^2 h$$

Substitute what we know

$$314 = 3.14 * (5)^2 h$$

$$314 = 3.14 * 25 * h$$

Simplify

$$314 = 78.5h$$

$$\frac{314}{78.5} = \frac{78.5h}{78.5}$$

Solve

$$4 = h$$

So the height of the cylinder is $4\ in$. Similarly, we can solve for the height if we know the volume of a cone. The difference is only the fraction. In most cases it will be easier to eliminate the fraction at the beginning. For example, consider a cone with a volume of $37.68\ in^3$ and a radius of $2\ in$. Follow the same process as above to solve for the height.

$$V = \frac{1}{3}\pi r^2 h$$

Substitute what we know

$$37.68 = \frac{1}{3} * 3.14 * (2)^2 h$$

$$3 * 37.68 = 3 * \frac{1}{3} * 3.14 * (2)^2 h$$

Multiply by 3 to eliminate the fraction.

$$113.04 = 3.14 * 4 * h$$

$$113.04 = 12.56h$$

$$\frac{113.04}{12.56} = \frac{12.56h}{12.56}$$

Solve

$$9 = h$$

Solving for the Radius

When solving for the radius, we'll have to think back to our knowledge of square and cube roots. Since the radius is either squared or cubed in the volume formulas, we will need to apply either the square root or cube root as one of our inverse operations.

Let's look at a sphere with a volume of $904.32\ m^3$. We know the formula for volume, so we will substitute, simplify and solve.

$$V = \frac{4}{3}\pi r^3$$

$$904.32 = \frac{4}{3} * 3.14 * r^3$$

Substitute what we know

$$\frac{3}{4} * 904.32 = \frac{3}{4} * \frac{4}{3} * 3.14 * r^3$$

$$678.24 = 3.14 * r^3$$

Multiply by $\frac{3}{4}$ to eliminate the fraction.

$$\frac{678.24}{3.14} = \frac{3.14 * r^3}{3.14}$$

$$216 = r^3$$

$$\sqrt[3]{216} = \sqrt[3]{r^3}$$

Inverse Operations

$$6 = r$$

In this case the radius was $6\ m$, and one of the steps necessary in solving this was using the cube root. If we were solving for the radius in a cylinder or cone, we would need the square root.

Lesson 8.6

Answer the following questions using $\pi \approx 3.14$. Round your answer to the nearest hundredth where necessary.

1. Find the height of a cylinder with a volume of 30 in^3 and a radius of 1 in.

2. Find the height of a cylinder with a volume of 100 cm^3 and a radius of 2 cm.

3. Find the height of a cylinder with a volume of $720\pi \text{ ft}^3$ and a radius of 6 ft.

4. Find the height of a cylinder with a volume of $1215\pi \text{ mm}^3$ and a radius of 9 mm.

5. Find the radius of a cylinder with a volume of 950 in^3 and a height of 10 in.

6. Find the radius of a cylinder with a volume of 208 cm^3 and a height of 4 cm.

7. Find the radius of a cylinder with a volume of $108\pi \text{ ft}^3$ and a height of 12 ft.

8. Find the radius of a cylinder with a volume 686 mm^3 and a height of 14 mm.

9. Find the height of a cone with a volume of 150 in^3 and a radius of 10 in.

10. Find the height of a cone with a volume of 21 ft^3 and a radius of 4 ft.

11. Find the radius of a cone with a volume of 175 cm^3 and a height of 21 cm.

12. Find the radius of a cone with a volume of $196\pi \text{ mm}^3$ and a height of 12 mm.

13. Find the radius of a sphere with volume $\approx 113.04 \text{ in}^3$.

14. Find the radius of a sphere with volume $\approx 904.32 \text{ cm}^3$.

15. Find the radius of a sphere with volume $\approx 3052.08 \text{ m}^3$.

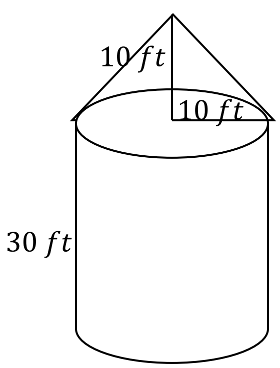
16. Find the radius of a sphere with volume $\approx 4.18\overline{6} \text{ ft}^3$.

8.7 Volume of Composite Shapes

Up to this point we have been dealing with a single shape at a time. However, there are many times when we have more than one shape we need to find the volume of. For example, a typical house is a rectangular prism with a triangular prism on top. To install a medical grade air filter in a house, you must know the volume of the house. We can do this by finding the volume of the rectangular prism and adding it to the volume of the triangular prism. This is a simple example of finding the volume of composite shapes with prisms, but now we will move on to composite shapes with rounded objects.

Two Shapes Put Together

Imagine a grain silo, the place where farmers store their grain after a harvest. They are usually a cylinder with a cone roof, and the grain is dropped into the silo at the very top of the cone. To know how much room farmers have for storing their harvest, they must calculate the volume of the grain silo. To do this, simply find the volume of the cylinder, find the volume of the cone, and add the two together. Let's assume the radius of the silo is $10\ ft$, the height of the cylinder part is $30\ ft$, and the height of the cone part is $10\ ft$.



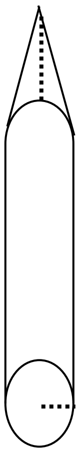
Cone	Cylinder
$V = \frac{1}{3}\pi r^2 h$	$V = \pi r^2 h$
$V \approx \frac{1}{3} * 3.14 * 10^2 * 10$	$V \approx 3.14 * 10^2 * 10$
$V \approx 1046.67\ ft^3$	$V \approx 3140\ ft^3$

Total volume is: $V \approx 1046.67 + 3140 \approx 4186.67\ ft^3$

Regardless of what shapes we put together, or how many, we can always find the total volume by finding the volume of parts and adding them together.

Lesson 8.7

Answer the following questions using $\pi \approx 3.14$ and rounding your answer to the nearest hundredth where necessary.



Visual for questions 1-10

1. Find the volume of a cone used for the tip of a rocket with a diameter of 12 yds and a height of 15 yds.

2. Find the volume of a pencil with a radius of 0.5 cm, a cone height of 3 cm, and a cylinder height of 14 cm.

3. Find the volume of a model rocket with a radius of 1 in, a cone height of 3 in, and a cylinder height of 8 in.

4. Find the volume of a caulking gun with a radius of 2 cm, a cone height of 3 cm, and a cylinder height of 20 cm.

5. Find the volume of a crayon with a radius of 2 mm, a cone height of 21 mm, and a cylinder height of 80 mm.

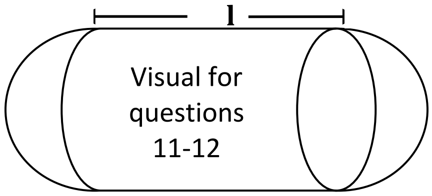
6. Find the volume of a model jet with a radius of 1 ft, a cone height of 3 ft, and a cylinder height of 6 ft.

7. Find the radius of a pencil with a volume of $110\pi\ \text{mm}^3$, a cone height of 3 mm, and a cylinder height of 10 mm.

8. Find the radius of a model rocket with a volume of $2500\ \text{in}^3$, a cone height of 6 in, and a cylinder height of 25 in.

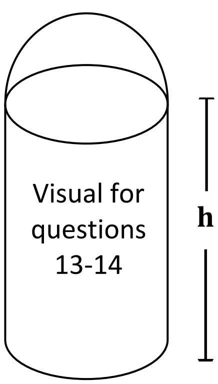
9. Find the cylinder height of a caulking gun with a volume of $300\ \text{in}^3$, a cone height of 6 in, and a radius of 2 in.

10. Find the cone height of a pencil with a volume of $750\ \text{mm}^3$, a radius of 3 mm, and a cylinder height of 25 mm.



11. Find the volume of a propane gas tank with half spheres on either end that has a radius of 3 ft and a length (l) of 7 ft.

12. Find the volume of a submarine with half spheres on either end that has a radius of 6 m and a length (l) of 15 m.



13. Find the volume of a grain silo with a half sphere on one end that has a diameter of 6 m and a height (h) of 15 m.

14. Find the volume of a grain silo with a half sphere on one end that has a diameter of 6 ft and a height (h) of 35 ft.

Review Unit 8: Geometry Applications

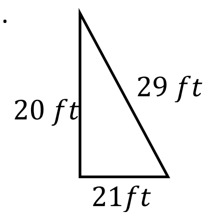
You may use a calculator.

Unit 8 Goals

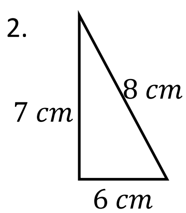
- *Explain a proof of the Pythagorean Theorem and its converse. (8.G.6)*
- *Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions. (8.G.7)*
- *Apply the Pythagorean Theorem to find the distance between two points in a coordinate system. (8.G.8)*
- *Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems. (8.G.9)*

Determine if the following triangles are right triangles or not using the Pythagorean Theorem.

1.

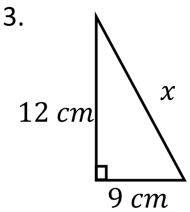


2.

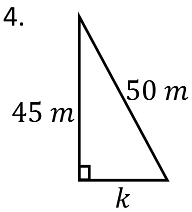


Find the length of the missing side of each right triangle. Round to three decimal places if necessary.

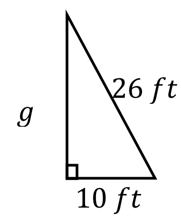
3.



4.

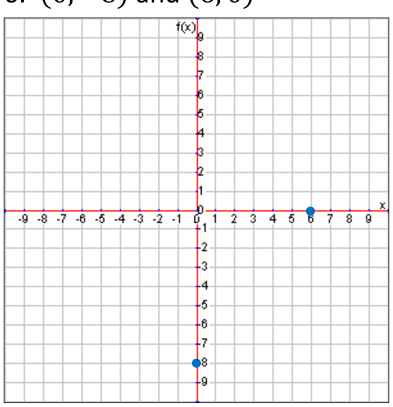


5.

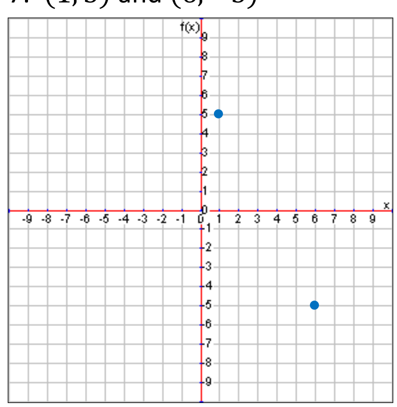


Determine the distance between the given points. Round to three decimal places if necessary.

6. $(0, -8)$ and $(6, 0)$

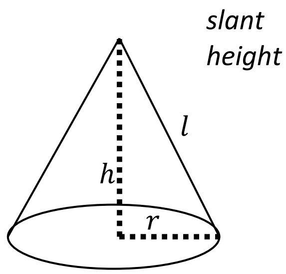


7. $(1, 5)$ and $(6, -5)$

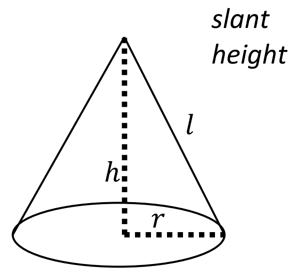


Find the value of the variable.

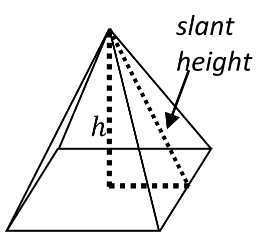
8. The following cone has a radius of $11\ mm$ and a slant height of $61\ mm$. What is h, the height?



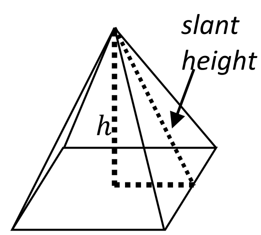
9. The following cone has a height of $20\ cm$ and a slant height of $29\ cm$. What is r, the radius?



10. The following pyramid has a square base that is $30\ ft$ on each side. The height is $8\ ft$. What is l, the slant height of the pyramid?



11. The following pyramid has a square base. The height is $12\ in$ and the slant height is $20\ in$. What is s, the side length of the base of the pyramid?



Solve the following problems.

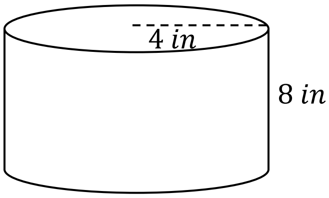
12. Firefighters position an 85-foot ladder 13 feet away from the building. The top of the ladder just reaches a window on the fourth floor. How high off the ground is the window?

13. The school is located 9 meters north and 40 meters west of Kiley's house. Kiley walks through her neighbors' yards, so she can take the shortest route possible (a straight line). How far does she have to travel if she walks to and from school?

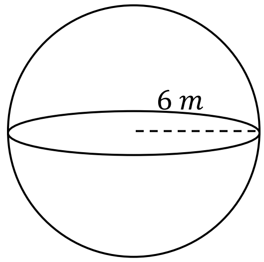
14. An open field is 85 meters wide and 105 meters long. The owner wants to put spray paint along both diagonals of the field. If the spray paint costs approximately $2 per meter of coverage, how much should the owner budget for spray paint?

Find the volume of the given shapes using $\pi \approx 3.14$ for your answers or in terms of π.

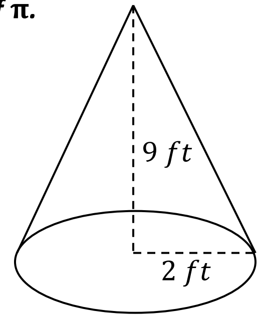
15.



16.



17.



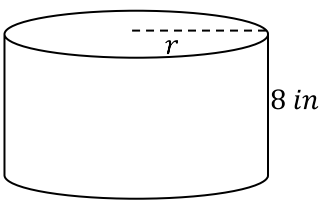
18. Cylinder
 $h = 12 \text{ in}$
 $r = 2\ in$

19. Cone
 $h = 15 \text{ cm}$
 $r = 3\ cm$

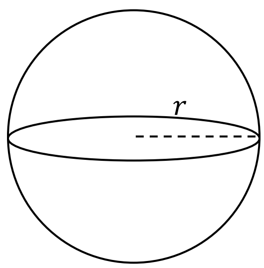
20. Sphere
 $r = 3 \text{ m}$

Find the missing dimension of the given shapes using $\pi \approx 3.14$ for your answers or in terms of π.

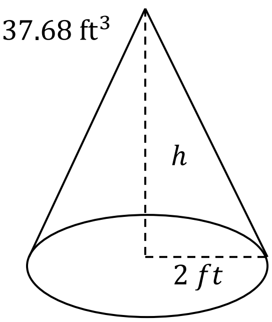
21. $V \approx 401.92 \text{ in}^3$



22. $V \approx 3052.08 \text{ m}^3$



23. $V \approx 37.68 \text{ ft}^3$



24. Cylinder
 $h = 12 \text{ in}$
 $r = ?$
 $V \approx 150.72 \text{ in}^3$

25. Cone
 $h = ?$
 $r = 3\ cm$
 $V \approx 141.3 \text{ cm}^3$

26. Sphere
 $r = ?$
 $V \approx 113.04 \text{ mm}^3$

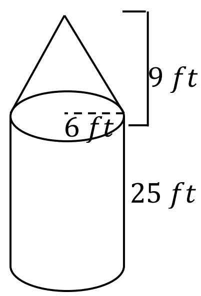
Find the volume of the given shapes in terms of π or using $\pi \approx 3.14$.

27. A spherical volleyball with a radius of 15 cm.

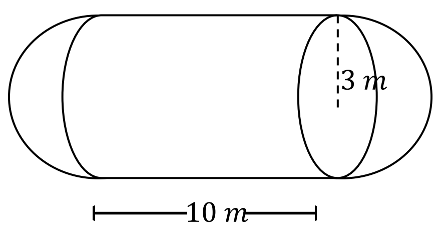
28. A cylindrical water bottle with a height of 10 in and a radius of 1 in.

