The Kosterlitz-Thouless Transition

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1 Introduction

Matter in the universe is organised in a hierarchical structure. At the bottom we have elementary particles: quarks, gluons, electrons etc. At present we don't know what these particles are made off. But we believe we know that quarks, gluons etc. make up protons, neutrons, etc. These then go together to make atoms, that are the building blocks of molecules. Atoms and molecules gives us gas, liquids and solids from which we get stars and planets which are grouped together in galaxies, that then form clusters and eventually we arrive at the entire universe. Or from atoms and molecules we get macro molecules like proteins and DNA, that are the building blocks of organelles, which together form the cells. From cells we get organs, that put together form organisms: animals and plants of a great variety of species. The totality of individuals and species constitute the entire ecology.

One branch of science is concerned with the breaking-up of systems into smaller and smaller parts. The behaviour and properties are studied at each respective level. Statistical Mechanics is concerned with the opposite quest. Namely, from the interactions between the components, say atoms, at one given level the aim is to understand the collective coherent behaviour which emerges as many atoms are but together and the next level if formed. Often the microscopic details of the properties of the *individual* building blocks are not so crucial. Rather it happens that the collective behaviour is controlled by general properties of the *interaction* between the building "atoms".

In these lectures we shall discuss a particular case, where it is possible to follow in detail, how components at one level go together and form certain collective coherent structures: topological defects or topological charges. These charges can be Coulomb charges in two dimensions, dislocations in two dimensional crystals, vortices in two dimensional superconductors and more. The interaction between the topological charges depends in all cases logarithmically on the spatial separation and this leads to some very general collective behaviour, most spectacular it causes a certain type of phase transition: the Kosterlitz-Thouless transition [1].

2 The Two Dimensional XY-Model

We will use the 2d XY-model as our reference model. The model consists of planar rotors of unit length arranged on a two dimensional square lattice. The Hamilatonian of the system is given by

$$H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j). \tag{1}$$

Here $\langle i, j \rangle$ denotes summation over all nearest neighbour sites in the lattice, and θ_i denotes the angle of the rotor on site i with respect to some (arbitrary) polar direction in the two dimensional vector space containing the rotors.

If we assume that the direction of the rotors varies smoothly from site to site, we can approximate $\cos(\theta_i - \theta_j)$ by the first two terms $1 - \frac{1}{2}(\theta_i - \theta_j)^2$ in the Taylor expansion of cos. The sum over the nearest neighbours corresponds to the discrete Laplace operator, which we can express in terms of partial derivatives through $\theta_i - \theta_j = \partial_x \theta$ for two site i and j which differs by one lattice spacing in the x-direction. This leads to the continuum Hamiltonian

$$H = E_0 + \frac{J}{2} \int d\mathbf{r} (\nabla \theta)^2. \tag{2}$$

Here $E_0 = 2JN$ is the energy of the completely aligned ground state of N rotors.

The thermodynamics of the system is obtained from the partition function

$$Z = e^{-\beta E_0} \int D[\theta] \exp\{-\beta \frac{J}{2} \int d\mathbf{r} (\nabla \theta)^2\}, \tag{3}$$

a functional integral over all possible configurations of the director field $\theta(\mathbf{r})$. The integral over $\theta(\mathbf{r})$ can be divided into a sum over the local minima θ_{vor} of $H[\theta]$ plus fluctuations θ_{sw} around the minima

$$Z = e^{-\beta E_0} \sum_{\theta_{vor}} \int D[\theta_{sw}] \exp\{-\beta(H[\theta_{vor}] + \frac{1}{2} \int d\mathbf{r}_1 \int d\mathbf{r}_2 \theta_{sw}(\mathbf{r}_1) \frac{\delta^2 H}{\delta\theta(\mathbf{r}_1)\delta\theta(\mathbf{r}_2)} \theta_{sw}(\mathbf{r}_2))\}. \tag{4}$$

The field configurations corresponding to local minima of H are solutions to the extremal condition

$$\frac{\delta H}{\delta \theta(\mathbf{r})} = 0 \;\Rightarrow\; \nabla^2 \theta(\mathbf{r}) = 0. \tag{5}$$

