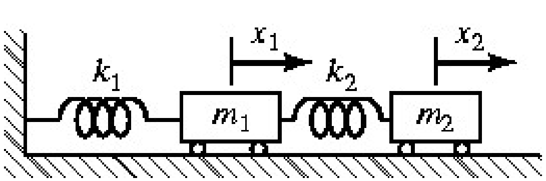
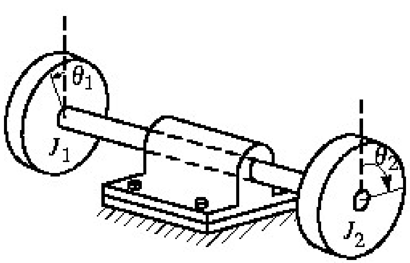
Response of MDOF systems

Degree of freedom (DOF): The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time.

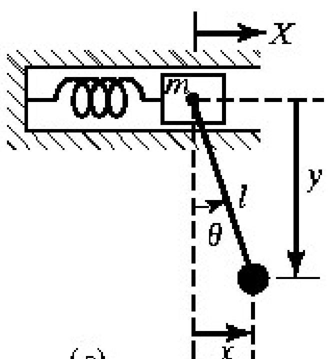
Two DOF systems



(a)

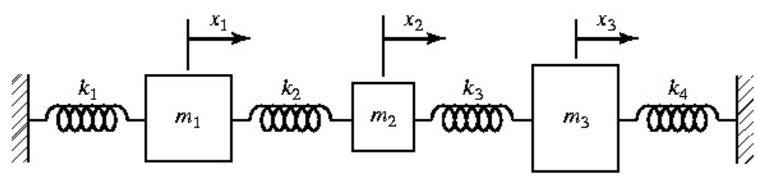


(b)

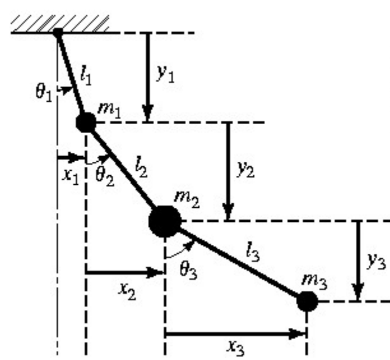


(c)

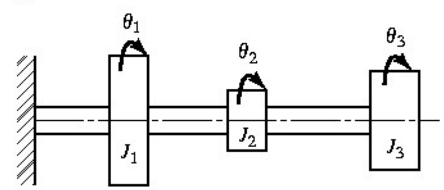
Three DOF systems



(a)



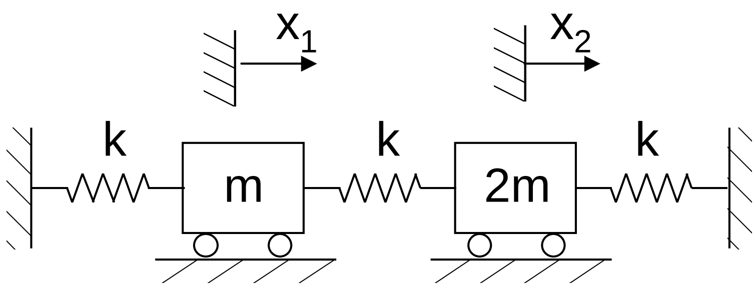
(b)



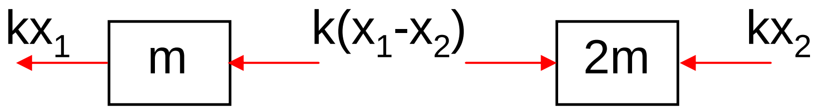
(c)

The normal mode analysis (EOM-1)

Example: Response of 2 DOF system



FBD



EOM

$$-kx_1 - k(x_1 - x_2) = m\ddot{x}_1$$

$$k(x_1 - x_2) - kx_2 = 2m\ddot{x}_2$$

In matrix form, EOM is

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

EOM -2 (example)

EOM

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{\mathbf{x}}} + \underbrace{\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{F}}$$

In general form

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t)$$

M is the inertia of mass matrix (*n x n*)

C is the damping matrix (*n x n*)

K is the stiffness matrix (*n x n*)

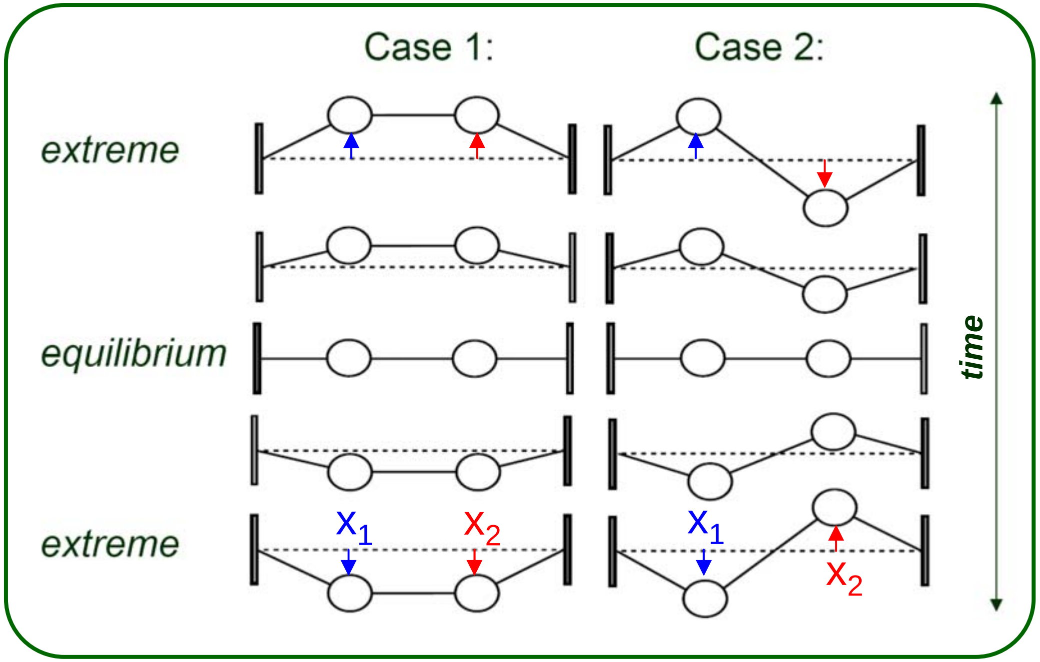
F is the external force vector (*n x 1*)

x is the position vector (*n x 1*)

Synchronous motion

From observations, free vibration of undamped MDOF system is a synchronous motion.