29. An ice cream cone with a height of 6 in and radius of 2 in.

30. A grain silo as pictured:



31. A propane tank as pictured:



Unit 9: Bivariate Data

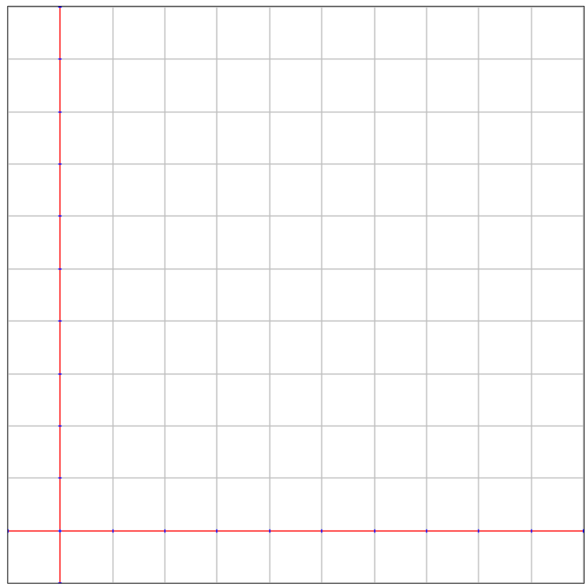
Pre-Test Unit 9: Scatter Plots

You may use a calculator.

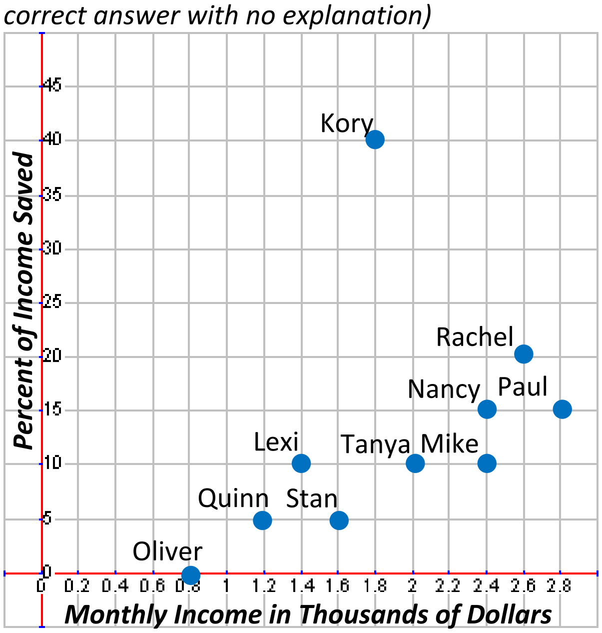
Construct a scatter plot for the following data set using appropriate scale for both the x- and y-axis.
(10 pts; 2 pts for each axis scale/interval, 1 pt for deciding to break each axis or not, 2 pts for correct independent/dependent axes, 2 pts for correctly plotted points)

1. This table shows the number of hours students slept the night before their math test and their scores.

	Hours Slept	Test Score
Anna	8	95
Bob	7	90
Carly	8	85
Damien	6	75
Esther	5	65
Franco	8	90
Georgia	8	80
Hank	9	95
Innya	7	80
Jacob	6	70



Use the following scatter plot to answer each question. The scatter plot shows the monthly income of each person in hundreds of dollars versus the percent of their income that they save each month. *(5 pts; 2 pts for correct answer with no explanation)*



2. Does this scatter plot represent a positive association, negative association, or no association? Why?

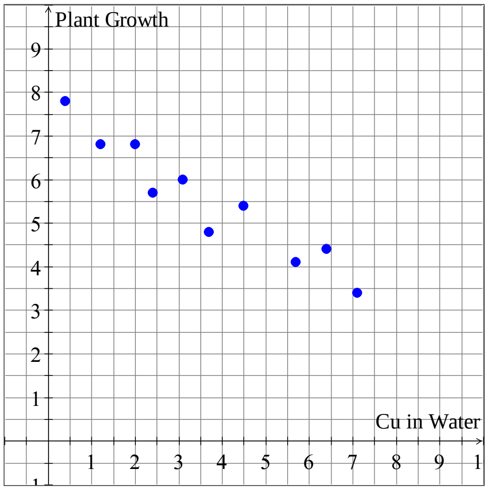
3. Which person makes the most money per month? How much do they make?

4. Does this appear to a linear or non-linear association? Why?

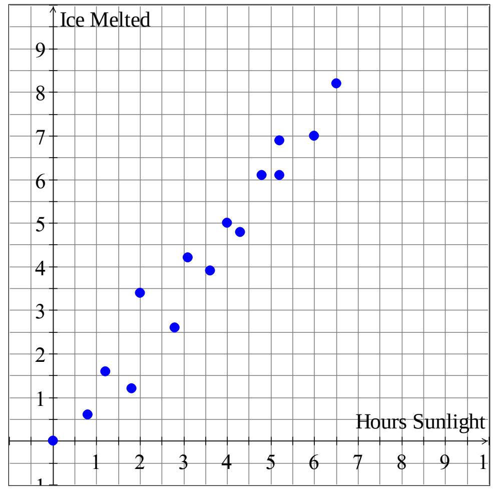
5. Which person is the outlier in this data set? Why?

Draw an informal line of best for the given scatter plots. *(5 pts; partial credit at teacher discretion)*

6. This scatter plot shows the amount copper in water in ppm versus plant growth in cm over three months.

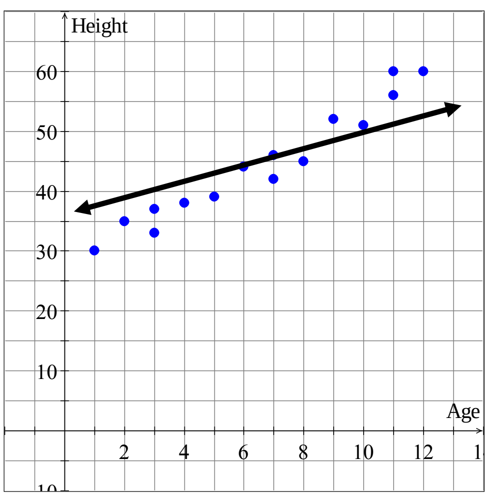


7. This scatter plot shows the hours a cubic foot of ice was exposed to sunlight versus the amount of ice that melted in cubic inches.

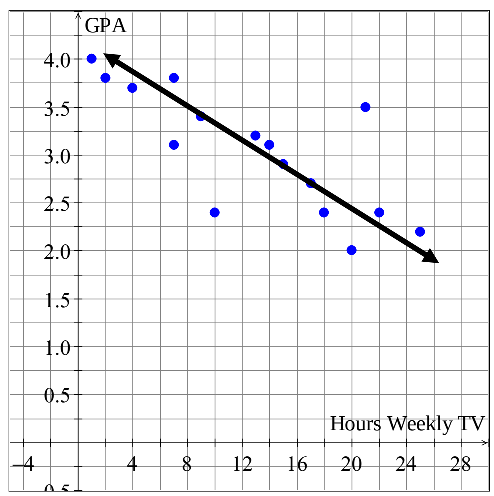


Explain why the drawn line of best fit is accurate or why not. *(5 pts; partial credit at teacher discretion)*

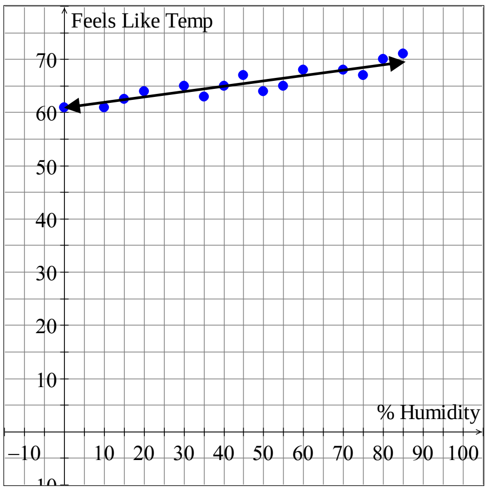
8. This scatter plot shows the age in years versus the height in inches of a group of children.



9. This scatter plot shows the hours of TV watched per week versus the GPA on a 4.0 scale for a group of students.



The scatter plot shows what people think the temperature "feels like" as the humidity varies when the room is actually at 68° F. The equation of the line of best fit is $\mathbf{y = \frac{1}{10}x + 61}$***.*** *(5 pts; 3 pts for equation answer, 2 pts for graph answer)*



10. Predict what a person would say the temperature "feels like" when the humidity is at 80% using both the equation and graph.

Equation Work:	Graph Prediction:

11. Predict what the humidity would be if someone said that it "feels like" $65°$ F in that room using both the equation and graph.

Equation Work:	Graph Prediction:

Using the same scatter plot and equation of the line of best fit of $\mathbf{y = \frac{1}{10}x + 61}$***, answer the following questions.*** *(5 pts; partial credit at teacher discretion)*

12. What does the slope of this equation mean in terms of the given situation? In other words, explain what the rise and run mean for this problem.

13. What does the y-intercept of this equation mean in terms of the given situation? In other words, explain what the y-intercept means when considering the humidity and "feels like" temperature.

Answer the following questions about two-way tables. *(5 pts; partial credit at teacher discretion)*

14. Construct a two-way table from the following data about whether people are democrats or republicans and whether or not they support stricter gun control laws.

Democrat or Republican?	D	R	R	R	D	D	R	D	D	D	R	D	R	R	D	R	D	D	R	R
Support Strict Gun Control?	Y	N	N	N	N	Y	N	Y	Y	Y	N	Y	N	Y	N	N	Y	Y	Y	N

15. Do you think there is a relationship between party affiliation and gun control laws? Based on the data, why or why not? *(no credit without explanation of why, partial credit at teacher discretion for explanation)*

Answer the following questions using the given two-way table. *(5 pts; no partial credit)*

	Support School Uniforms	Do Not Support School Uniforms
Students	278	1726
Teachers	82	23

16. How many students were surveyed?

17. How many people support school uniforms?

18. How many students do not support school uniforms?

19. As a percent to the nearest hundredth (two decimal places) what is the relative frequency of students who support school uniforms?

9.1 Constructing Scatter Plots

A **scatter plot** is a plot on the coordinate plane used to compare two sets of data and look for a correlation between those data sets. An **association** is a relationship or dependence between data. For example, the price of oil and the price of gasoline have a strong association. The daily price of oil and the number of penguins swimming in the ocean on that day most likely have no association at all. However, to find this association we need to make a scatter plot.

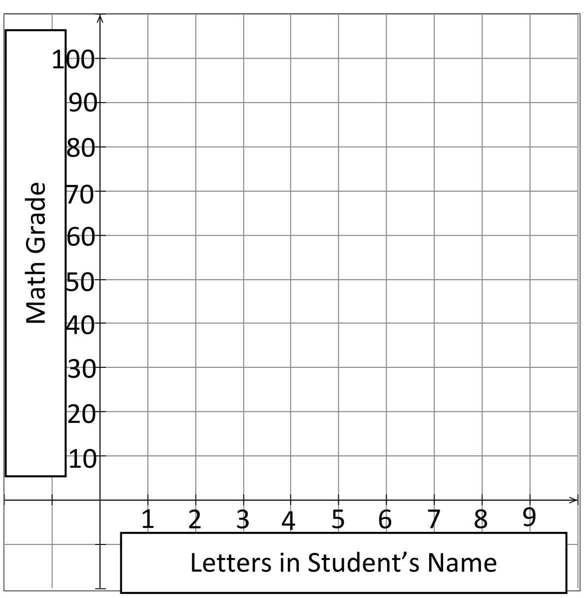
Start with the Data

Before we can make a scatter plot, we need two sets of data that we want to compare. For example, we might compare the number of letters in a student's first name and their math grade. Do people with shorter names tend to score higher in math? Do people with the lowest grades have longer names? These are questions of relationship, or correlation, that we can explore with a scatter plot once have some data. That data set might look like this:

Name	Nichole	Josiah	Kame	Gungar	Roberto	Frank	John	Herman	Sami	Daimon
Letters	7	6	4	6	7	5	4	6	4	6
Grade	58	83	61	70	31	76	81	70	72	57

Name	Yolina	Johanne	Karolinea	Kurt	Addison	Ian	Dennis	Ophelia	Kristina	Bradford
Letters	6	7	9	4	7	3	6	7	8	8
Grade	77	90	87	83	76	78	87	87	80	41

Prepare the Coordinate Plane



Now that we have our data, we need to decide how to put this data on the coordinate plane. We can let the x-axis be the number of letters in a student's name and the y-axis be the students overall math grade. Once we have decided this we should label our axes.

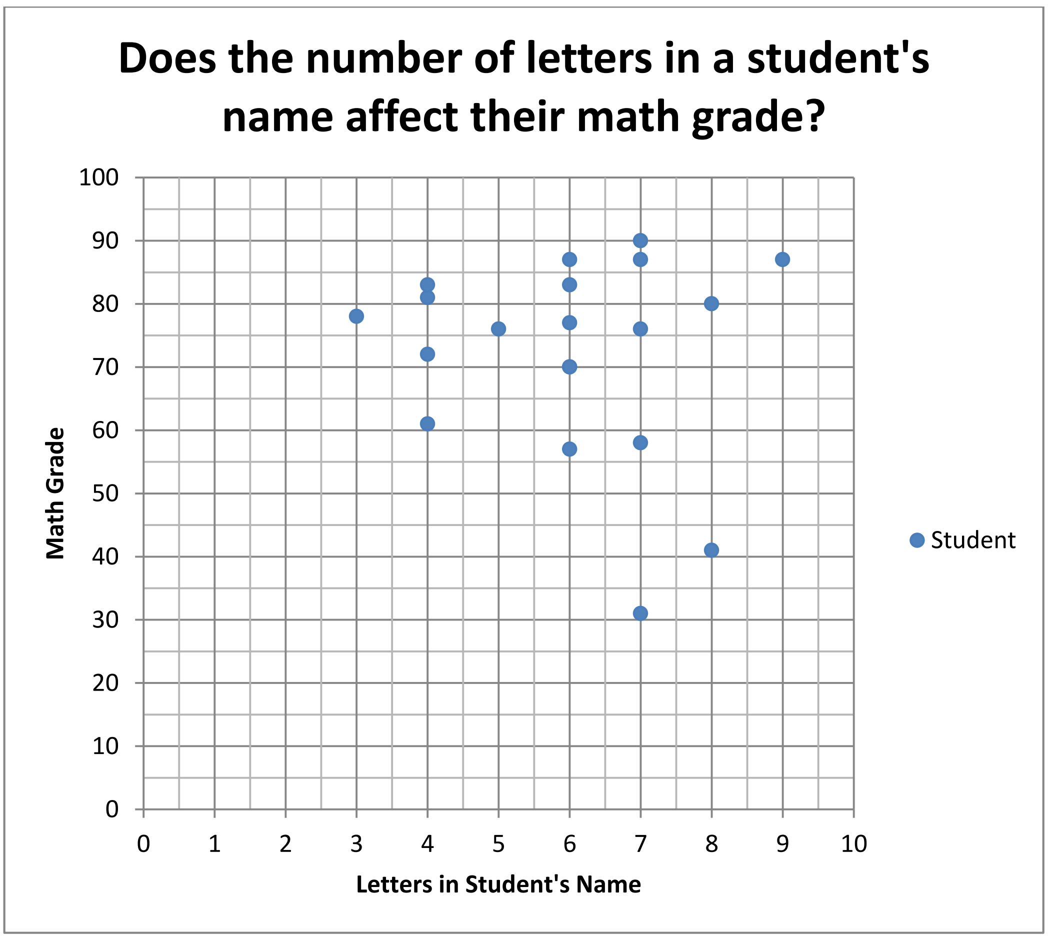
Next we'll need to decide on a scale and interval. The scale is the low to high number on the axis and the interval is what we count by. Notice first of all that we're only looking at Quadrant I because we won't have negative amounts of letters or negative grades. Since the grades can be from zero to one hundred, we might choose to count by tens on the y-axis giving us a scale of 0-100 and an interval of 10. Since the letters range from three to nine, we might count by ones on the x-axis. This gives us a scale of 0-10 with an interval of 1.