There are two types of solutions to this equation. The first consists of the ground state $\theta(\mathbf{r}) = constant$. The second type of solutions consist of vortices in the director field (see Fig. 1) and are obtained by imposing the following set of boundary conditions on the circulation integral of $\theta(\mathbf{r})$:

1) For all closed curves encircling the position $\mathbf{r}_0$ of the centre of the vortex

$$\oint \nabla \theta(\mathbf{r}) \cdot d\mathbf{l} = 2\pi n. \tag{6}$$

2) For all paths that don't encircle the vortex position $\mathbf{r}_0$

$$\oint \nabla \theta(\mathbf{r}) \cdot d\mathbf{l} = 0. \tag{7}$$

Condition 1) imposes a singularity in the director field. Note the circulation integral *must* be equal to an integer times 2π since we circle a closed path and therefore $\theta(\mathbf{r})$ has to point in the same direction after traversing the path as it did when we started.

We can estimate the energy of a vortex in the following way. The problem is spherical symmetric, hence the vortex field θ_{vor} must be of the form $\theta(\mathbf{r}) = \theta(r)$. The dependence on r can be found from Eq. 6. We calculate the circulation integral along a circle of radius r centred at the position $\mathbf{r}_0$ of the vortex

$$2\pi n = \oint \nabla\theta(\mathbf{r}) \cdot d\mathbf{l} = 2\pi r|\nabla\theta|. \tag{8}$$

We solve and obtain $|\nabla\theta(r)| = n/r$. Substitute this result into the Hamiltonian Eq. 2

$$\begin{aligned} E_{vor} - E_0 &= \frac{J}{2}\int d\mathbf{r}[\nabla\theta(\mathbf{r})]^2 && (9)\\ &= \frac{Jn^2}{2}\int_0^{2\pi}\int_a^L r dr \frac{1}{r^2} && (10)\\ &= \pi n^2 J \ln(\frac{L}{a}). && (11)\end{aligned}$$

The circulation condition Eq. 6 creates a distortion in the phase field $\theta(\mathbf{r})$ that persists infinitely far from the centre of the vortex. $|\nabla\theta|$ decays only as $1/r$ leading to a logarithmic divergence of the energy. Hence we need to take into account that the integral over r in Eq. 10 is cut-off for large r-values by the finite system size L and for small r-values by the lattice spacing a. We recall that our continuum Hamiltonian is an approximation to the lattice Hamiltonian in Eq. 1. A vortex with the factor n in Eq. 6 larger than one is called multiple charged. We notice that the energy of the vortex is quadratic in the charge. In an macroscopically large system even the energy of a single charge vortex will be large.

Consider now a pair of single charged vortex and an anti-vortex. When we encircle the vortex we pick up $\oint d\mathbf{l}\cdot\nabla\theta = 2\pi$ and when we encircle the anti-vortex we pick up $\oint d\mathbf{l}\cdot\nabla\theta = -2\pi$. Hence, if we choose a path large enough to enclose both vortices we pick up a circulation of the phase equal to $2\pi + (-2\pi) = 0$. I.e. the distortion of the phase field $\theta(\mathbf{r})$ from the vortex–anti-vortex pair is able to cancel out at distances from the centre of the two vortices large compared to the separation R between the vortex and the anti-vortex, see Fig. 2. This explains why the energy of the vortex pair is of the form

$$E_{2vor}(R) = 2E_c + E_1 \ln(R/a). \tag{12}$$

Where E_c is the energy of the vortex cores and E_1 is proportional to J. In detail, the phase field $\theta_{2vor}(\mathbf{r})$ of a vortex located at $\mathbf{r} = (-a, 0)$ and an anti-vortex located at $\mathbf{r} = (a, 0)$ is given by

$$\theta_{2vor}(\mathbf{r}) = \text{arctg}\left(\frac{2ay}{a^2 - r^2}\right). \tag{13}$$

3 Lack of Ordering in Two Dimensions

In order to highlight the peculiarity of two dimensions we consider the d-dimensional XY-model. We imagine a d-dimensional cubic lattice. Each lattice site contains a planar rotor or a phase. In the continuum limit the Hamiltonian is still given by Eq. 2 except the integral over $\mathbf{r}$ is now a d-dimensional integral and therefore the factor J is replaced by Ja^{2-d}. The average size of the projection of the rotors along the x-direction in $\mathbf{S}$ space, i.e. the magnetisation, is

$$\begin{aligned}\langle S_x\rangle &= \langle\cos\theta(\mathbf{r})\rangle && (14)\\ &= \langle\cos\theta(0)\rangle. && (15)\end{aligned}$$