- All coordinates pass the equilibrium points at the same time
- All coordinates reach extreme positions at the same time
- Relative shape does not change with time



$x_1 / x_2 =$ constant

No phase diff. between x_1 and x_2

$$x_1 = A_1 \sin(\omega t + \phi) \text{ or } = A_1 e^{j(\omega t + \phi)}$$
$$x_2 = A_2 \sin(\omega t + \phi) \text{ or } = A_2 e^{j(\omega t + \phi)}$$

Response of 2DOF system (example-1)

EOM
$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Synchronous motion
$$x_1 = A_1 \sin(\omega t + \phi) \quad \text{or} \quad = A_1 e^{j(\omega t + \phi)}$$
$$x_2 = A_2 \sin(\omega t + \phi) \quad \text{or} \quad = A_2 e^{j(\omega t + \phi)}$$

Sub. into EOM
$$\begin{bmatrix} -\omega^2 m & 0 \\ 0 & -2\omega^2 m \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -\omega^2 \mathbf{M}\mathbf{x}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0}$$

$$\begin{bmatrix} 2k - \omega^2 m & -k \\ -k & 2k - 2\omega^2 m \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow (\mathbf{K} - \omega^2 \mathbf{M})\mathbf{x}(t) = \mathbf{0}$$

$$\begin{vmatrix} 2k - \omega^2 m & -k \\ -k & 2k - 2\omega^2 m \end{vmatrix} = 0 \Rightarrow \det(\mathbf{K} - \omega^2 \mathbf{M}) = 0$$

Characteristic equation (CHE)

Response of 2DOF system (example-2)

CHE $\begin{vmatrix} 2k-\omega^2 m & -k \\ -k & 2k-2\omega^2 m \end{vmatrix} = 0 \Rightarrow \omega^4 - \left(3\frac{k}{m}\right)\omega^2 + \frac{3}{2}\left(\frac{k}{m}\right)^2 = 0$

Solve the CHE $\omega_1 = \sqrt{0.634\frac{k}{m}}\ ;\ \omega_2 = \sqrt{2.366\frac{k}{m}}$

Natural frequencies of the system

From

$$\begin{bmatrix} 2k-\omega^2 m & -k \\ -k & 2k-2\omega^2 m \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \frac{A_1}{A_2} = \frac{k}{2k-\omega^2 m} = \frac{2k-2\omega^2 m}{k}$$

$\omega = \omega_1$

$$\left(\frac{A_1}{A_2}\right)^{(1)} = \frac{k}{2k-(0.634\frac{k}{m})m} = 0.731$$

$\omega = \omega_2$

$$\left(\frac{A_1}{A_2}\right)^{(2)} = \frac{k}{2k-(2.366\frac{k}{m})m} = -2.73$$

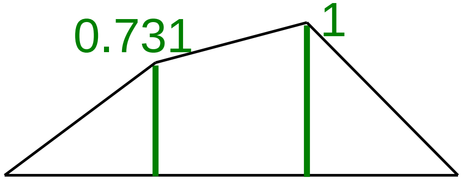
Response of 2DOF system (example-3)

$\omega = \omega_1$

Amp. ratio $\left(\dfrac{A_1}{A_2}\right)^{(1)} = 0.731$

The first mode shape

$$\phi_1(x) = \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix}$$



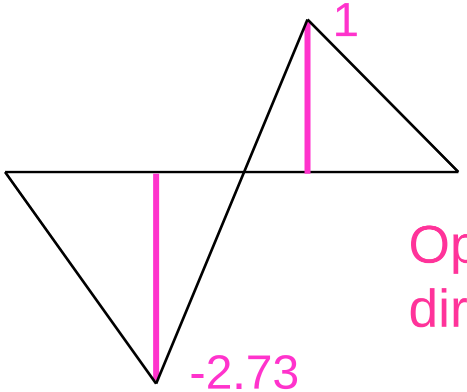
same direction

$\omega = \omega_2$

Amp. ratio $\left(\dfrac{A_1}{A_2}\right)^{(2)} = -2.73$

The second mode shape

$$\phi_2(x) = \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix}$$



Opposite direction

Response of 2DOF system (example-4)

In general, the free vibration contains both modes simultaneously (vibrate at both frequencies simultaneously)

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = c_1 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \sin(\omega_1 t + \psi_1) + c_2 \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \sin(\omega_2 t + \psi_2)$$

c_1, c_2, ψ_1, ψ_2 are constants (depended on initial conditions)

Initial conditions (1)

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = c_1 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \sin(\omega_1 t + \psi_1) + c_2 \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \sin(\omega_2 t + \psi_2)$$

Initial conditions $\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 2 \\ 4 \end{Bmatrix}$ and $\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Velocity response

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \omega_1 c_1 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \cos(\omega_1 t + \psi_1) + \omega_2 c_2 \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \cos(\omega_2 t + \psi_2)$$

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 2 \\ 4 \end{Bmatrix} \Rightarrow \begin{Bmatrix} 2 \\ 4 \end{Bmatrix} = c_1 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \sin\psi_1 + c_2 \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \sin\psi_2$$

$$\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \omega_1 c_1 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \cos\psi_1 + \omega_2 c_2 \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \cos\psi_2$$

Initial conditions (2)

$$\begin{Bmatrix} 2 \\ 4 \end{Bmatrix} = c_1 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \sin\psi_1 + c_2 \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \sin\psi_2$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \omega_1 c_1 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \cos\psi_1 + \omega_2 c_2 \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \cos\psi_2$$