When to use a broken axis

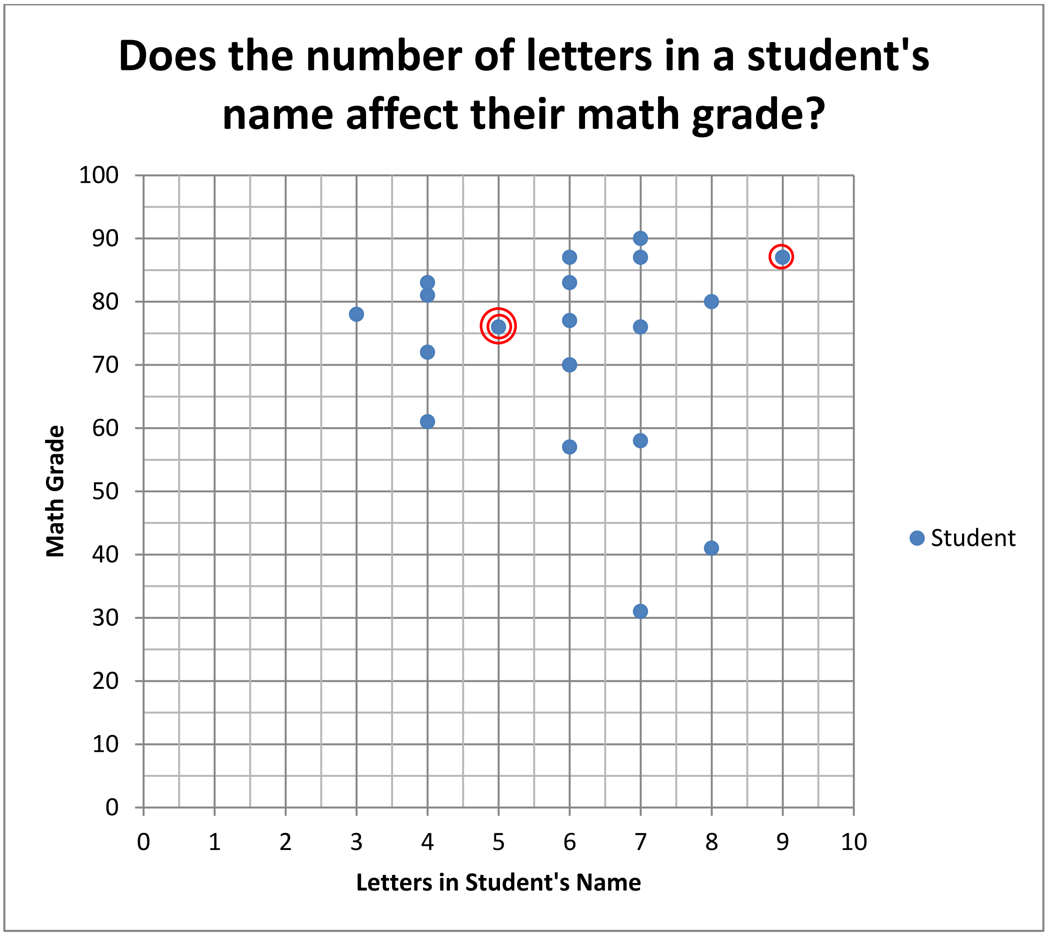
A broken axis is useful whenever more than half of the area of the scatter plot will be blank. Nobody likes to see a blank graph with all the data in one tiny area. So instead, we zoom use by using a broken axis. If the range of your data is less than the lowest data point, a broken axis may be useful. For example, in our math test situation above if everyone scored above a 60%, then we might break the y-axis and begin counting at 60. We could then count by 4's to make it up to 100%.

Plot the Points

Finally we would then plot each person on the graph. So Nicholas will be the point $(7, 58)$, Josiah the point $(6, 83)$, and so forth. Using Excel to make our scatter plot, the final scatter plot might look like the following. Notice that each dot on the graph represents a person. While the labeling is not necessary, it may be useful in some circumstances.



Many times on a scatter plot you may have the same data point multiple times. One way to represent this fact is to put another circle around the data point. Let's add a few new students to our data set: Johnathan (9 letters and 87 math score), Jacob (5 letters and 76 math score), and Helga (5 letters and 76 math score). The new graph could look like this:



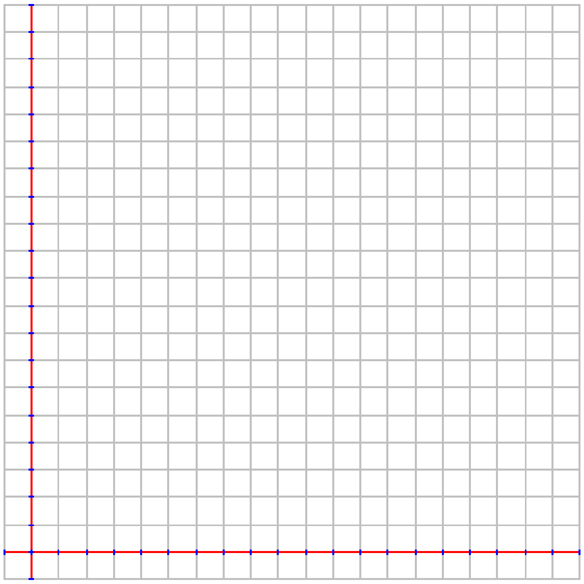
While this practice is not necessarily standard, it can be useful as a visual representation of what is happening with the data. We can more easily see the multiple data points this way. In Excel, you wouldn't get the red circles. Those would have to be put in by hand.

Lesson 9.1

Use the given data to answer the questions and construct the scatter plots.

Pathfinder Character Level vs. Total Experience Points

Level	2	3	6	9	10	11	14	15	17	20
XP	15	35	150	500	710	1050	2950	4250	8500	24000



1. Which variable should be the independent variable (x-axis) and which should be the dependent variable (y-axis)?

2. Should you use a broken axis? Why or why not?

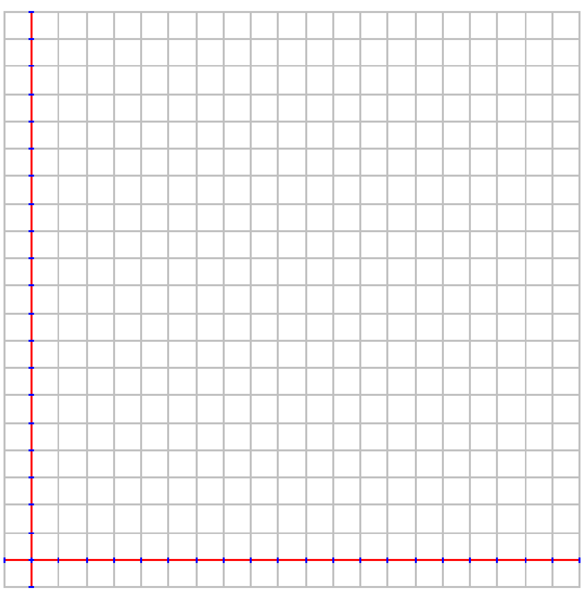
3. What scale and interval should you use for the x-axis?

4. What scale and interval should you use for the y-axis?

5. Construct the scatter plot.

Age vs. Weekly Allowance

Age	12	12	13	13	14	14	15	15	16	16
Allowance	0	5	5	8	10	15	20	20	25	30



6. Which variable should be the independent variable (x-axis) and which should be the dependent variable (y-axis)?

7. Should you use a broken axis? Why or why not?

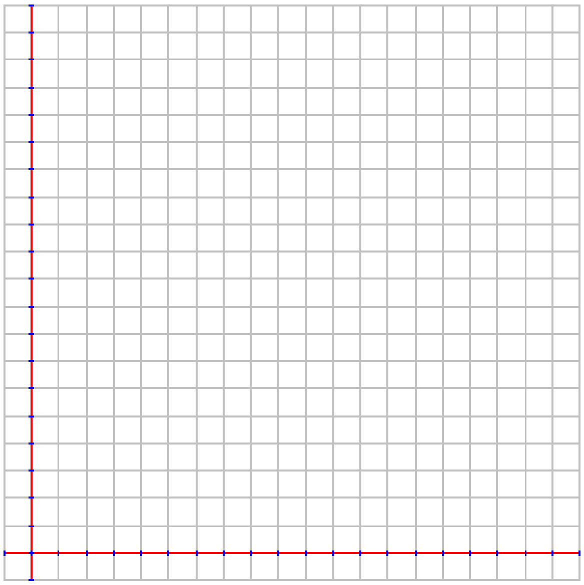
8. What scale and interval should you use for the x-axis?

9. What scale and interval should you use for the y-axis?

10. Construct the scatter plot.

Age vs. Number of Baby Teeth

Age	5	6	7	7	8	9	10	11	11	12
Baby Teeth	20	19	17	15	10	10	8	4	2	2



11. Which variable should be the independent variable (x-axis) and which should be the dependent variable (y-axis)?

12. Should you use a broken axis? Why or why not?

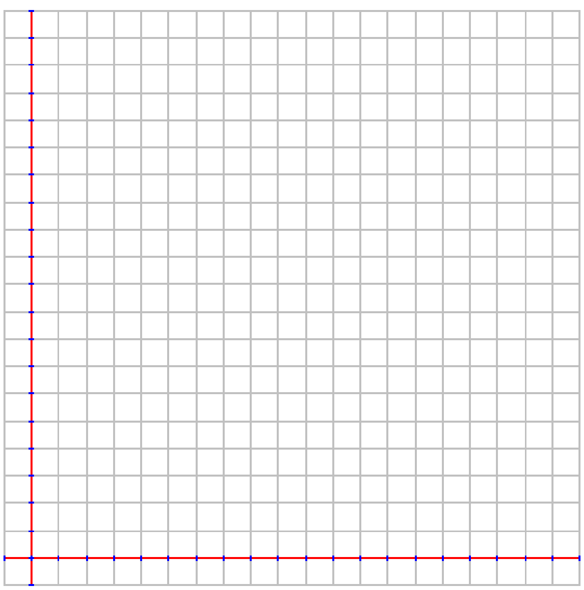
13. What scale and interval should you use for the x-axis?

14. What scale and interval should you use for the y-axis?

15. Construct the scatter plot.

Car Speed (in mph) vs. Gas Mileage (in mpg)

Speed	20	25	35	40	45	55	65	80	90	100
Mileage	25	27	28	30	31	32	30	29	25	22



16. Which variable should be the independent variable (x-axis) and which should be the dependent variable (y-axis)?

17. Should you use a broken axis? Why or why not?

18. What scale and interval should you use for the x-axis?

19. What scale and interval should you use for the y-axis?

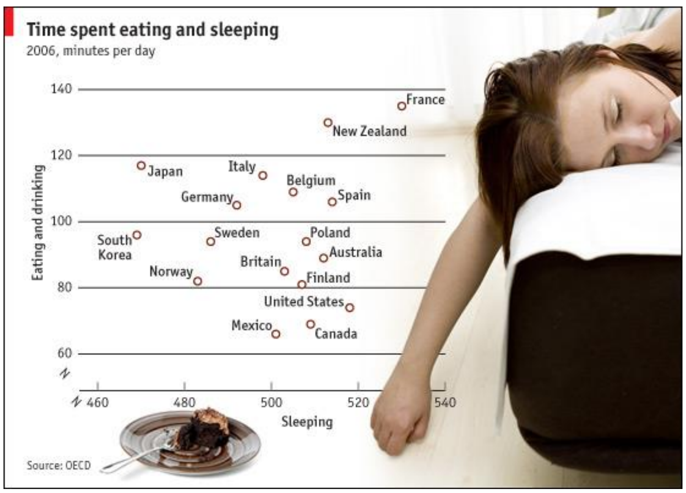
20. Construct the scatter plot.

9.2 Analyzing Scatter Plots

Now that we know how to draw scatter plots, we need to know how to interpret them. A scatter plot graph can give us lots of important information about how data sets are related if we understand what each part of the graph means.

Reading Data Points

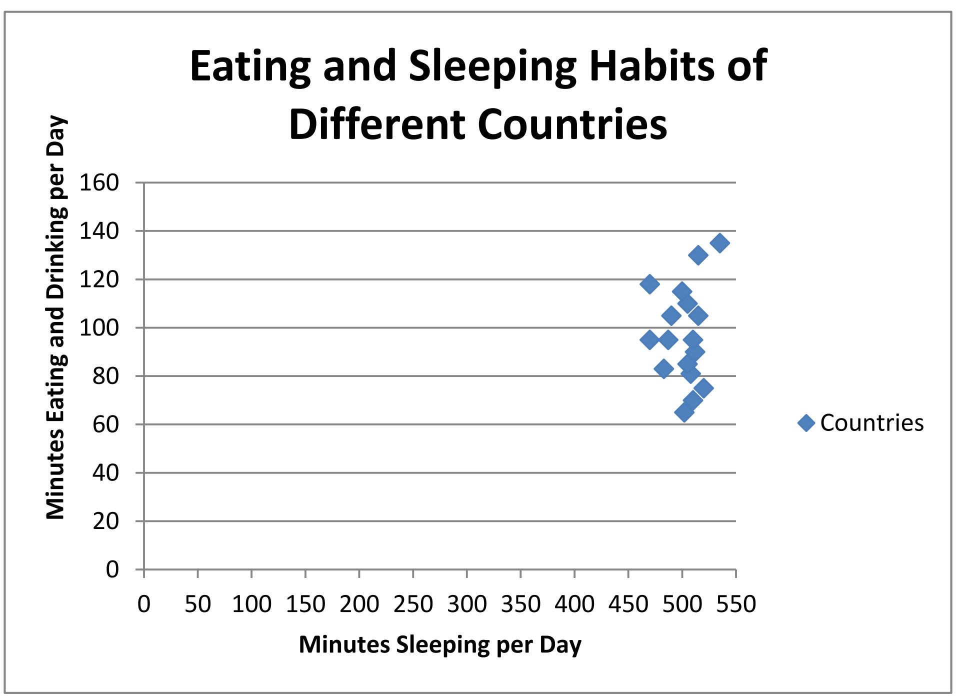
Each individual point on a scatter plot represents a single idea. For example, in the picture below each point represents a country. The axes tell us information about that country. The y-axis tells us about how many minutes per day that country spends eating and drinking. The x-axis tells us about how many minutes per day that country spends sleeping. Can you find the United States on this scatter plot? About how many minutes do we sleep per day? About how many minutes we spend eating and drinking per day? Are these numbers reasonable to you?



http://www.visualquest.in/2010/09/severalsimple-and-very-useful.html

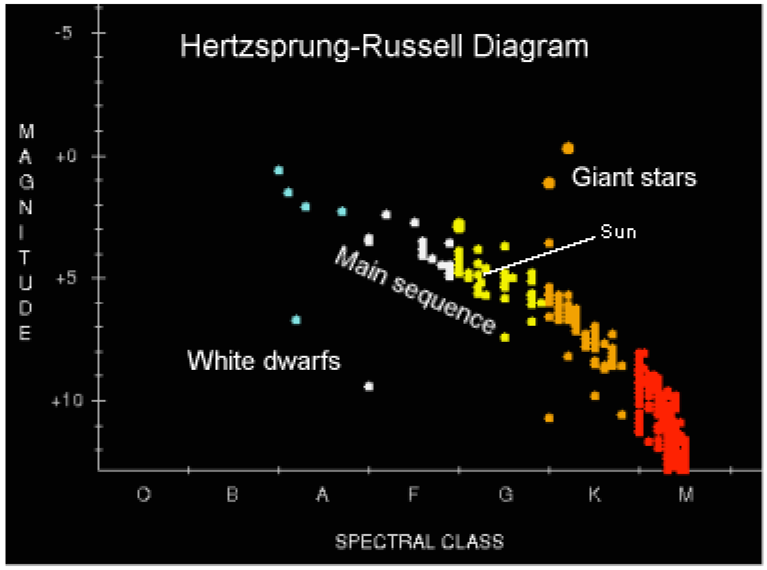
Another thing to notice about this scatter plot is that it uses the broken axis symbol (that little Z looking thing). This means that they don't start counting from zero on either axis. They skip ahead to a reasonable starting point but still apply a scale after that point. Even with the broken axis they must count by something in each direction. In this case, they count by 20 minutes on the x-axis and the y-axis as well.