We neglect the singular vortex contributions (which is perfectly safe at low temperature) and Fourier transform the phase field

$$\begin{aligned}\theta(\mathbf{r}) &= \int\frac{d\mathbf{k}}{(2\pi)^d}\hat{\theta}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}} && (16)\\ \theta(0) &= \int\frac{d\mathbf{k}}{(2\pi)^d}\hat{\theta}(\mathbf{k}) && (17)\\ \int d\mathbf{r}(\nabla\theta)^2 &= \int\frac{d\mathbf{k}}{(2\pi)^d}k^2\theta(\hat{\mathbf{k}})\theta(\hat{-\mathbf{k}}). && (18)\end{aligned}$$

These eqs. are substituted into the expression

$$\langle S_x\rangle = \frac{\int D[\theta]\cos(\theta(0))e^{-\beta H}}{\int D[\theta]e^{-\beta H}} = Re\left(\int D[\theta]\cos(\theta(0))e^{-\beta H+i\theta(0)}/Z\right). \tag{19}$$

After some algebra one obtains the following expression

$$\langle S_x\rangle = \exp\left(-\frac{T}{2Ja^{2-d}}\mathcal{S}_d\int_{\pi/L}^{\pi/a}dkk^{d-3}\right). \tag{20}$$

The behaviour of $\langle S_x\rangle$ is controlled by the integral

$$I(L) = \int_{\pi/L}^{\pi/a}dkk^{d-3}. \tag{21}$$

The behaviour of $I(L)$ strongly depends on the dimension d. For $d<2$ we have $I(L)\sim L^{2-d}\to\infty$ as $\to\infty$. Hence, $\langle S_x\rangle = 0$ in the limit of large systems for dimensions less than 2. For $d>2$ we have that

$$I(L)\to A = \frac{1}{d-2}\left(\frac{\pi}{a}\right)^{d-2} \tag{22}$$

and therefore

$$\langle S_x\rangle = \exp(-\frac{\mathcal{S}_d}{2Ja^{2-d}}AT) > 0. \tag{23}$$

Finally for $d=2$ the integral $I(L)$ is logarithmically divergent $I(L)=\ln(L/a)$ which is sufficient to force $\langle S_x\rangle$ to zero for any non-zero temperature.

We conclude that there is no ordered low temperature phase for $d\leq 2$. For $d<2$ this means that there is no phase transition. The situation is different for $d=2$. Although for any non-zero temperature the rotors are unable to order along a common direction the vortices are able to induce a phase transition. Though a transition of a special type. One for which there is no local order parameter, there is no magnetization that goes to zero at a critical temperature. This is the Kosterlitz-Thouless transition which we shall return to in a little while. First we want to investigate the correlations in the XY-model for different dimensions. We will again neglect vortex excitations and repeat the calculation we did for $\langle S_x \rangle$. This time we calculate the correlation function $\langle \mathbf{S}(\mathbf{r})\mathbf{S}(0)\rangle$ and obtain

$$\begin{aligned}\langle \mathbf{S}(\mathbf{r})\mathbf{S}(0)\rangle &= \langle \cos(\theta(\mathbf{r})\theta(0)\rangle && (24)\\ &= Re\langle \exp(i(\theta(\mathbf{r}) - \theta(0))\rangle && (25)\\ &= \exp[g(r)]. && (26)\end{aligned}$$

Where the integral

$$g(r) = Tja^{2-d}\int \frac{d\mathbf{k}}{(2\pi)^d}\frac{1 - e^{-i\mathbf{k}\cdot\mathbf{r}}}{k^2} \tag{27}$$

behaves asymptotically for $|r| \to \infty$ in the following way

$$g(r) \simeq \begin{cases} \frac{\mathcal{S}_d}{d-2}\left(\frac{\pi}{L}\right)^{d-2} & \text{for } d > 2 \\ \frac{1}{2\pi}\ln(r/L) & \text{for } d = 2 \\ r/2 & \text{for } d = 1 \end{cases} \tag{28}$$