4 Eqs.,
4 unknowns

Solve for four unknowns

$$c_1 = 3.732, \qquad c_2 = 0.268, \qquad \psi_1 = \psi_2 = \pi/2$$

The response is

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 3.732 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \sin(\omega_1 t + \frac{\pi}{2}) + 0.268 \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \sin(\omega_2 t + \frac{\pi}{2})$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2.732 \\ 3.732 \end{Bmatrix} \cos\omega_1 t + \begin{Bmatrix} -0.732 \\ 0.268 \end{Bmatrix} \cos\omega_2 t$$

Initial conditions (3)

Try to do

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = c_1 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \sin(\omega_1 t + \psi_1) + c_2 \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \sin(\omega_2 t + \psi_2)$$

(a) Initial conditions $\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 1.464 \\ 2 \end{Bmatrix}$ and $\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

(b) Initial conditions $\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix}$ and $\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Summary (Free-undamped) (1)

Direct Method

1 EOM $\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0}$

2 The motion is synchronous: constant ω and ϕ ⇒ $\mathbf{x} = \mathbf{A}\sin(\omega t + \phi)$ or $= \mathbf{A}e^{j(\omega t + \phi)}$

3 Eigen value problem

$$-\omega^2\mathbf{M}\mathbf{x}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0}$$

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{x}(t) = \mathbf{0}$$

4 Characteristics equation

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$$

ω_n^2 Eigen value

$\omega_{n1}, \omega_{n2}, \ldots, \omega_{nN}$ *N* natural freq.

5

$$(\mathbf{K} - \omega_{ni}^2\mathbf{M})\mathbf{x}_i = \mathbf{0}$$

$\mathbf{x}_i$ Eigen vector

$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ N mode shapes

Summary (Free-undamped) (2)

6 Free-undamped response

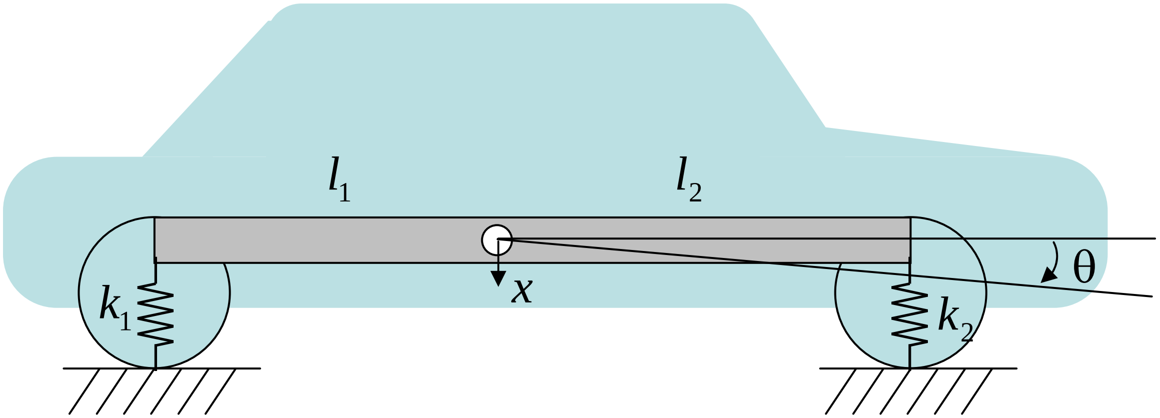
Direct Method

$$\mathbf{x}(t) = \mathbf{x}_1 A_1 \sin(\omega_1 t + \phi_1) + \mathbf{x}_2 A_2 \sin(\omega_2 t + \phi_2) + \ldots \mathbf{x}_N A_N \sin(\omega_N t + \phi_N)$$

$$\mathbf{x}(t) = \sum_{i=1}^{N} \mathbf{x}_i A_i \sin(\omega_i t + \phi_i)$$

where A and ϕ are from initial condition $\mathbf{x}(0)$ and $\mathbf{v}(0)$

Example



Determine the normal modes of vibration of an automobile simulated by simplified 2-dof system with the following numerical values

$$W = 3220 \text{ lb} \qquad J_C = \frac{W}{g} r^2 \qquad r = 4 \text{ ft}$$

$$l_1 = 4.5 \text{ ft} \qquad k_1 = 2400 \text{ lb/ft}$$

$$l_2 = 5.5 \text{ ft} \qquad k_2 = 2600 \text{ lb/ft}$$

Forced harmonic vibration (1)

Example

EOM
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} F_1 \\ 0 \end{bmatrix}\sin\omega t$$

System is undamped, the solution can be assumed as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\sin\omega t$$

Sub. into EOM
$$\begin{bmatrix} k_{11}-m_1\omega^2 & k_{12} \\ k_{21} & k_{22}-m_2\omega^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}=\begin{bmatrix} F_1 \\ 0 \end{bmatrix}$$

Simpler notation,
$$[Z(\omega)]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}=\begin{bmatrix} F_1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}=[Z(\omega)]^{-1}\begin{bmatrix} F_1 \\ 0 \end{bmatrix}$$

Forced harmonic vibration (2)

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = [Z(\omega)]^{-1} \begin{bmatrix} F_1 \\ 0 \end{bmatrix} = \frac{\text{adj}[Z(\omega)]}{|Z(\omega)|} \begin{bmatrix} F_1 \\ 0 \end{bmatrix}$$

Where $$|Z(\omega)| = m_1 m_2 (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)$$