If we did not use the broken axis, it might look more like the scatter plot below. To be able to label the data points, it is useful in this case to use the broken axes.



Outliers

An outlier is a data point that is significantly far away from the majority of the data. There is no precise mathematical definition for what makes a data point an outlier. It's usually somewhat obvious. For example, notice that White Dwarf Stars and Giant Stars are both outliers in the below scatter plot showing a star's spectral class (temperature) versus its magnitude (brightness).

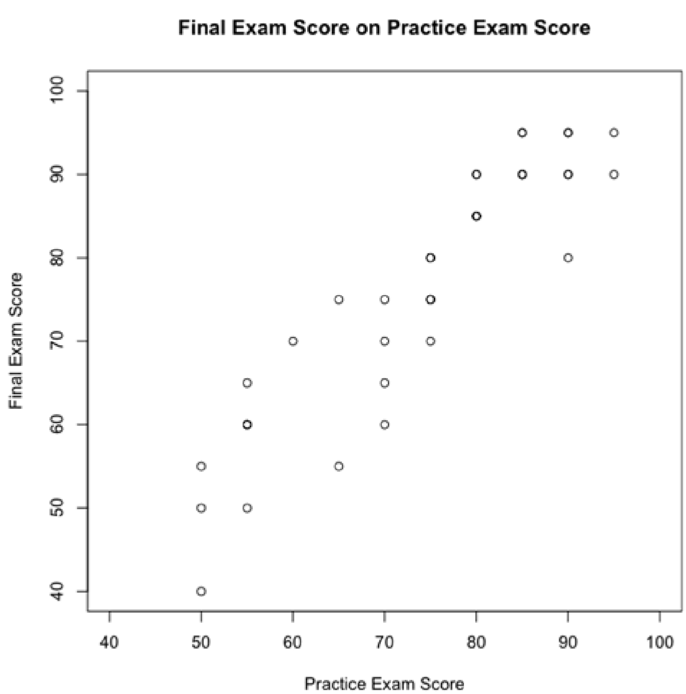


http://starplot.org/docs/ch1.html

Why do we care about outliers? We care because outliers often throw off the analysis of the data set. For example, let's say you have three test grades in math class: 80%, 80%, and 80%. Your current class average is, you guessed it, 80%. However, if we throw in an outlier, like a 0%, for the next test, your class average drops down to 60%. You have dropped two letter grades from a B- to a D-. Yikes! The outlier sure hurt your grade.

Positive and Negative Associations

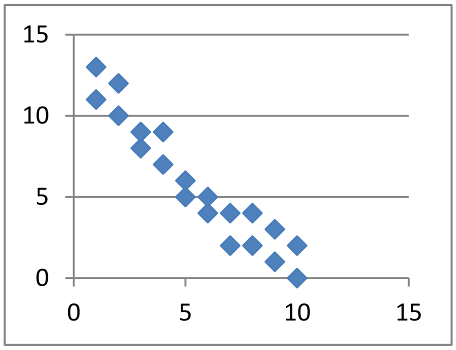
An **association**, sometimes called a correlation, is a relationship between two data sets. For example, in the above star scatter plot, there appears to be a relationship between a star's temperature and brightness. We'd have to know more about the science of stars to fully interpret the graph, but we can see there is an association because most of the data follows a pattern (except for those pesky outliers).



http://www.r-bloggers.com/r-tutorial-series-basic-polynomial-regression/

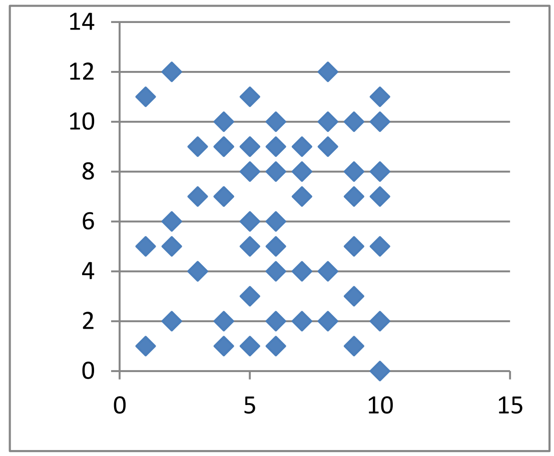
In fact, the more tightly clumped the data is, the stronger the association is. We might say that there is a strong association between the brightness and temperature of a star. In the scatter plot to the left, we see a slightly weaker association between scores on a practice exam and scores of the final exam.

We would also say that the scatter plot to the left has a positive association because it appears that the students who scored higher on the practice exam also scored higher on the final exam. As one variable (practice exam score) increased, the other variable (final exam score) also increased. We call this a positive association.



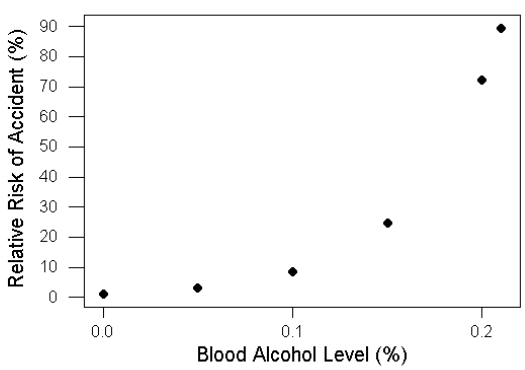
There are also negative associations. These associations are recognized by the fact that as one variable increases, the other decreases. For example, as the supply of oil increases, the cost of gasoline decreases. They have a negative association. A scatter plot with a negative association might look like the graph to the left.

No association would mean that there appears to be no relationship between the two data sets (or variables). For example, we might consider the daily price of tea and the daily number of fruit flies born. There is likely no relationship between those two things which would produce a graph similar to the one to the right.



Linear or Non-Linear Associations

Whether the association is positive or negative, it may appear linear or non-linear. A linear association would be a scatter plot where the data points clump together around what appears to be a line. The negative association graph above and to the left is an example of a linear association. The scatter plot about practice and final exams is an example of a positive linear association.

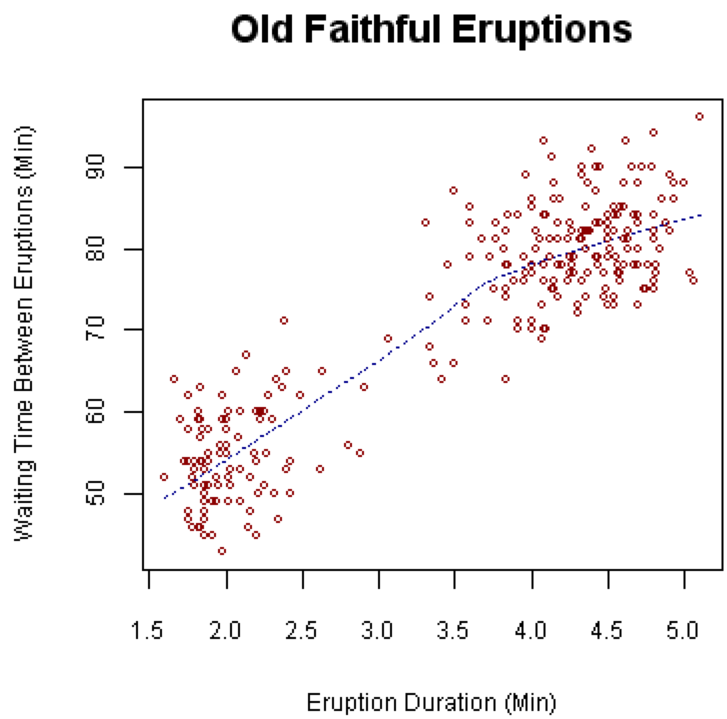


http://wps.prenhall.com/esm_walpole_probstats_7/55/14203/3635978.cw/content/index.html

A non-linear association is usually curved to some extent. There are many types of curves that it could fit, but we'll just focus on the fact that it doesn't look a line and therefore is non-linear. Consider the graph to the left showing the relative risk of an accident compared to the blood alcohol level. As you can see, the graph curves sharply up when there is more alcohol in the blood stream. This should not only serve as an example of non-linear scatter plot, but also the risks of drinking and driving.

Clustering

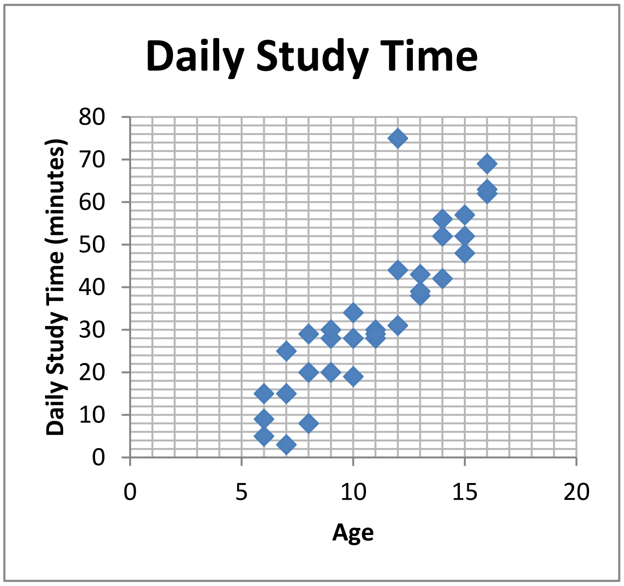
Clustering is when there is an association, but it appears to come in clumps. Consider the following scatter plot that shows the time between eruptions and eruption duration of Old Faithful. Notice how the points cluster towards the lower left and upper right. While this does show us a positive association (meaning the longer between eruptions, the longer the next eruption will last), it also shows us that there are not very many medium length eruptions. They are either short eruptions with short wait times or long eruptions with long wait times.



http://en.wikipedia.org/wiki/Scatter_plot

Lesson 9.2

Use the given scatter plots to answer the questions.

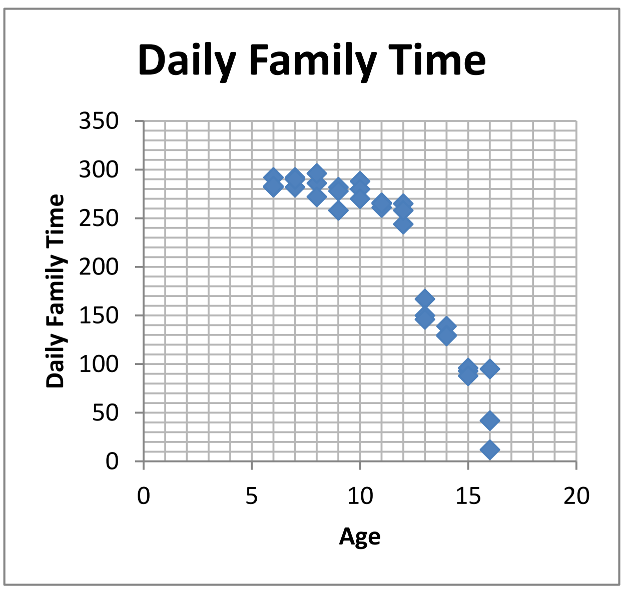


1. Does this scatter plot show a positive association, negative association, or no association? Explain why.

2. Is there an outlier in this data set? If so, approximately how old is the outlier and how about many minutes does he or she study per day?

3. Is this association linear or non-linear? Explain why.

4. What can you say about the relationship between your age and the amount that you study?

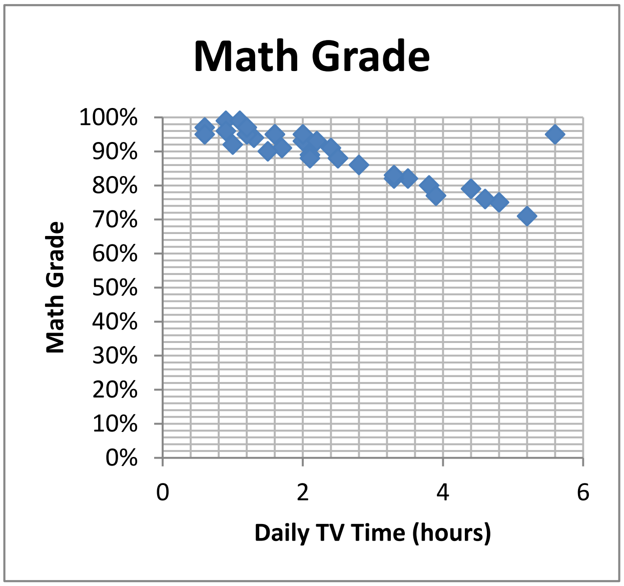


5. Does this scatter plot show a positive association, negative association, or no association? Explain why.

6. Is there an outlier in this data set? If so, approximately how old is the outlier and about how many minutes does he or she spend with family per day?

7. Is this association linear or non-linear? Explain why.

8. What can you say about the relationship between your age and the amount of time that you spend with family?

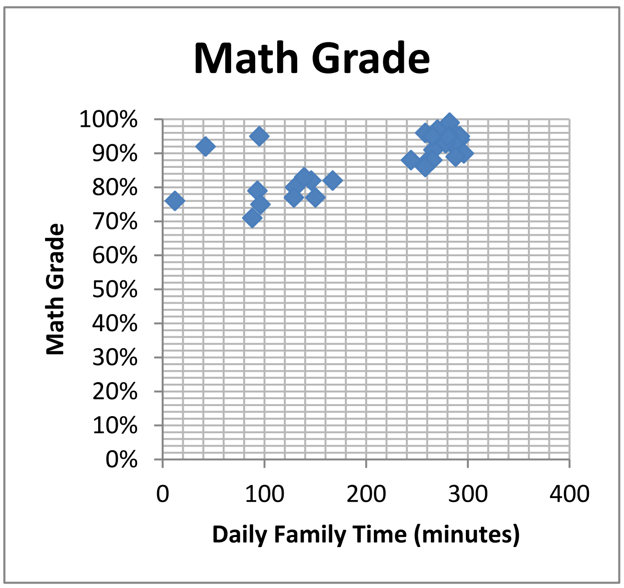


9. Does this scatter plot show a positive association, negative association, or no association? Explain why.

10. Is there an outlier in this data set? If so, approximately how much does that person watch TV daily and what is his or her approximate math grade?

11. Is this association linear or non-linear? Explain why.

12. What can you say about the relationship between the amount of time you watch TV and your math grade?



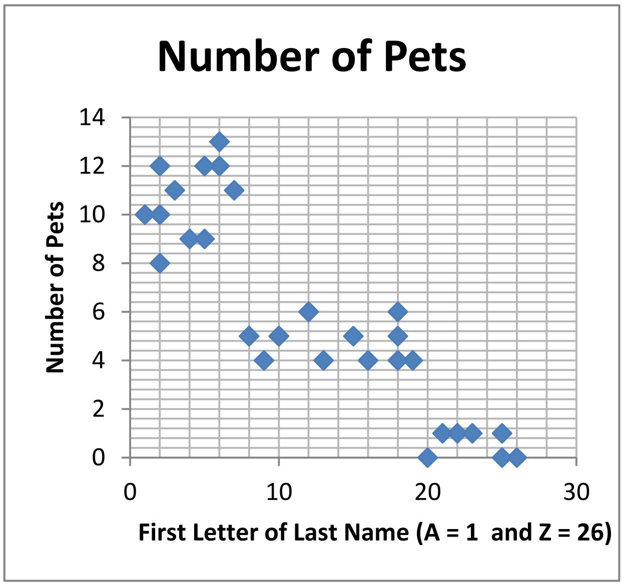
13. Does this scatter plot show a positive association, negative association, or no association? Explain why.