From this it follows that the long distance behaviour of the correlation function is

$$\langle \mathbf{S}(\mathbf{r})\mathbf{S}(0)\rangle \simeq \begin{cases} e^{-\text{const.}T} & \text{for } d > 2 \\ \left(\frac{r}{L}\right)^{-\eta} & \text{for } d = 2 \\ \exp(-\frac{T}{2Ja}r) & \text{for } d = 1. \end{cases} \tag{29}$$

We notice that for $d > 2$ the correlations survive, $\langle \mathbf{S}(\mathbf{r})\mathbf{S}(0)\rangle$ decays to a non-zero constant as $r \to \infty$. This indicates long range order, a certain degree of alignment of the rotors, i.e. the model posses an ordered phase at low temperature. In $d = 1$ the correlation function decays to zero exponentially over a correlation length $\xi = 2Ta/T$ that diverges in the limit of $T \to 0$.

The situation is very different in two dimensions. Here the correlation function depends algebraically on r with an exponent $\eta = T/2\pi J$ that continuously changes with temperature. Algebraic decay of the correlation function is what we expect when the temperature is tuned to the critical temperature of a continuous phase transition. In the 2d XY-model we find critical algebraic correlations for all temperatures for which our calculation is valid. In the calculation we have neglected vortices, so we expect our results to break down when the temperature becomes high enough to excite vortex pairs.

We conclude, that although in two dimensions there is no long range order with a non-zero value of $\langle S_x \rangle$ for any temperature above zero, the correlations of the two dimensional model are algebraic. This is the usual case, say in Ising systems, precisely at the critical temperature where the correlations are algebraic and the order parameter is still zero, though it will become non-zero if the temperature is lowered and infinitesimal amount.

4 Vortex Unbinding

We mentioned at the end of the previous section that we expect vortices to become important as the temperature is increased. To see this we estimate the free energy of a single vortex.

The Helmholtz free energy is given by the difference between the energy and the entropy multiplied by the temperature $F = E - TS$. The energy is given by Eq. 11. We estimate the entropy from the number of places where we can position the vortex centre, namely on each of the L^2 plaquette of the square lattice, i.e., $S = k_B \ln(L^2/a^2)$. Accordingly the free energy is given by

$$F = E_0 + (\pi J - 2k_B T)\ln(L/a). \tag{30}$$

For $T < \pi J/2k_B$ the free energy will diverge to plus infinity as $L \to \infty$. At temperatures $T > \pi J/2k_B$ the system can lower its free energy by producing vortices: $F \to -\infty$ as $L \to \infty$. This simple heuristic argument points to the fact that the logarithmic dependence on system size of the energy of the vortex combines with the logarithmic dependence of the entropy to produce the subtleties of the vortex unbinding transition. Assume a different dependence of the energy on systems size and one will either have thermal activation of vortices at all temperatures (in case $E_{vor} \to const. < \infty$) or vortices will not be activated at any temperature (in case $E_{vor} \sim (L/a)^b$ with $b > 0$). It is the logarithmic size dependence of the 2d vortex energy that allows the outcome of the competition between the entropy and the energy to change qualitatively at a certain finite temperature T_{KT}.

In reality it is not single vortices of the same sign that proliferates at a certain temperature. What happens is that the larger vortex pairs which are bound together for temperatures below T_{KT} unbind at T_{KT}. This is a collective effect. The vortex pairs induced as one approaches T_{KT} disturbs the phase field so much that the effective value of the vortex binding term E_1 in the vortex pair free energy [1] is driven to zero for large vortex separations. In the next section we shall see in detail how this happens, but preliminary insight can be obtained from the following

Exercise: Use the expression in Eq. 12 for the energy of a vortex pair to calculate, as function of temperature, the average separation $\langle R \rangle$ of a vortex pair.