ω_1 and ω_2 are natural frequencies

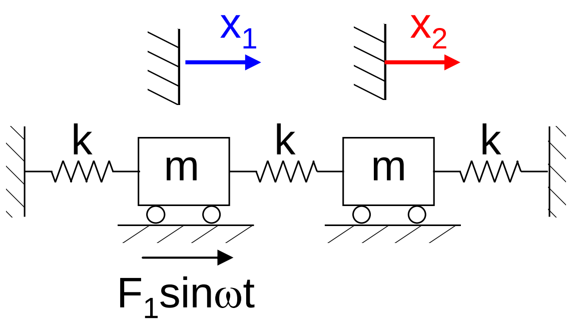
$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{|Z(\omega)|} \begin{bmatrix} k_{22} - m_2\omega^2 & -k_{12} \\ -k_{21} & k_{11} - m_1\omega^2 \end{bmatrix} \begin{bmatrix} F_1 \\ 0 \end{bmatrix}$$

The amplitudes are

$$X_1 = \frac{(k_{22} - m_2\omega^2)F_1}{m_1 m_2 (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

$$X_2 = \frac{-k_{21}F_1}{m_1 m_2 (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

Forced harmonic vibration (3)



EOM

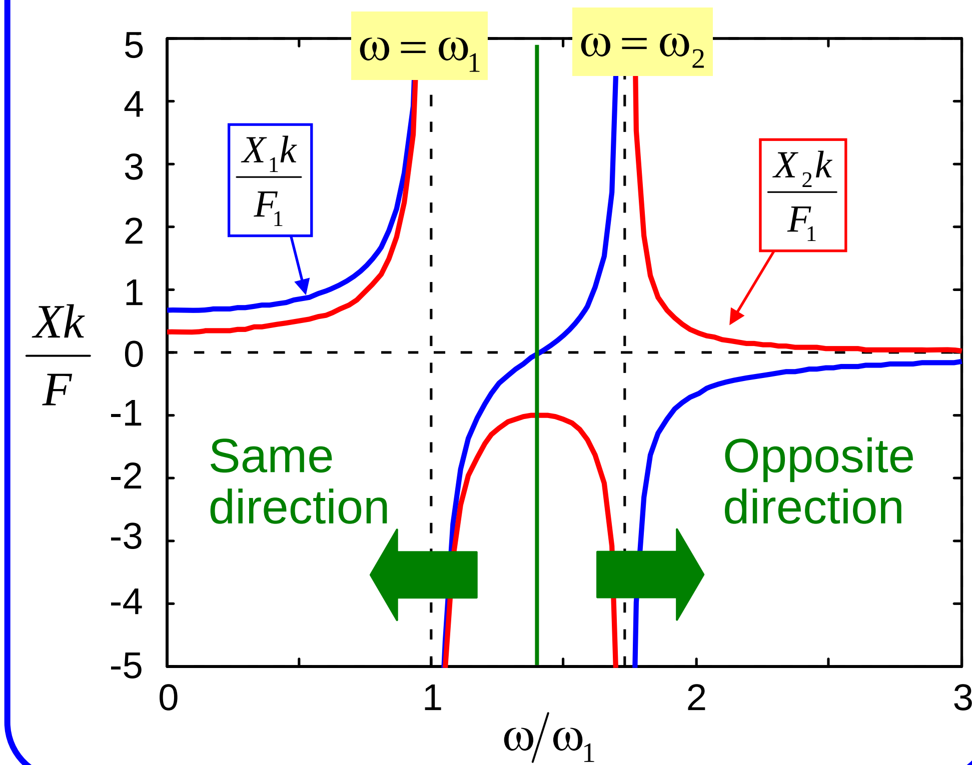
$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \sin\omega t$$

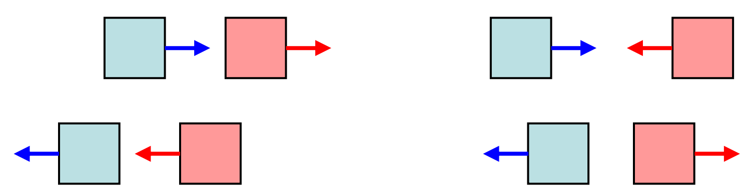
$$\omega_1 = \sqrt{\frac{k}{m}}, \qquad \omega_2 = \sqrt{\frac{3k}{m}}$$

$$X_1 = \frac{(2k - m\omega^2)F_1}{m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

$$X_2 = \frac{kF_1}{m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

Force response of a 2 DOF system





Solving methods

Solving Methods to be taught in this class:

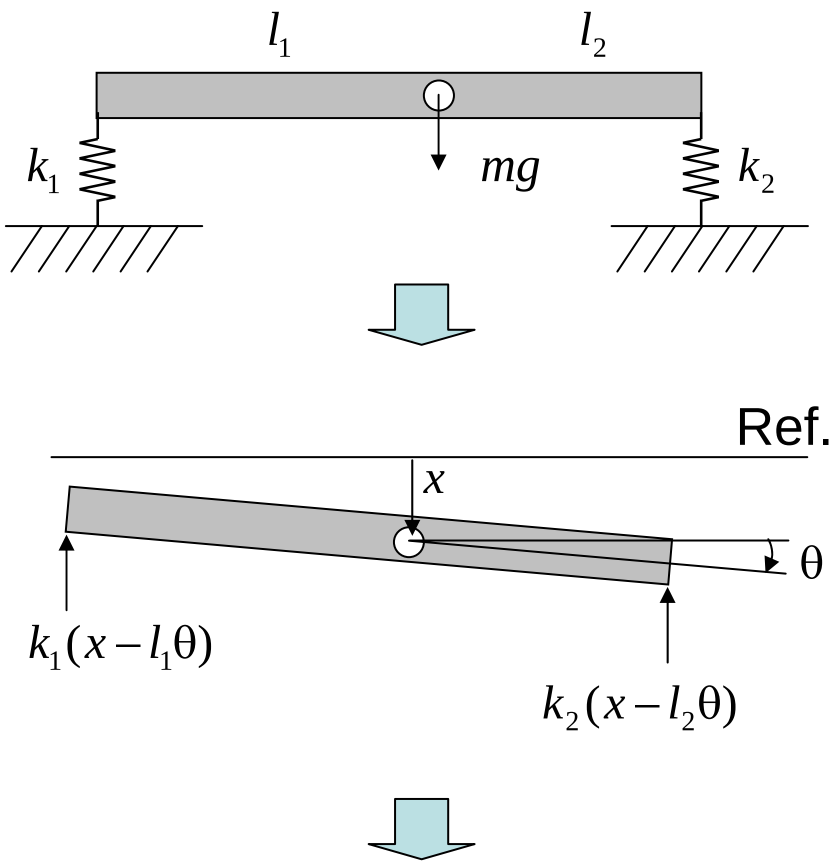
	Free Vibration	Forced Vibration
Undamped	1. Direct 2. Modal Analysis	1. Modal Analysis
Damped	1. Modal Analysis (only some systems)	1. Modal Analysis (only some systems)