14. Is there an outlier(s) in this data set? If so, approximately how much time does that person(s) spend with his or her family daily and what is his or her approximate math grade?

15. Is this association linear or non-linear? Explain why.

16. What can you say about the relationship between the amount of time that you spend with your family and your math grade?

17. Are there any other patterns that you notice in this data?



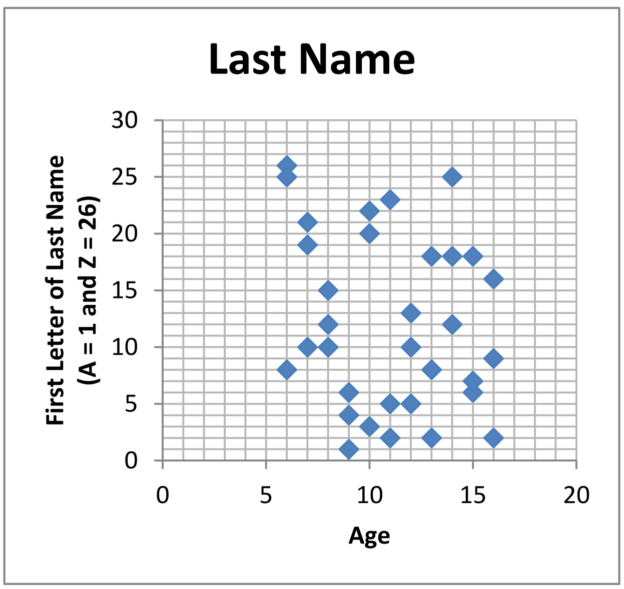
18. Does this scatter plot show a positive association, negative association, or no association? Explain why.

19. Is there an outlier(s) in this data set? If so, approximately how many pets does that person(s) have?

20. Is this association linear or non-linear? Explain why.

21. What can you say about the relationship between your last name and the number of pets you have?

22. Are there other patterns that you notice about people's last names and how many pets they have?

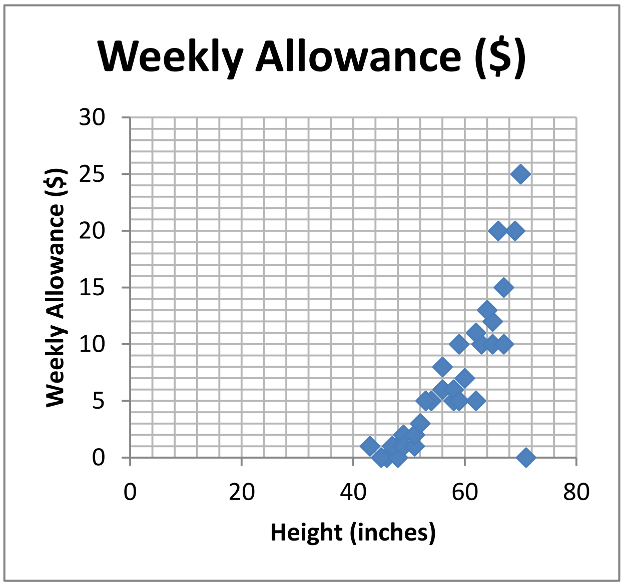


23. Does this scatter plot show a positive association, negative association, or no association? Explain why.

24. Is there an outlier(s) in this data set? If so, approximately how old is that person?

25. Is this association linear or non-linear? Explain why.

26. What can you say about the relationship between your last name and your age?



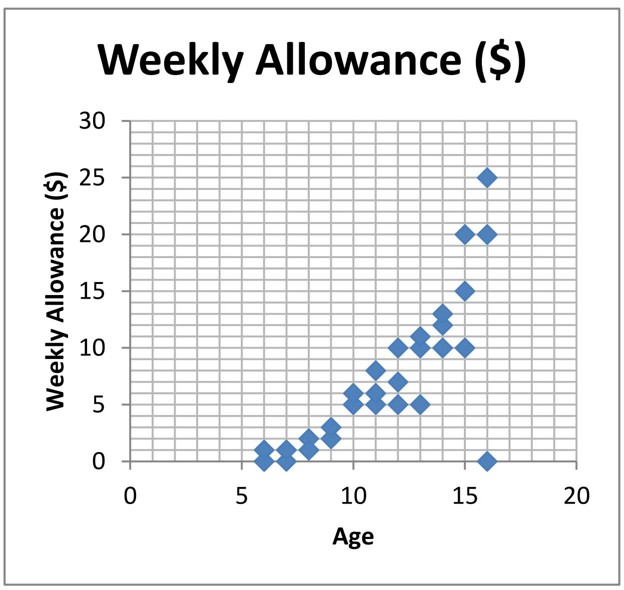
27. Does this scatter plot show a positive association, negative association, or no association? Explain why.

28. Is there an outlier(s) in this data set? If so, approximately how tall is that person and how much does he or she make in allowance each week?

29. Is this association linear or non-linear? Explain why.

30. What can you say about the relationship between your height and your allowance?

31. Do you think that being taller means that you will get more allowance? In other words, do you think this relationship is a causation or a correlation?



32. Does this scatter plot show a positive association, negative association, or no association? Explain why.

33. Is there an outlier(s) in this data set? If so, approximately how old is that person and how much does he or she make in allowance each week?

34. Is this association linear or non-linear? Explain why.

35. What can you say about the relationship between your age and your allowance?

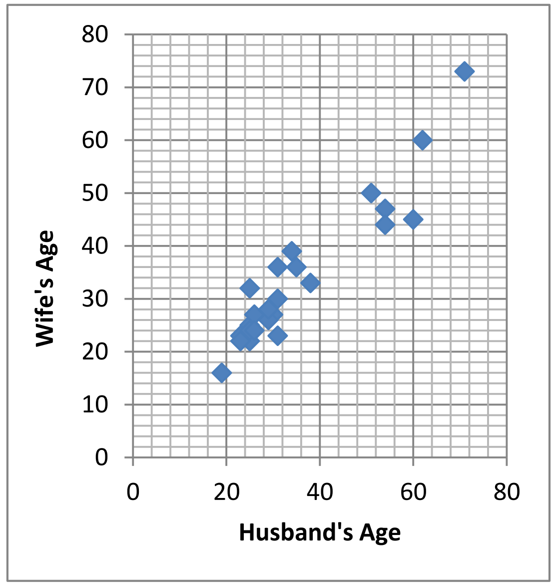
36. Do you think that being older means that you will get more allowance? In other words, do think this relationship is a causation or a correlation?

9.3 The Line of Best Fit

When we have a scatter plot that suggests a linear association, it is often useful to draw in a line of best fit to help us interpret the data more accurately. A **line of best fit** is a line drawn on the scatter plot such that the distance between each of the points and the line are minimized. Let's look at some examples.

Drawing the Line of Best Fit

Finding the true line of best fit is quite an involved task if we do it by hand. While programs like Excel will automatically draw in the line of best fit for us, for now we will focus on informally drawing a line of best fit. In other words, we know that our line is not the exact line of best fit, but it will be a nice estimate. Consider the scatter plot to the right.

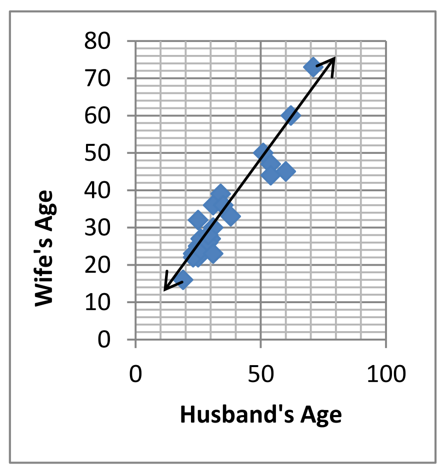


In this scatter plot there are 24 couples represented and it appears that there is a positive linear association between their ages. Generally speaking it looks like the older the husband is, the older the wife is. If we wanted to informally draw a line of best fit in this scatter plot, we would look for a place where we the line would roughly split data in half and have the same general rate of change (or slope) as the data.

Now consider the three scatter plots below. Which line of best fit seems most appropriate? The first attempted line of best fit does appear to cut the data roughly in half, but it definitely doesn't match the rate of change that the data seems to represent. The second attempted line of best fit seems to match the rate of change but doesn't roughly cut the data in half. The third one is our best option for an informal line of best fit.

First Attempt

Second Attempt



Third Attempt

Now for the sake of comparison, let's see the actual line of best fit that Excel comes up with. It looks like our line of best fit is very close to the true line of best fit.

Before drawing in the line of best fit on a given data set, it may be useful to lay down a pen or pencil on the scatter plot and try to arrange the pen where the line of best fit should be. Once you have visualized where the line of best fit should be, then draw it in.

Extrapolating with the Line of Best Fit

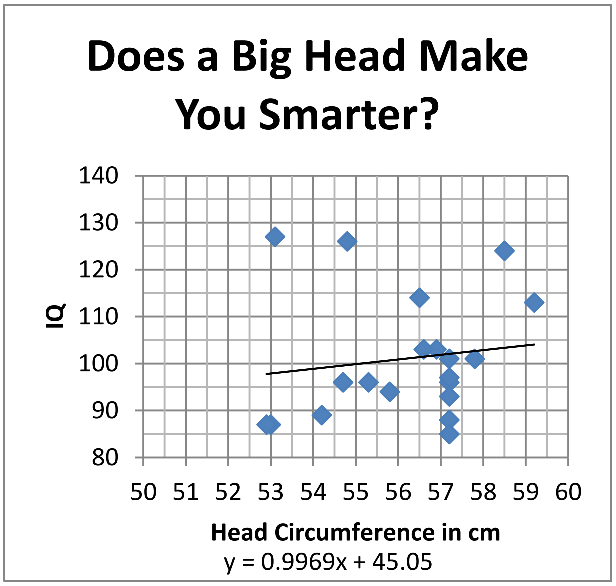
To **extrapolate** means to estimate or predict an answer in an unknown situation. We can use the line of best fit to make these predictions from the data. For example, using the above line of best fit, how old would we expect the wife to be of a husband that was 45 years old? We don't have a data point there, so we don't know what the answer to this would be, but we can extrapolate using the line of best fit. Go to 45 years old on the husband axis and go up to the line of best fit. Note that the line of best fit is at a height of 42 years old for the wife meaning this would be a good estimate for how old we would expect the wife to be.

If the wife were 60 years old, how old would we expect the husband to be? This time go to the height of 60 on the wife axis and travel over to the line of best fit. It appears to be at about 65 years old on the husband axis, so we would expect the husband to be near that age.

Notice that these are only estimates and would not necessarily be exactly what we would find in real life, but it is useful as a guideline.

The Equation of the Line of Best Fit

Since we have a *line* of best fit, we know that a line can be expressed as an equation. In fact, we are most familiar with the slope intercept form of an equation. We can use this line to extrapolate data further, learn more about the rate of change, and more. Let's look at some data taken from sets of twins where they were studying if there was an association between the size of a person's skull and his or her IQ.



First of all, notice that all the data is clustered between the 50 cm and 60 cm mark, so Excel decided it would be beneficial to use a broken axis on this graph. Secondly, notice that Excel has drawn in the line of best fit and given us the equation for that line.

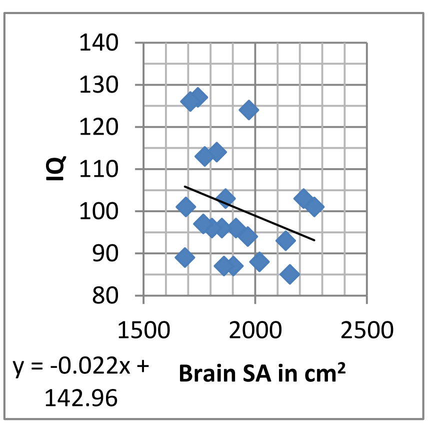
At first glance it appears that there may be no association between the size of your skull and your IQ. The line of best fit is nearly flat suggesting either a constant association or no association at all. However, because of the broken axes, this is misleading.

Let's first approximate the equation for ease of analysis. The slope of 0.9969 is very close 1 and the y-intercept is very close to 45, so let's approximate the line of best fit to be $y = x + 45$.

What does the slope mean in this context? The slope is approximately 1, which means that for every one centimeter increase in skull size we would expect a one point increase in IQ. So maybe there is something to that old "egg head" comment, as mean as it is.

What does the y-intercept mean in this context? The y-intercept is about 45, which tells us that no matter the size of a person's head, their IQ is very unlikely to drop below 45. Even a skull size of zero centimeters in circumference would supposedly have an IQ of 45, but we know this isn't possible.

What would be the expected IQ if a person had a head circumference of 80 cm? In our equation, the y represents the IQ and x represents the head circumference. Simply plug in and solve like this: $y = 80 + 45 = 125$ to see that the expected IQ would be about 125. If a person had an IQ of 150, what would we expect their head circumference to be using our line of best fit? $150 = x + 45$ and then subtract 45 from both sides to see that $x = 115\ cm$. That's a big head!



This final graph shows the surface area of the brain compared to IQ. If we rounded the slope and intercept, the equation of the line of best fit is approximately $y = -\frac{1}{50}x + 143$. This means that the IQ goes down one for every additional 50 cm^2 in surface area. In context that means that the more of your brain that is "exposed", the lower your IQ.

Data Used

Brain data came from: http://lib.stat.cmu.edu/datasets/IQ_Brain_Size

Husband and wife data came from: http://www.statcrunch.com/5.0/viewreport.php?reportid=10183

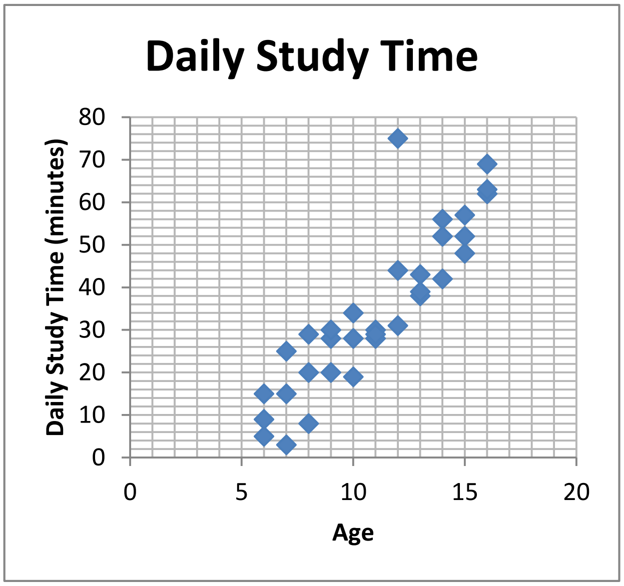
Head Circumference	Brain Surface Area	Brain Volume	Body Weight	IQ
54.7	1913.88	1005	57.607	96
54.2	1684.89	963	58.968	89
53	1902.36	1035	64.184	87
52.9	1860.24	1027	58.514	87
57.8	2264.25	1281	63.958	101
56.9	2216.4	1272	61.69	103
56.6	1866.99	1051	133.358	103
55.3	1850.64	1079	107.503	96
53.1	1743.04	1034	62.143	127
54.8	1709.3	1070	83.009	126
57.2	1689.6	1173	61.236	101
57.2	1806.31	1079	61.236	96
57.2	2136.37	1067	83.916	93
57.2	2018.92	1104	79.38	88
55.8	1966.81	1347	97.524	94
57.2	2154.67	1439	99.792	85
57.2	1767.56	1029	81.648	97
56.5	1827.92	1100	88.452	114
59.2	1773.83	1204	79.38	113
58.5	1971.63	1160	72.576	124

Couple	Husband's Age	Wife's Age
1	25	22
2	25	32
3	51	50
4	25	25
5	38	33
6	30	27
7	60	45
8	54	47
9	31	30
10	54	44
11	23	23
12	34	39
13	25	24
14	23	22
15	19	16
16	71	73
17	26	27
18	31	36
19	26	24
20	62	60
21	29	26
22	31	23
23	29	28

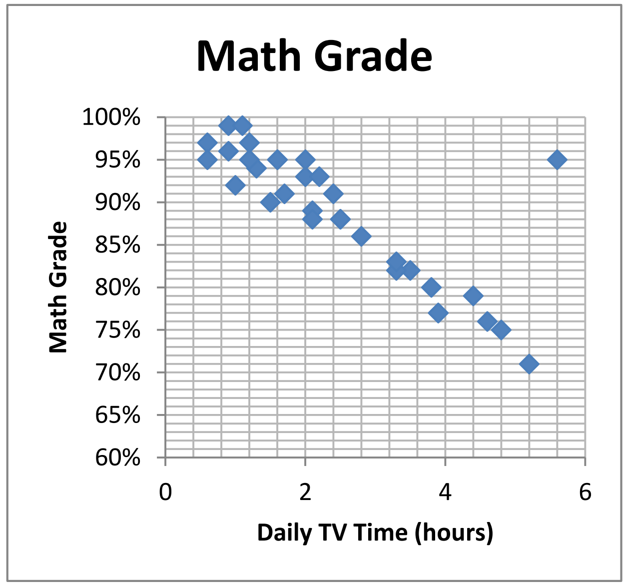
Lesson 9.3

Draw an informal line of best fit on the given scatter plot and explain why you drew the line where you did.