4.1 The Spin Wave Stiffness

The effect of the thermally activated vortex pairs is describe by the temperature dependent *spin wave stiffness* ρ_s^R. This is an example of what Philip W Anderson calls a generalised rigidity[2]. The spin wave stiffness describes how much free energy it costs to apply a twist, or gradient, to the rotors (also called spins):

$$\theta(\mathbf{r}) = \theta_0(\mathbf{r}) + \mathbf{v}_{ex} \cdot \mathbf{r}, \tag{31}$$

here $\theta_0(\mathbf{r})$ is allowed to vary according to the canonical ensemble. The increase in the free energy is given by

$$F(\mathbf{v}_{ex}) - F(0) = \frac{1}{2} V \rho_s^R v_{ex}^2. \tag{32}$$

An number of comments concerning the notation are illuminating. The notation $\mathbf{v}_{ex}$ for the gradient applied to the phase field $\theta(\mathbf{r})$ has its origin in the fact that the same physics, as we describes here, applies to superfluid films and superconducting films. In these cases the

[1] That is Eq. 12 generalised to non-zero temperature

field $\theta(\mathbf{r})$ is the phase of the complex order parameter, the wave function of the super-fluid. Being the phase of a quantum mechanical wave function the gradient of $\theta(\mathbf{r})$ is related to a probability current and thereby to the velocity field of the super-fluid. The notation ρ_s^R is meant to remind one that this phase rigidity, is determined by the **density** of superfluid in the case of a superfluid or a superconductor. The superscript R in ρ_s^R indicates that thermal excitations renormalise the quantity. It follows immediately from the Hamiltonian in Eq. 2 that at zero temperature $\rho_s^R = J = \rho_s$. The spin wave stiffness is similar to the shear constant of a material. The shear constant determines how the (free) energy increase of a shear deformation. As temperature is increased the shear constant decreases and drops abruptly to zero when the solid melts into a liquid.

To obtain ρ_s^R one calculates the left hand side of Eq. 32. Details can be found in the wonderful book by Chaikin and Lubensky[3]. The phase field is split into two parts

$$\theta_0(\mathbf{r}) = \theta_s(\mathbf{r}) + \theta_v(\mathbf{r}), \tag{33}$$

where the first term describes smooth spin waves and the second term contains the singular vortex contribution. The free energy is obtained from $F = k_b T \ln Z$ and the partition function in Eq. 3 by introducing Fourier transforms of the phase field. After quite a bit of algebra one arrives at the following simple expression

$$\rho_s^R = \rho_s - \frac{1}{2}\frac{\rho_s^2}{T}\lim_{k\to 0}\frac{\langle \hat{n}(\mathbf{k})\hat{n}(-\mathbf{k})\rangle_)}{k^2}. \tag{34}$$

which expresses the renormalized stiffness in terms of the correlation function of the Fourier transform of the vortex density function

$$n(\mathbf{r}) = \sum_\alpha n_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha), \tag{35}$$

for a collection of vortices of charge n_α (see Eq. 6) with centres located at positions $\mathbf{r}_\alpha$. The thermodynamic average is performed over the canonical ensemble with no twist imposed, hence the subscript 0. Eq. 34 can be used to determine how the spin wave stiffness behave at large distances as a function of temperature. We will discuss how in the next section.

Exercise: A very enlightening and stimulating activity for a quiet afternoon is to go thorough the details leading to Eq. 34. The simplest way to do this is to study Chaikin and Lubensky's book [3], but it is also strongly recommendable to dig out the original papers by Kosterlitz and Thouless [1, 7].

4.2 The KT transition

Let us first summarise the phenomenology of the Kosterlitz-Thouless transition. As the temperature is increased more and more vortex pairs are thermally activated. This makes ρ_s^R decrease, see Eq. 34. This corresponds to a decrease in the increment of the free energy induced by a certain twist $\mathbf{v}_{ex}$. We can understand this from the fact that the phase field $\theta(\mathbf{r})$ becomes more and more distorted as the temperature is increased, hence the extra perturbation caused by $\mathbf{v}_{ex}$ becomes relatively less important. Quantitatively one finds

$$\rho_s^R = \begin{cases} \rho_s^R(T_{KT}^-)[1 + const.(T_{KT} - T)^{1/2}] & \text{for } T < T_{KT} \\ 0 & \text{for } T > T_{KT}\ . \end{cases} \tag{36}$$