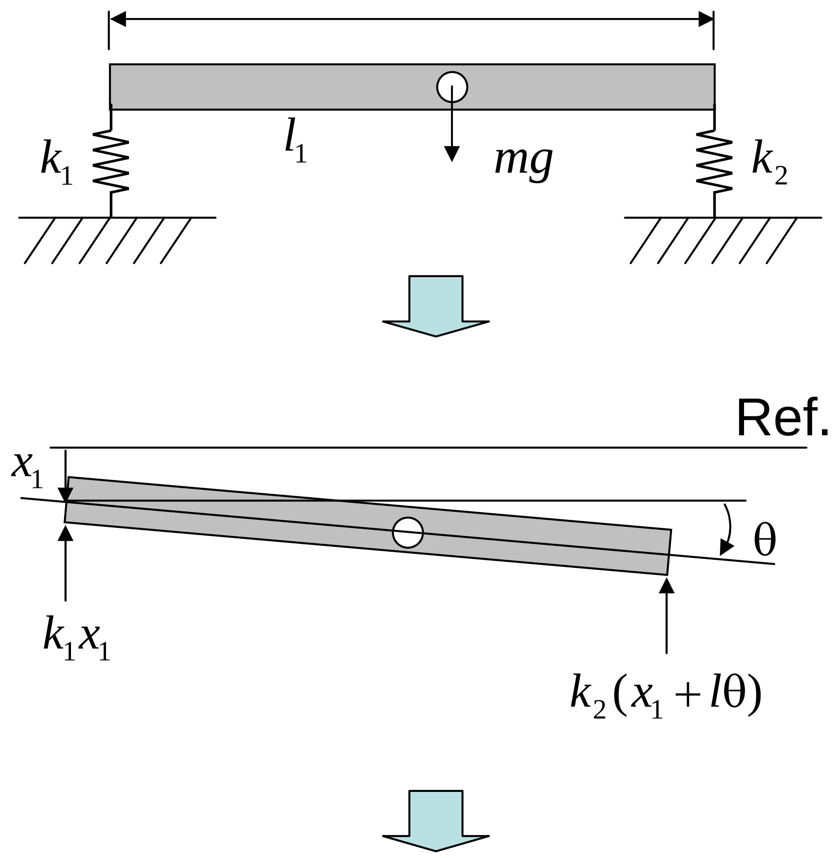
Modal analysis

Introduction

- is a method for solving for both transient and steady state responses of free and forced MDOF systems through analytical approaches.
- Uses the orthogonality property of the modes to "decouple" the EOM breaking EOM into independent SDOF equations, which can be solved for response separately.

Coordinate coupling





$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} m & m l_1 \\ m l_1 & J_1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l \\ k_2 l & k_2 l^2 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Concept of modal analysis

EOM in physical coordinate (Coordinates are coupled)

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} F(t) \\ 0 \end{bmatrix}$$

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t)$$

EOM in modal coordinate (Independent SDOF equations)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \end{bmatrix} + \begin{bmatrix} \omega_{n1}^2 & 0 \\ 0 & \omega_{n2}^2 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} N_1(t) \\ N_2(t) \end{bmatrix}$$

$$\ddot{\mathbf{r}}(t) + \mathbf{\Lambda}\mathbf{r}(t) = \mathbf{N}(t)$$

➡ Solve for $\mathbf{r}(t)$

Transform $\mathbf{r}(t)$ back to $\mathbf{x}(\mathrm{t})$

Orthogonality

$\mathbf{x}$ = eigen vector (vector of mode shape)

If $\mathbf{M}$ and $\mathbf{K}$ are symmetric and $\omega_{ni} \neq \omega_{nj}$ then $\mathbf{x}_i$ and $\mathbf{x}_j$ are said to be "orthogonal" to each other.

$$\mathbf{x}_j^T \mathbf{M} \mathbf{x}_i = 0, \qquad i \neq j$$

$$\mathbf{x}_i^T \mathbf{M} \mathbf{x}_i = M_{ii}$$

$$\mathbf{x}_j^T \mathbf{K} \mathbf{x}_i = 0, \qquad i \neq j$$

$$\mathbf{x}_i^T \mathbf{K} \mathbf{x}_i = K_{ii}$$

Normalization

$\mathbf{u}$ = normalized eigen vector (respect to mass matrix)

$$\mathbf{u}_j^T \mathbf{M} \mathbf{u}_i = 0, \qquad i \neq j$$

$$\mathbf{u}_i^T \mathbf{M} \mathbf{u}_i = 1$$

$$\mathbf{u}_i = C \mathbf{x}_i, \qquad C \text{ is constant}$$

From eigen value problem

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{x}(t) = \mathbf{0}$$

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{u}(t) = \mathbf{0}$$

or

$$\mathbf{K} \mathbf{u}_i = \omega_i^2 \mathbf{M} \mathbf{u}_i$$

$$\mathbf{u}_i^T \mathbf{K} \mathbf{u}_i = \omega_i^2 \mathbf{u}_i^T \mathbf{M} \mathbf{u}_i = \omega_i^2$$

Modal matrix

Modal matrix is the matrix that its columns are the mode shape of the system

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \dots \mathbf{u}_n \end{bmatrix}$$