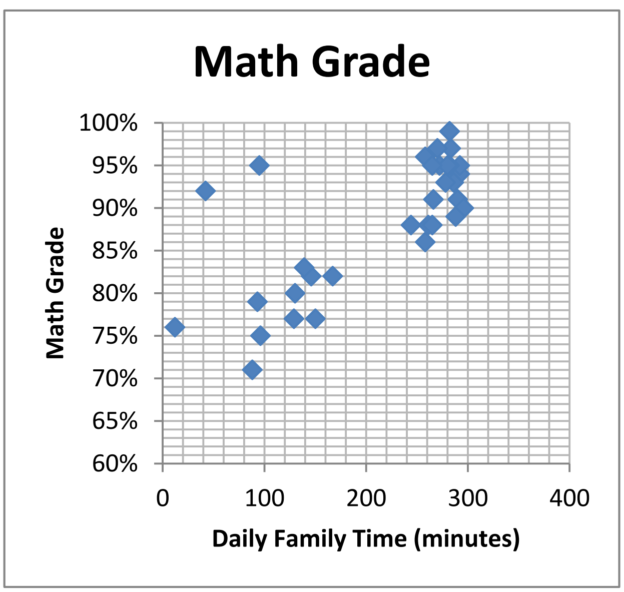
1.



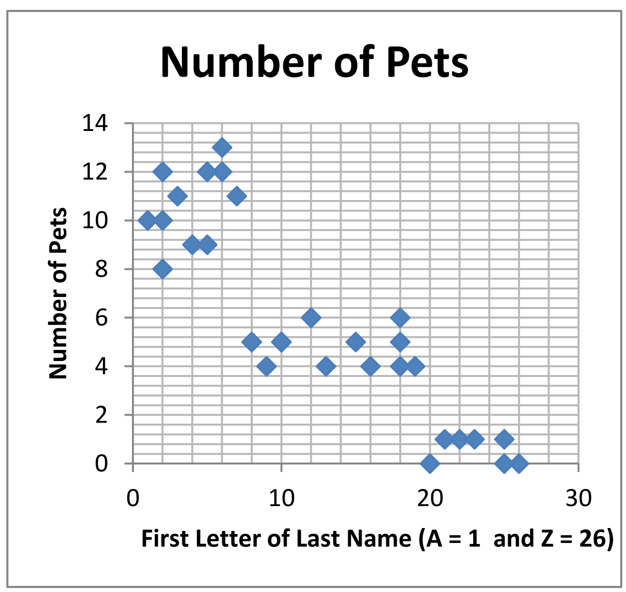
2.



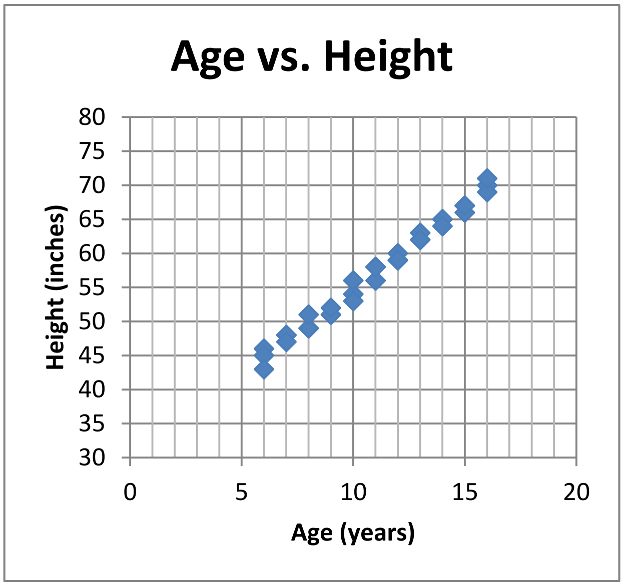
3.



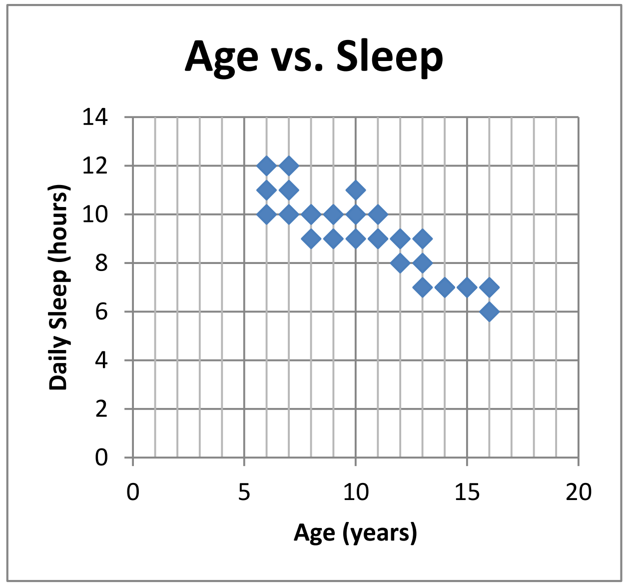
4.



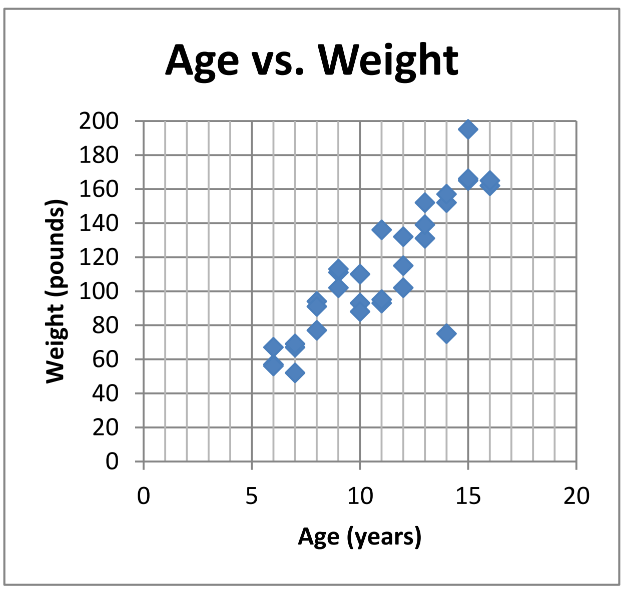
5.


Age vs. Height
80
75
70
65
60
55
50
45
40
35
30
Height (inches)
0
5
10
15
20
Age (years)

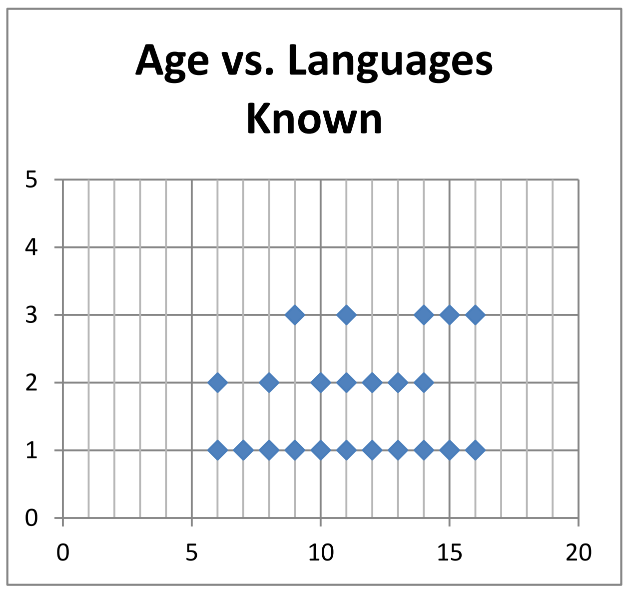
6.


Age vs. Sleep
14
12
10
8
6
4
2
0
Daily Sleep (hours)
0
5
10
15
20
Age (years)

7.

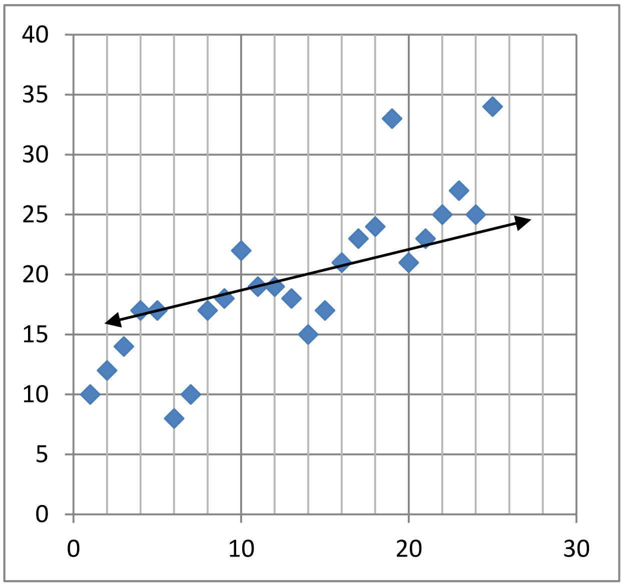

Age vs. Weight
200
180
160
140
120
100
80
60
40
20
0
Weight (pounds)
0
5
10
15
20
Age (years)

8.


Age vs. Languages Known
5
4
3
2
1
0
0
5
10
15
20

Determine whether the drawn line of best fit is accurate or not. Explain why you think your position is true.

9.



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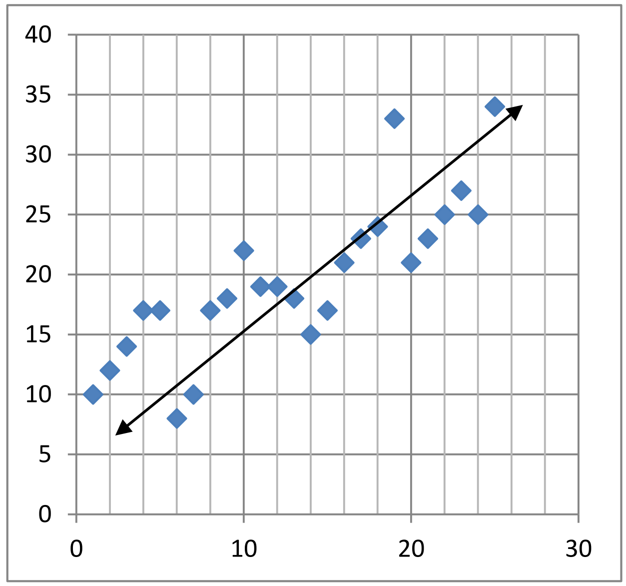
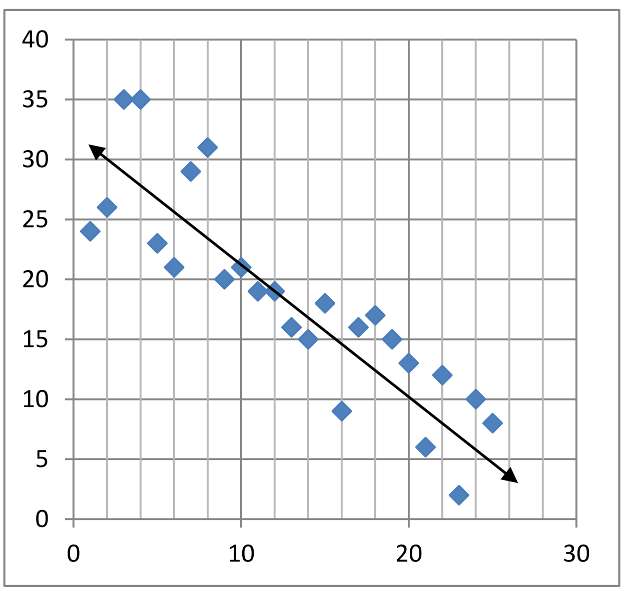
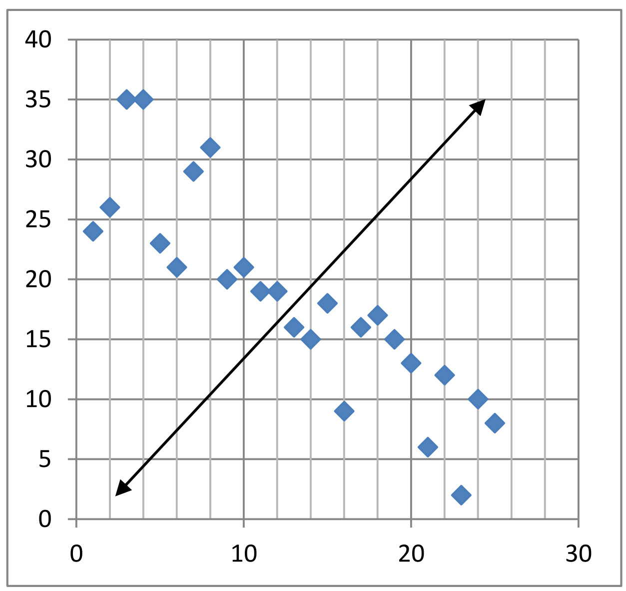


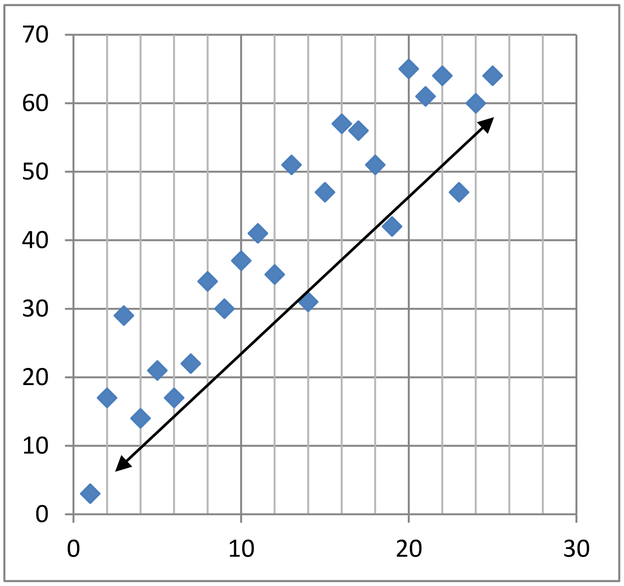
11.



12.