Here, T_{KT} is the Kosterlitz-Thouless temperature at which vortex pairs unbind. The value of T_{KT} differ from one system to another. In the 2d XY-model $T_{KT}/J \simeq 0.893 \pm 0.002$ [4]. The remarkable thing is, as we shall see below, that the ratio

$$\rho_s^R(T_{KT}^-)/T_{KT} = 2/\pi \tag{37}$$

is universal for all systems that undergoes a KT-transition. Since $\rho_s^R(T_{KT}^+) = 0$ Eq. 37 is referred to as the universal jump. The correlation length $\xi(T)$ behaves in a very unusual way as one approaches T_{KT} from above. We are used to a relatively slow algebraic divergence of the correlation length as the critical temperature is approached. For the KT-transition the divergence is, however, much faster

$$\xi(T) \sim \exp\left(\frac{const.}{(T - T_{KT})^{1/2}}\right) \text{for } T > T_{KT}. \tag{38}$$

Can we in a simple way understand this exponential divergence. Yes, we can. The phase field is significantly distorted by unbound vortices, since these vortices are not screened by a nearby anti-vortex. I.e. the phases $\theta(\mathbf{r})$ can remain correlated over distances shorter than the typical distance $\langle D\rangle = 1/\sqrt{n_{ub}}$ between unbound vortices of density n_{ub} [5]. Or in other words, we expect the correlation length $\xi \sim D$. The vortices are thermally induced and therefor their density is expected to depend on the temperature through a Boltzmann factor $\exp(-E_{vor}/T)$.[2] This argument can indicate the cause of the exponential dependence of ξ. But it is no more than an indication since the exponential dependence in Eq. 38 is significantly different from a simple Boltmann factor. This difference is due to corrective renormalization effects.

Continuous phase transitions are accompanied by divergences in thermodynamic quantities caused by the divergence of the correlation length as the critical temperature T_c is approached. The singular part of the free energy density f can be estimated as the amount of thermal energy T_c within a correlated volume ξ^d of $f \sim T_c/\xi^d$. The specific heat c_V is given by the second derivative of the free energy $c_V = -T\partial^2 f/\partial T^2 \sim \partial^2\xi^{-d}\partial T^2$. For the KT-transition the exponential divergence of $\xi(T)$ in Eq. 38 is so rapid and occur over such a narrow temperature range that the divergence in c_V cannot be resolved in simulations or in experiment. The measured c_V is sketched in the Fig. 3 and is smooth through T_{KT} with a broad peak above T_{KT} induced by the entropy released by the unbounding of the vortices (see [3]).

4.3 The Green's function for the vortex-vortex interaction

The behaviour describe in the previous subsection is obtained by a real space renormalization procedure first devised by Kosterlitz [7]. A detailed and readable presentation of this calculation can be found in Chaikin and Lubensky's book [3]. Here we only briefly mention the main ingredients of this calculation and leave it as an

[2] The situation described here is exactly what happens in the one dimensional so-called ϕ^4 model. This model supports thermally activated solitons. The correlation length is set by the inverse of the soliton density and diverges exponentially as the temperature goes to zero [6].

Exercise: To go through in details the calculations described in Kosterlitz's 1974 paper [7], the 1978 paper by Josè, Kadanoff, Kirkpatrick, and Nelson [8] and spelled out in more detail in eg. Chaikin and Lubensky's book [3].

The aim of the calculation is to study the behaviour of ρ_s^R in Eq. 34. We want to extract the behaviour of ρ_s^R in the limit where the applied twist in Eq. 32 varies very slowly. One can think of this as probing the rigidity between two rotors very far apart. According to Eq. 34 the crucial quantity we need to analyse is the vortex density-density correlation function. This is done by rewriting the Hamiltonian in a way that allows a more direct interpretation of how the vortices interact.

The phase field is separated into a spin wave and a vortex part as in Eq. 33. This makes the Hamiltonian in Eq. 2 split into to terms

$$H = H_{sw} + H_{vor} \tag{39}$$

where the vortex part can be expressed in terms of the Fourier transform of the vortex density function given in Eq. 35

$$\frac{1}{T}H_{vor} = \frac{K}{2}\int d\mathbf{k}\frac{\hat{n}(\mathbf{k})\hat{n}(-\mathbf{k})}{k^2}. \tag{40}$$

Here the temperature dependent "coupling constant" $K = \rho_s/T = J/T$ is introduced in accordance with standard stat. mech. tradition. It turns out to be advantageous to transform the Hamiltonian in Eq. 40 into real space. One obtains

$$\frac{1}{T}H = \frac{K}{2}(2\pi)^2\int d\mathbf{r}_1\int d\mathbf{r}_2 n(\mathbf{r}_1)G(\mathbf{r}_1-\mathbf{r}_2)n(\mathbf{r}_2). \tag{41}$$