Then

$$\mathbf{U}^T\mathbf{M}\mathbf{U} = \mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\mathbf{U}^T\mathbf{K}\mathbf{U} = \underbrace{\begin{bmatrix} \omega_{n1}^2 & 0 & \dots & 0 \\ 0 & \omega_{n2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega_{nN}^2 \end{bmatrix}}_{\mathbf{\Lambda}}$$

$\mathbf{\Lambda}$ (Spectral matrix)

Modal analysis (undamped systems)-1

Procedures

1. Draw FBD, apply Newton's law to obtain EOM → $\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{F}(t)$
2. Solve for natural frequencies through CHE → $\det(\mathbf{K}-\omega^2\mathbf{M})=0$
3. Determine mode shapes through EVP → $(\mathbf{K}-\omega^2\mathbf{M})\mathbf{x}(t)=\mathbf{0}$
4. Construct modal matrix (normalized)

$$
\begin{cases}
\mathbf{U}=[\mathbf{u}_1 \quad \mathbf{u}_2 \ldots \mathbf{u}_n] \\
\mathbf{U}^T\mathbf{M}\mathbf{U}=\mathbf{I} \\
\mathbf{U}^T\mathbf{K}\mathbf{U}=\boldsymbol{\Lambda}
\end{cases}
$$

5. Perform a coordinate transformation $\mathbf{x}(t)=\mathbf{U}\mathbf{r}(t)$

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{F}(t) \longrightarrow \mathbf{M}\mathbf{U}\ddot{\mathbf{r}}(t)+\mathbf{K}\mathbf{U}\mathbf{r}(t)=\mathbf{F}(t)$$

$$\mathbf{U}^T\mathbf{M}\mathbf{U}\ddot{\mathbf{r}}(t)+\mathbf{U}^T\mathbf{K}\mathbf{U}\mathbf{r}(t)=\mathbf{U}^T\mathbf{F}(t)$$

$$\ddot{\mathbf{r}}(t)+\boldsymbol{\Lambda}\mathbf{r}(t)=\mathbf{U}^T\mathbf{F}(t)$$

Modal analysis (undamped systems)-2

$$\ddot{\mathbf{r}}(t) + \boldsymbol{\Lambda}\mathbf{r}(t) = \mathbf{U}^T \mathbf{F}(t)$$

$$\begin{bmatrix} \ddot{r}_1(t) \\ \ddot{r}_2(t) \\ \vdots \\ \ddot{r}_N(t) \end{bmatrix} + \begin{bmatrix} \omega_{n1}^2 & 0 & \dots & 0 \\ 0 & \omega_{n2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega_{nN}^2 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_N(t) \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1N} \\ u_{21} & u_{22} & \dots & u_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{N1} & u_{N2} & \dots & u_{NN} \end{bmatrix}^T \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \\ F_4(t) \end{bmatrix} = \begin{bmatrix} N_1(t) \\ N_2(t) \\ N_3(t) \\ N_4(t) \end{bmatrix}$$

→ Independent SDOF equations, can be solve for $\mathbf{r}(t)$

6. Transform the initial conditions to modal coordinates

From $\mathbf{x}(t) = \mathbf{U}\mathbf{r}(t)$ and $\mathbf{U}^T\mathbf{M}\mathbf{U} = \mathbf{I}$ }

$$\mathbf{x}(0) = \mathbf{U}\mathbf{r}(0)$$

$$\mathbf{U}^T\mathbf{M}\mathbf{x}(0) = \mathbf{U}^T\mathbf{M}\mathbf{U}\mathbf{r}(0)$$

$$\mathbf{r}(0) = \mathbf{U}^T\mathbf{M}\mathbf{x}(0)$$

and

$$\dot{\mathbf{r}}(0) = \mathbf{U}^T\mathbf{M}\dot{\mathbf{x}}(0)$$

Modal analysis (undamped systems)-3

7. Find the response in modal coordinates → $\mathbf{r}(t) = \ldots$
8. Transform the response in modal coordinate $\mathbf{r}(t)$ back to that in original coordinate $\mathbf{x}(t)$ → $\mathbf{x}(t) = \mathbf{U}\mathbf{r}(t)$

Example (Modal analysis) -1

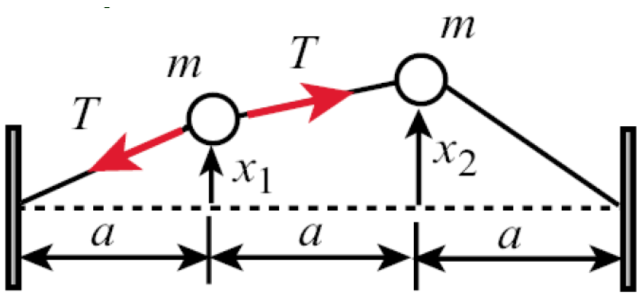
EOM

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Initial conditions $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Example (Modal analysis) -2

2dof string-bead system



EOM

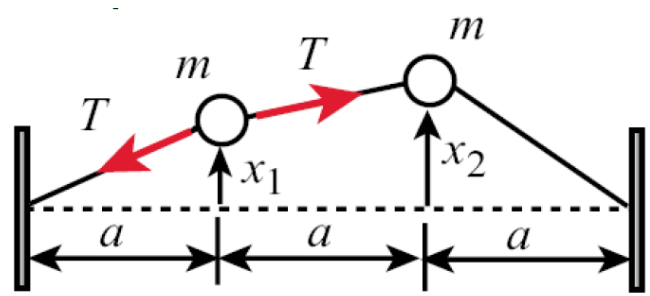
$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \frac{T}{a} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