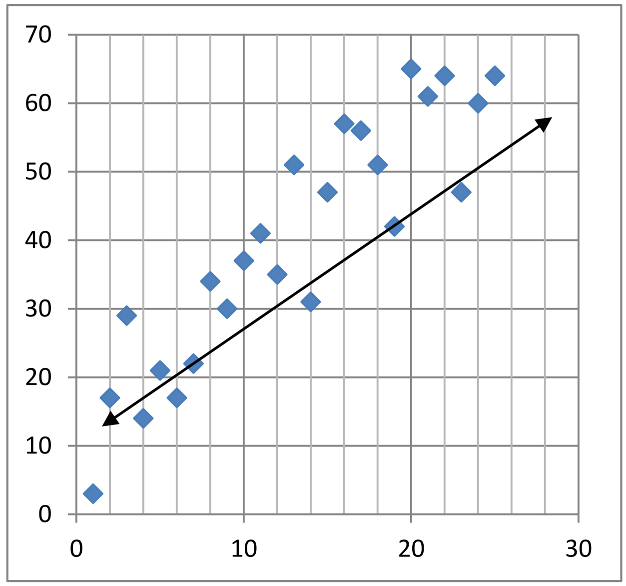


13.



70
60
50
40
30
20
10
0
0
10
20
30

14.



70
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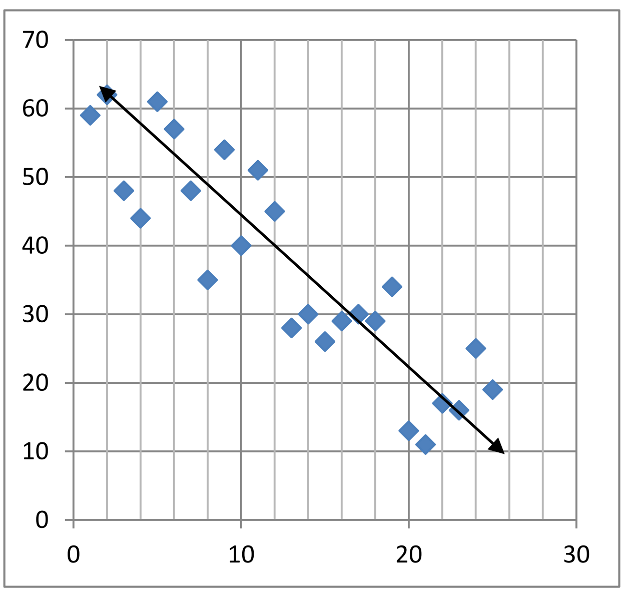
15.



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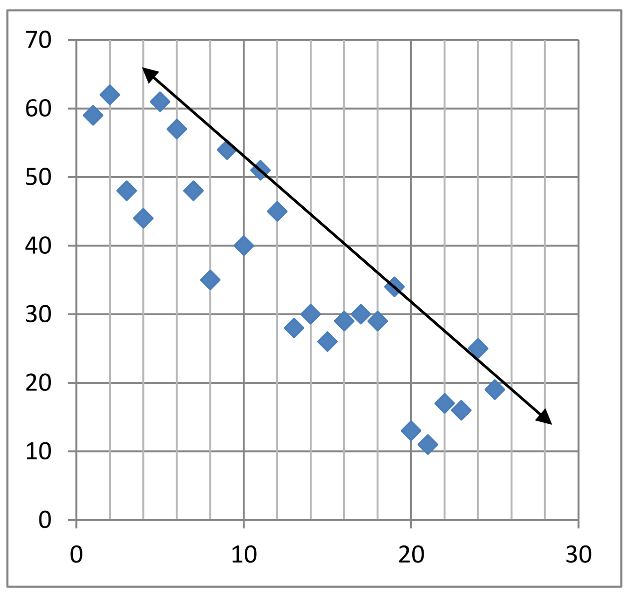
16.



70
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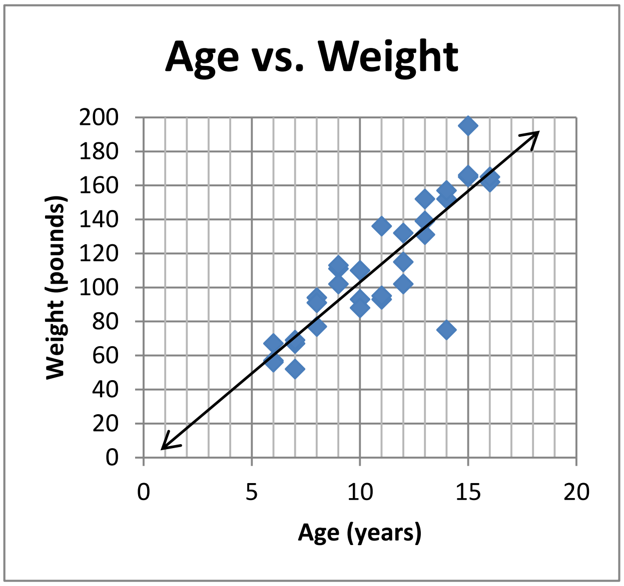
10
20
30

Use the given line of best fit or equation of the line of best fit to answer the following questions.



17. Using the graph only, about how much would you expect an 18 year old to weigh?

18. Using the graph only, about how much would you expect a 4 year old to weigh?

19. Using the graph only, if a person weighed 80 pounds, how old would you expect them to be?

20. Using the graph only, if a person weighed 120 pounds, how old would you expect them to be?

The line of best fit for the scatter plot showing age (x-value) compared to weight (y-value) is approximately:

$$y = \frac{21}{2}x - \frac{7}{2}$$

21. Using the line of best fit equation (show your work), about how much would you expect an 18 year old to weigh? How does this answer compare to the answer you gave to problem number 17?

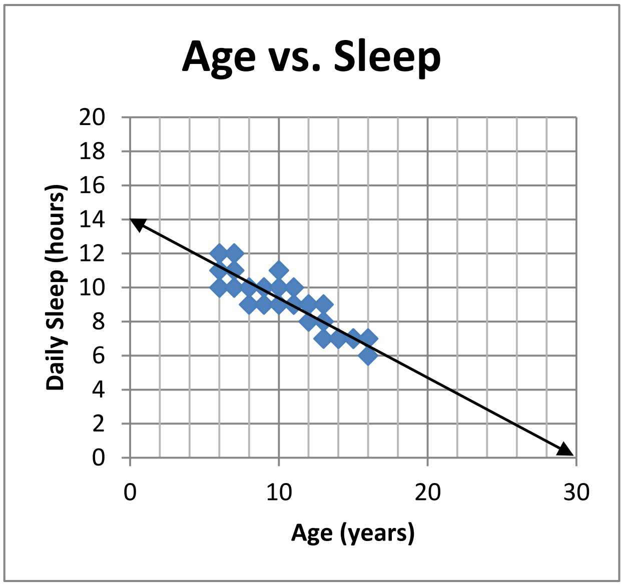
22. Using the line of best fit equation (show your work), about how much would you expect an 4 year old to weigh? How does this answer compare to the answer you gave to problem number 18?

23. Using the line of best fit equation (show your work), about how old would you expect a person to be if they weighed 80 pounds? How does this answer compare to the answer you gave to problem number 19?

24. Using the line of best fit equation (show your work), about how old would you expect a person to be if they weighed 120 pounds? How does this answer compare to the answer you gave to problem number 20?

25. What is the rate of change (slope) of the line of best fit? What does the slope represent in this context and does that make sense?

26. What is the initial value (y-intercept) of the line of best fit? What does it represent in this context and does that make sense?



27. Using the graph only, about how much would you expect a 22 year old to sleep?

28. Using the graph only, about how much would you expect a 4 year old to sleep?

29. Using the graph only, if a person slept 6 hours, how old would you expect them to be?

30. Using the graph only, if a person slept 13 hours, how old would you expect them to be?

The line of best fit for the scatter plot showing age (x-value) compared to daily hours of sleep (y-value) is approximately:

$$y = -\frac{1}{2}x + 14$$

31. Using the line of best fit equation (show your work), about how much would you expect a 22 year old to sleep? How does this answer compare to the answer you gave to problem number 27?

32. Using the line of best fit equation (show your work), about how much would you expect a 4 year old to sleep? How does this answer compare to the answer you gave to problem number 28?

33. Using the line of best fit equation (show your work), about how old would you expect a person to be if they slept 6 hours? How does this answer compare to the answer you gave to problem number 29?

34. Using the line of best fit equation (show your work), about how old would you expect a person to be if they slept 13 hours? How does this answer compare to the answer you gave to problem number 30?

35. What is the rate of change (slope) of the line of best fit? What does the slope represent in this context and does that make sense?

36. What is the initial value (y-intercept) of the line of best fit? What does it represent in this context and does that make sense?

9.4 Two-Way Tables

Sometimes we need to compare two sets of data where the data is a yes/no type answer. In this case a scatter plot doesn't make sense since we don't have numerical data. We use what is called a two-way table to analyze this type of data.

Constructing a Two-Way Table

To construct a two-way table, we first need some data. Let's look at the following fictional table where we asked a class of 22 students a series of questions:

	Anne	Brad	Cathy	Devin	Edith	Frank	Gabby	Hannah	Ignus	Jake	Koty
Democrat/ Republican?	D	R	R	R	D	D	R	R	R	D	D
Do you eat McDonald's weekly?	Y	Y	Y	N	N	Y	Y	Y	Y	N	N
Want higher taxes?	Y	N	N	N	N	Y	N	N	N	Y	N
Do you own smartphone?	N	Y	Y	N	Y	Y	N	Y	Y	N	N

	Lisa	Mo	Nancy	Opy	Peggy	Quira	Ron	Shela	Toni	Ula	Vanna
Democrat/ Republican?	D	D	D	R	R	D	R	D	R	D	D
Do you eat McDonald's weekly?	N	Y	N	Y	N	N	N	N	Y	N	N
Want higher taxes?	N	Y	Y	N	N	N	Y	N	N	Y	Y
Do you own smartphone?	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y

Now that we have our data, we will consolidate some of it into a two-way table. Let's first compare the students' political view to their eating habits at McDonald's. A two-way table for this comparison would look like this:

	Democrat	Republican
Eat McD's weekly	3	7
Don't eat McD's weekly	9	3

How did we fill this out? We counted the number of Democrats that eat at McDonald's weekly, the number of Republicans that eat at McDonald's weekly, the number of Democrats that don't eat at McDonald's weekly, and the number of Republicans that don't eat at McDonald's weekly. Each of those numbers we filled in the table in the appropriate place. Obviously one of the advantages of the two-way table is the fact that it takes up so much less space than the original data. We could make a similar two-way table comparing political affiliation with tax views or comparing tax views with owning a smart phone.

Analyzing a Two-Way Table

There are many things that a two-way table can tell us. Let's look at another example of how the U.S. House of Representatives voted on a recent bill that would force the national budget to be balanced.

	Democrat	Republican
In favor	25	236
Against	161	4

How many Democrats voted on this bill? If 25 voted in favor of the bill and 161 voted against, that means that a total of 186 Democrats voted on this bill. In essence we are finding the frequency of being a Democrat by adding the numbers in the Democrat column. **Frequency** is how often something occurs.

Similarly we can see how many Republicans voted on this bill, which is 240. How many total representatives voted on this bill? We can find this by adding all the numbers together. This means that 426 representatives in total voted on this bill. Since there are 435 representatives (we know this from our social studies class), we can then ask why the remaining 9 representatives didn't vote. Call them and ask.

We can also see that 261 voted in favor of the bill and 165 voted against the bill by adding the numbers in the rows. While this is a majority vote, it is not the required 290 votes needed to pass, so ultimately this bill failed.

At times it may be more useful to look at the relative frequency instead of the frequency. **Relative frequency** is the ratio of the frequency to the total number of data entries. So while the frequency of in favor votes was 261, it might be more useful to know that the relative frequency is $\frac{261}{426} \approx 0.61$. So about 61% of the House voted in favor of this bill and a vote of $\frac{2}{3}$ or $66.\overline{6}\%$ was needed for the bill to pass.

Are there any other conclusions we can make based on the information in the two-way table? For example, is there evidence that one party supported the bill over the other? It appears from the data table that there is a positive association between being a Republican and being in favor of the balanced budget bill. It appears that there is a negative association between being a Democrat and being in favor of the balanced budget bill. Notice that this doesn't mean that the Republicans are positive (or correct) and Democrats are negative (or wrong). Instead the positive and negative refer to the association or correlation in the data.

Lesson 9.4

Use the data set to answer the following questions. For this data set a class of middle school students was asked what they thought was most important in school: good grades or popularity.

Boy or Girl	B	B	G	G	G	B	G	B	B	G	G	B	G	B	G	B	B	G	G	B
Grades or Popularity	P	G	G	P	G	P	G	G	P	G	G	P	G	P	P	P	G	G	G	P

Boy or Girl	B	B	G	G	G	B	G	B	B	G	G	B	G	B	G	B	B	G	G	B
Grades or Popularity	P	G	P	G	G	P	G	P	P	G	G	G	G	P	P	P	G	P	G	G

1. Construct a two-way table of the data.

	Grades	Popularity
Boys		
Girls		

2. What is the frequency of students who believe grades are more important?

3. What is the *relative* frequency of students who believe grades are more important?

4. What is the frequency of students who believe popularity is more important?

5. What is the *relative* frequency of students who believe popularity is more important?

6. What is the frequency of girls who believe grades are more important?

7. What is the *relative* frequency of girls who believe grades are more important?

8. What is the frequency of boys who believe popularity is more important?

9. What is the *relative* frequency of boys who believe popularity is more important?

10. Based on this data, do you feel there is relationship between a student's gender and what they think is most important in school? What is that relationship and what evidence do you have that it exists?

Use the data set to answer the following questions. For this data set a class of middle school students was asked what hand was their dominant hand.

Boy or Girl	B	B	G	G	G	B	G	B	B	G	G	B	G	B	G	B	B	G	G	B
Right or Left	L	R	R	L	R	L	R	R	R	R	L	R	R	R	R	R	L	R	L	R

Boy or Girl	B	B	G	G	G	B	G	B	B	G	G	B	G	B	G	B	B	G	G	B
Right or Left	R	R	L	R	R	R	L	R	L	R	R	R	L	R	R	L	R	R	L	L

11. Construct a two-way table of the data.

	Right-handed	Left-handed
Boys		
Girls		

12. What is the frequency of students who are right-handed?

13. What is the *relative* frequency of students who are right-handed?

14. What is the frequency of students who are left-handed?

15. What is the *relative* frequency of students who are left-handed?

16. What is the frequency of girls who are right-handed?

17. What is the *relative* frequency of girls who are right-handed?

18. What is the frequency of boys who are right-handed?

19. What is the *relative* frequency of boys who are right-handed?

20. Based on this data, do you feel there is relationship between a student's gender and whether or not they are right-handed? What is that relationship and what evidence do you have that it exists?

Use the two-way tables representing surveys middle school students took to answer the following questions.

Survey 1:	Prefer Spicy Salsa	Prefer Mild Salsa
Boys	255	45
Girls	68	132

Survey 2:	Prefer Spicy Salsa	Prefer Mild Salsa
Right-handed	280	170
Left-handed	43	7

21. How many students were surveyed?

22. What is the *relative* frequency of students who prefer spicy salsa? Is it the same on both two-way tables?

23. How many boys were surveyed?

24. How many girls were surveyed?

25. What is the *relative* frequency of boys who prefer spicy salsa?

26. What is the *relative* frequency of girls who prefer spicy salsa?

27. Do you think there is a relationship between gender and salsa preference? What is that relationship and what evidence do you have that it exists?

28. How many right-handed students were surveyed?

29. How many left-handed students were surveyed?

30. What is the *relative* frequency of right-handed students who prefer mild salsa?

31. What is the *relative* frequency of left-handed students who prefer mild salsa?

32. Do you think there is a relationship between a student's dominant hand and salsa preference? What is that relationship and what evidence do you have that it exits?

Review Unit 9: Bivariate Data

You may use a calculator.

Unit 9 Goals

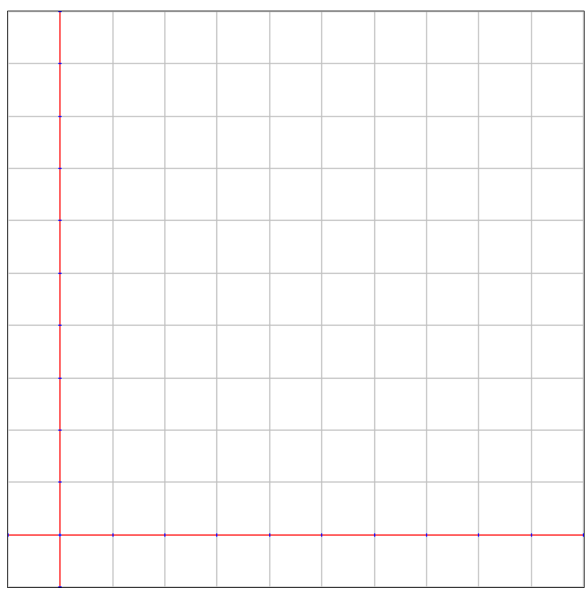
- *Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association. (8.SP.1)*
- *Know that straight lines are widely used to model relationships between to quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line. (8.SP.2)*
- *Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. (8.SP.3)*
- *Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables. (8.SP.4)*

You may use a calculator.

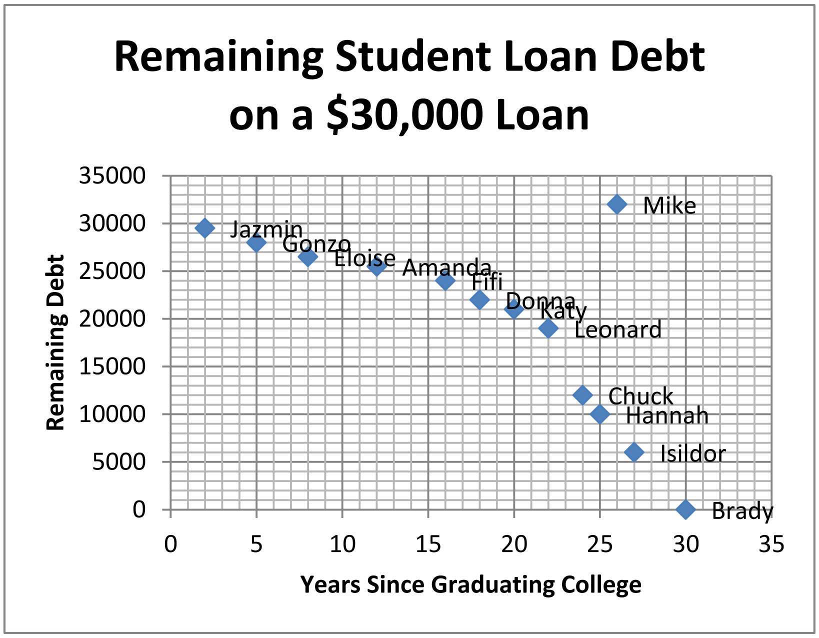
Construct a scatter plot for the following data set using appropriate scale for both the x- and y-axis.

1. This table shows the age of students and their scores on the MAP test.

	Age	MAP Score
Anna	8	180
Bob	10	200
Carly	11	215
Damien	12	220
Esther	9	195
Franco	15	235
Georgia	13	230
Hank	14	235
Innya	13	225
Jacob	14	225



Use the following scatter plot to answer each question. The scatter plot shows the number of years each person invested ten thousand dollars versus the end value of that investment in thousands of dollars.



2. Does this scatter plot represent a positive association, negative association, or no association? Why?

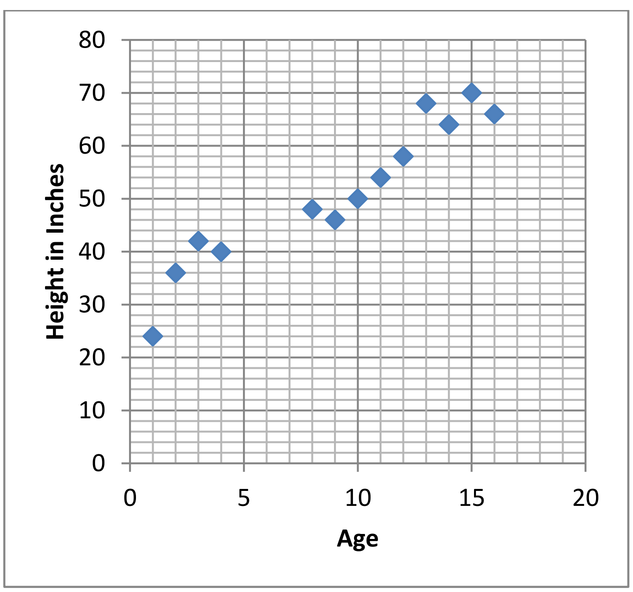
3. Which person paid off their debt? About how long did it take?

4. Does this appear to a linear or non-linear association? Why?

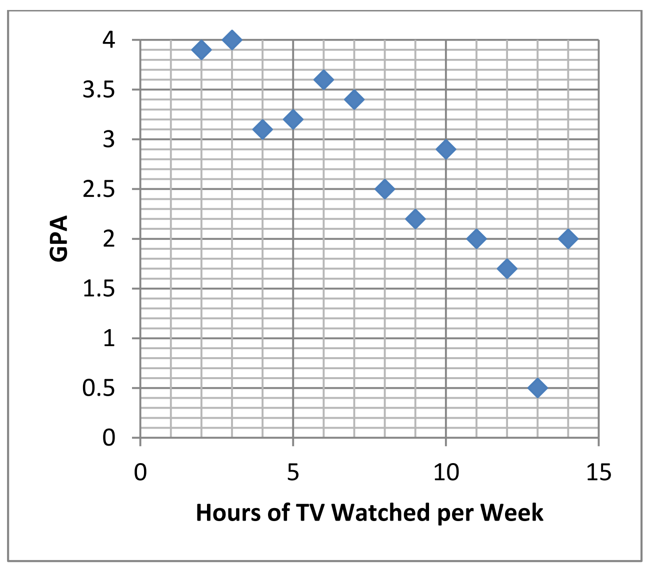
5. Which person is the outlier in this data set? Why?

Draw an informal line of best for the given scatter plots.

6. This scatter plot shows the age in years versus the height in inches of a group of children.

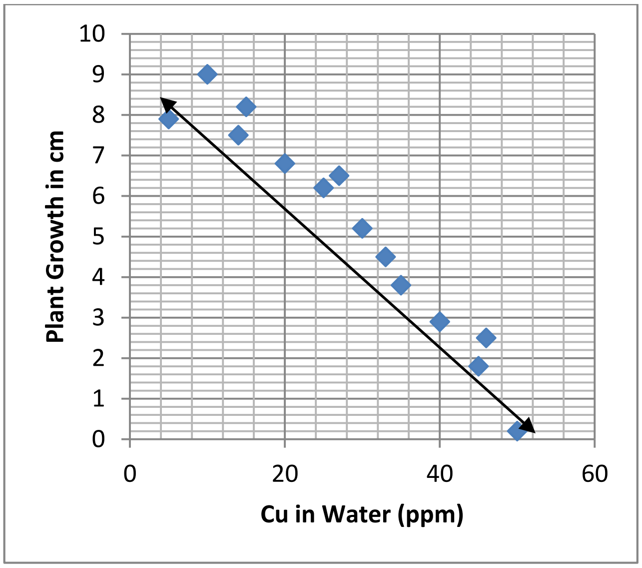


7. This scatter plot shows the hours of TV watched per week versus the GPA on a 4.0 scale for a group of students.

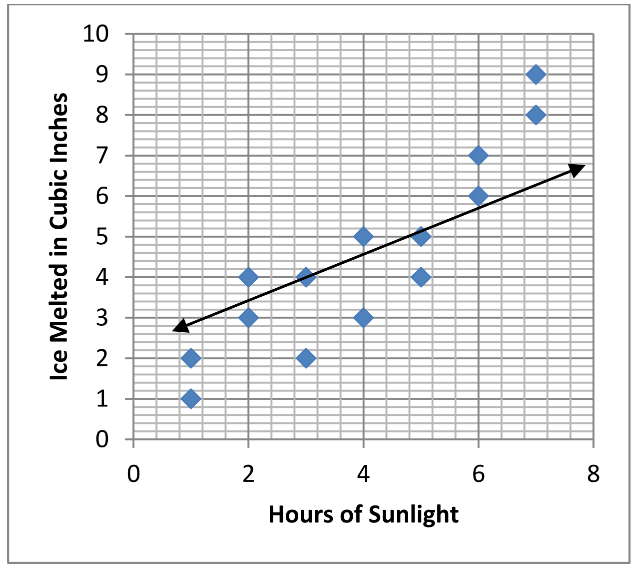


Explain why the drawn line of best fit is accurate or why not.

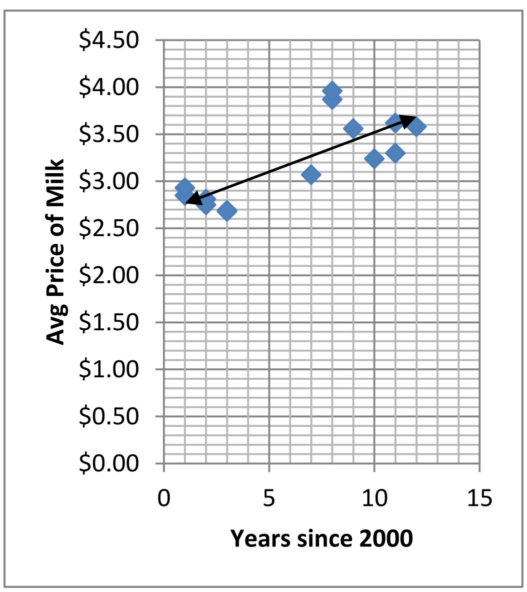
8. This scatter plot shows the amount copper in water in ppm versus plant growth in cm over three months.



9. This scatter plot shows the hours a cubic foot of ice was exposed to sunlight versus the amount of ice that melted in cubic inches.



The scatter plot shows the price of a gallon of milk from 2001 to 2012. The equation of the line of best fit is approximately $\mathbf{y = \frac{21}{250}x + 2.68}$.



10. Predict what price of a gallon of milk would have been in 2005 using both the equation and the graph.

Equation Work:	Graph Prediction:

11. Predict what year it would have been when a gallon of milk cost approximately $3.00 using both the equation and the graph.

Equation Work:	Graph Prediction:

Using the same scatter plot and equation of the line of best fit of $\mathbf{y = \frac{21}{250}x + 2.68}$***, answer the following questions.***

12. What does the slope of this equation mean in terms of the given situation? In other words, explain what the rise and run mean for this problem.

13. What does the y-intercept of this equation mean in terms of the given situation? In other words, explain what the y-intercept means when considering the price of a gallon of milk and the year.

Answer the following questions about two-way tables.

14. Construct a two-way table from the following data about whether or not students own an iPhone and whether or not they own an iPad.

Own an iPhone?	Y	N	Y	Y	N	Y	N	N	Y	Y	Y	N	N	Y	N	N	Y	N	Y	N
Own a iPad?	Y	N	Y	N	N	Y	Y	N	Y	N	Y	Y	N	Y	N	N	Y	Y	N	N

15. Do you think there is a relationship between owning a iPhone and owning an iPad? Based on the data, why or why not?

Answer the following questions using the given two-way table.

	Support Year-Round School	Do Not Support Year-Round School
Students	250	2150
Teachers	80	70

16. How many teachers were surveyed?

17. How many students were surveyed?

18. How many people support year-round school?

19. How many teachers do not support year-round school?

20. How many students do not support year-round school?

21. As a percent to the nearest hundredth (two decimal places) what is the relative frequency of the teachers compared to all those surveyed?

22. As a percent to the nearest hundredth (two decimal places) what is the relative frequency of the students who support year-round school compared to all students?

23. As a percent to the nearest hundredth (two decimal places) what is the relative frequency of the teachers who do not support year-round school compared to all teachers?

4^{th} Quarter Exam Review

You may use a calculator.

1. Solve $4g + 2g - 12 + 8 = 14$.

2. Solve $5(h + 2) = 30$.

3. Solve $3(a + 4) = 4a + 12 - a$.

4. Solve $2(b + 4) = 4(\frac{1}{2}b + 3)$.

5. Solve $x^2 = 81$

6. Solve $x^3 = 8$

7. Solve the following system of equations.
$$y = 2x - 4$$
$$y = 3x - 8$$

8. Solve the following system of equations.
$$x = 4$$
$$y = -2x + 6$$

9. Solve the following system of equations.
$$2x + y = -4$$
$$6x + 3y = -12$$

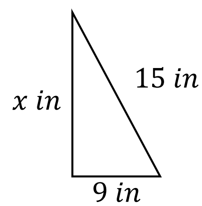
10. Sam sold snakes and snails at the Slithery and Slimy Pet Store. Snakes cost \$5 each and snails cost \$2 each. One afternoon Sam sold 10 pets and made \$29. How many snakes and how many snails did he sell?

11. Is the number $\sqrt{22}$ rational or irrational?

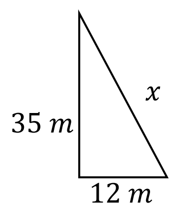
12. Approximate $\sqrt{40}$ to one decimal place?

13. List these numbers in order from least to greatest.
$$\frac{13}{3}, 6, \sqrt{56}, \pi$$

14. What is the length of the missing side?



15. What is the length of the missing side?



16. What is the distance between $(3,7)$ and $(-3,-1)$?

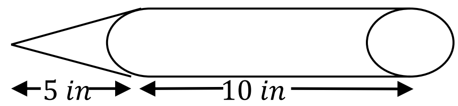
17. A fire truck pulls up to a burning building with a person trapped at the 2nd floor window which is $15\ ft$ off the ground. If the truck is parked $8\ ft$ away from the building, and it takes $2\ seconds$ to extend the ladder one foot, how long will it take the firefighters to extend the ladder to the window?

18. A cylindrical fire hose that is 100 ft long and has a radius of 0.5 ft. What is its volume both in terms of π and using $\pi \approx 3.14$?

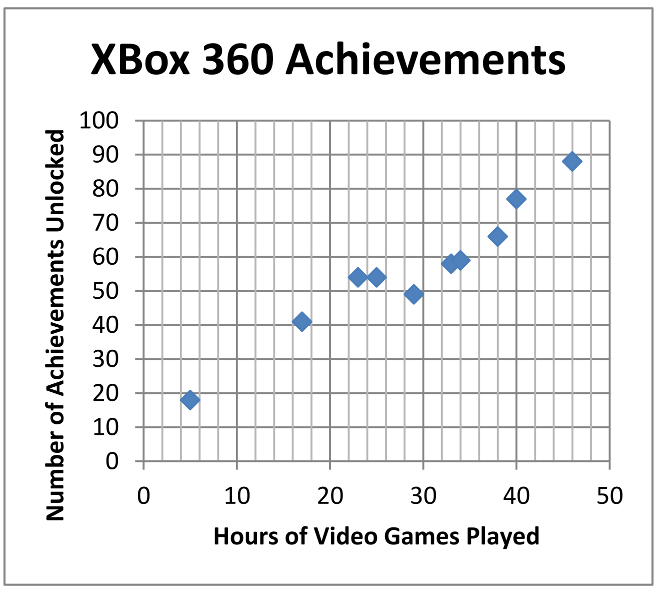
19. A spherical water balloon with a diameter of 20 mm. What is the volume?

20. What is the volume of a cone with height of 10 cm and radius of 3 cm?

21. What is the volume of a caulking gun with a radius of 1 inch, cone height of 5 inches, and cylinder height of 10 inches?



Use the following scatter plot to answer the next two questions. It shows the hours of video games students have played versus the number of achievements they have unlocked.



22. What type of associations does this graph show?

23. Which would be the best prediction for number of achievements unlocked for a student who has played 12 hours of video games each week using the line of best fit equation $y = \frac{3}{2}x + 12$ or the graph?

24. Give two variables that have a negative association

Use the following two-way table to answer the last two questions.

	Support School Uniforms	Do Not Support School Uniforms
Students	237	1928
Teachers	91	15

25. What is the relative frequency of students surveyed who do not support school uniforms?