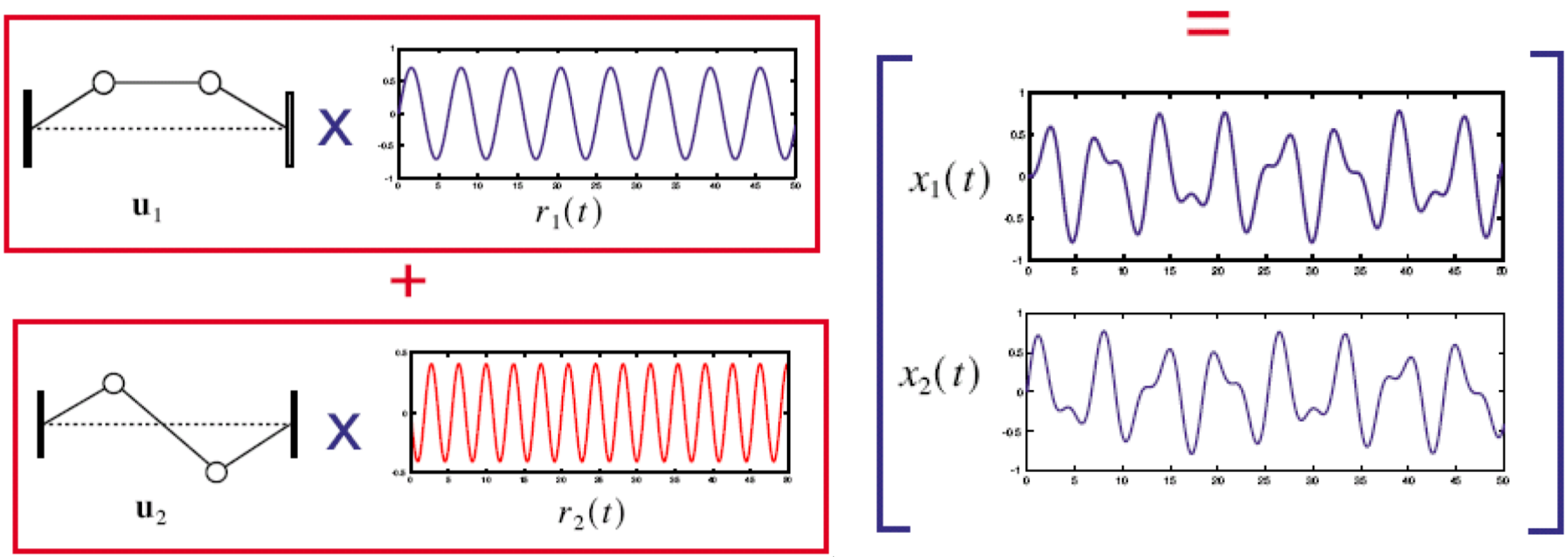
Initial conditions $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Example (Modal analysis) -2_2

$$x_1(t) = \frac{1}{2\omega_{n1}} \sin(\omega_{n1} t) - \frac{1}{2\omega_{n2}} \sin(\omega_{n2} t)$$

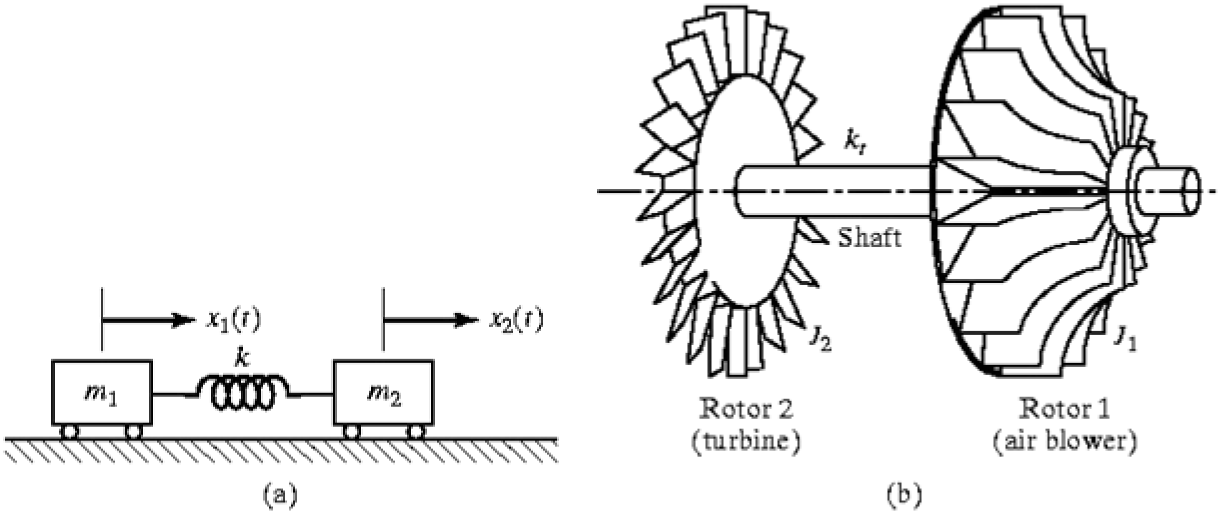
$$x_2(t) = \frac{1}{2\omega_{n1}} \sin(\omega_{n1} t) + \frac{1}{2\omega_{n2}} \sin(\omega_{n2} t)$$



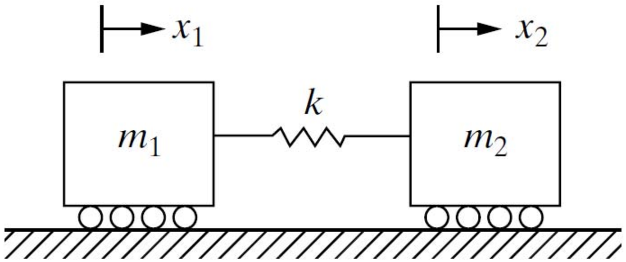


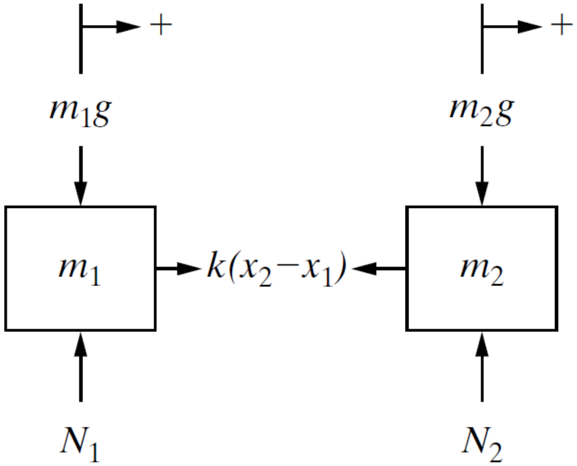
Rigid body mode

- Rigid body mode is the mode that the system moves as a rigid body.
- The system moves as a whole without any relative motion among masses.
- There is no oscillation. $\omega_n = 0$



Rigid-body modes





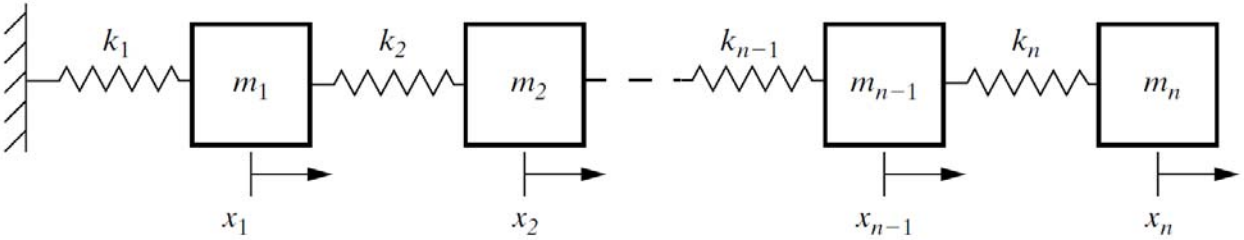
Compute the solution of the system.

Let m_1 = 1 kg, m_2 = 4 kg and k = 400 N/m.

Initial condition $x_0 = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}$ $v_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

More than two degrees of freedom

Calculate the solution of the n-degree-of-freedom system in the figure for $n = 3$ by modal analysis. Use the values $m_1 = m_2 = m_3 = 4$ kg and $k_1 = k_2 = k_4 = 4$ N/m, and the initial condition $x_1(0) = 1$ m with all other initial displacements and velocities zero.



Modal analysis on damped systems (1)

EOM $\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t)$

The original modal analysis can be applied to MDOF <u>damped</u> system if and only if

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}$$ ⟵ Necessary and sufficient condition

Such system is called "classically damped".

However, there are subsets of the above systems where C can be written as a linear combination of M and K.

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$$ ⟵ Sufficient but not necessary condition

α and β are constants. Such system is called "proportionally damped."

Modal analysis on damped systems (2)

For proportionally damped

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t)$$

$$\mathbf{M}\ddot{\mathbf{x}}(t) + (\alpha\mathbf{M} + \beta\mathbf{K})\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t)$$

Let $\mathbf{x}(t) = \mathbf{U}\mathbf{r}(t)$

$$\mathbf{M}\mathbf{U}\ddot{\mathbf{r}}(t) + (\alpha\mathbf{M} + \beta\mathbf{K})\mathbf{U}\dot{\mathbf{r}}(t) + \mathbf{K}\mathbf{U}\mathbf{r}(t) = \mathbf{F}(t)$$

Premultiply by $\mathbf{U}^{\mathrm{T}}$

$$\mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U}\ddot{\mathbf{r}}(t) + \mathbf{U}^{\mathrm{T}}(\alpha\mathbf{M} + \beta\mathbf{K})\mathbf{U}\dot{\mathbf{r}}(t) + \mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U}\mathbf{r}(t) = \mathbf{U}^{\mathrm{T}}\mathbf{F}(t)$$

$$\ddot{\mathbf{r}}(t) + (\alpha\mathbf{I} + \beta\mathbf{\Lambda})\dot{\mathbf{r}}(t) + \mathbf{\Lambda}\mathbf{r}(t) = \mathbf{U}^{\mathrm{T}}\mathbf{F}(t) = \mathbf{N}(t)$$

Thus, the system when it is written in modal coordinates r(*t*) can be decoupled into *k* sets of SDOF equations

$$\ddot{r}_1(t) + 2\zeta\omega_{n1}\dot{r}_1(t) + \omega_{n1}^2 r_1(t) = N_1(t)$$

$$\vdots$$

where $2\zeta\omega_{n1} = \alpha + \beta\omega_{n1}^2$

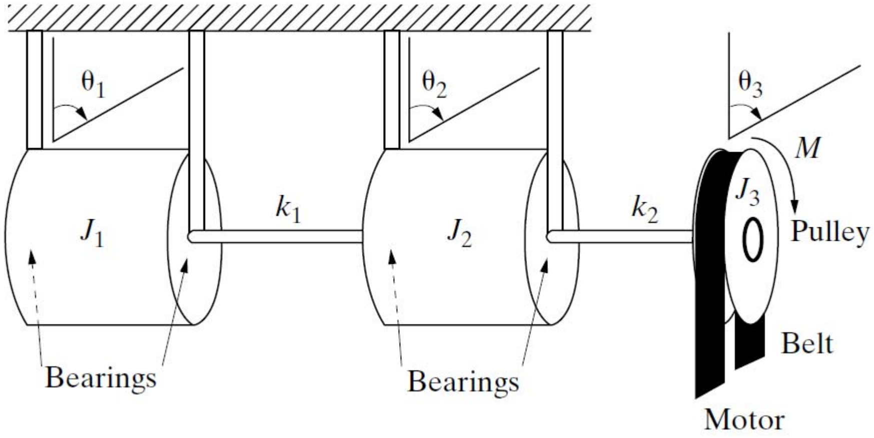
Modal analysis on damped systems (Ex.)

A belt-driven lathe
•bearings are modeled as providing viscous damping
•shafts provide stiffness
•belt drive provides and applied torque.

$$J_1 = J_2 = J_3 = 10 \text{ kg.m}^2\text{/rad}$$

$$k_1 = k_2 = 10^3 \text{ N.m/rad}$$

$$c = 2 \text{ N.m.s/rad}$$



- Zero initial conditions
- Applied moment $M(t)$ is a unit impulse function

Modal analysis on damped systems (Ex.)