Where the Green's function mediating the interaction between two vortices is obtained from

$$G(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2} \tag{42}$$

$$= \int_{\pi/L}^{\pi/a} dk\frac{1}{k}\int_0^{2\pi}\cos[kr\cos(\phi)]. \tag{43}$$

The integral needs to be regularized by the introduction of upper and lower cut-offs. The integral over ϕ introduces the zero order Bessel function

$$J_0(z) \simeq \begin{cases} 1-\frac{1}{2}z^2+\cdots & \text{for } |z|\ll 1 \\ \sqrt{\frac{2}{\pi z}}(\cos(z-\pi/4)+\cdots) & \text{for } |z|\gg 1 \end{cases} \tag{44}$$

The substitution $u = rk$ leads to

$$G(r) = \frac{1}{2\pi}\int_{\frac{\pi r}{L}}^{\frac{\pi r}{a}} du\frac{J_0(u)}{u}. \tag{45}$$

From the asymptotic behaviour listed in Eq. 44 we note that

$$\frac{1-J_0(u)}{u} \sim u \tag{46}$$

can be integrated in the limit $u \to 0$ and that

$$\frac{J_0(u)}{u} \sim \frac{1}{u^{3/2}} \tag{47}$$

can be integrated in the limit $u \to \infty$. This suggest that the integral over u should be split up in the following way

$$\begin{aligned}
G(r) &= \frac{1}{2\pi}\int_{\frac{\pi r}{L}}^{\frac{\pi r}{a}} du \left\{\frac{1}{u} - \frac{1-J_o(u)}{u}\right\} &(48)\\
&= \frac{1}{2\pi}[\ln\left(\frac{L}{a}\right) - \int_{\frac{\pi r}{L}}^{1} \frac{1-J_0(u)}{u} - \int_{1}^{\frac{\pi r}{a}} \frac{1-J_0(u)}{u} &(49)\\
&= \frac{1}{2\pi}[\ln\left(\frac{L}{a}\right) - \ln\left(\frac{r}{a}\right) + const. &(50)
\end{aligned}$$

We arrive at a logarithmic dependence of the vortex-vortex interaction as anticipated by the simple argument leading to Eq. 12. Next step is to substitute this expression for $G(r)$ into Eq. 41. In the limit $L \to \infty$ charge neutrality $\sum_\alpha n_\alpha = 0$ is needed to obtain a finite energy. We also need to include a core energy E_c associated with each vortex. When we put this together and return to the original lattice notation we obtain

$$\frac{1}{T}H_{vor} = -\pi K \sum_{\mathbf{r}_1 \neq \mathbf{r}_2} n(\mathbf{r}_1) \ln\left(\frac{|\mathbf{r}_1 - \mathbf{r}_2|}{a}\right) n(\mathbf{r}_2) + \frac{E_c}{T}\sum_{\mathbf{r}} n(\mathbf{r})^2. \tag{51}$$

The leading contributions to the thermodynamic averages come from configurations for which

$$n(\mathbf{r}) \in \{-1, 0, 1\}. \tag{52}$$

Multiple charges leads to much higher energies. The tick is now to use the Hamiltonian in Eq. 51 to perform the average of $\langle \hat{n}(\mathbf{k})\hat{n}(-\mathbf{k})\rangle_0$ over the canonical ensemble. The result of this calculation is substituted into Eq. 34 for the renormalized spin wave stiffness and leads to

$$\frac{1}{K_R} = \frac{1}{K} + 2\pi^3 y^2 \int_a^L \frac{dr}{a}\left(\frac{r}{a}\right)^{3-2\pi K}. \tag{53}$$

Where $K_R = \rho_s^R/T$ and $y = \exp(-\beta E_c)$ is called the vortex fugacity. Eq. 53 is derived under the assumption of low temperature, i.e. Eq. 52, or low density of vortices. Those people who did the first Exercise about the average size $\langle R\rangle$ of a vortex pair will recognize that the exponent $3-2\pi K$ may lead to interesting behaviour. At low temperature $K = \rho_s/T$ is large and the integral in Eq. 53 will converge as $L \to \infty$. The renormalized spin wave stiffness is accordingly corrected by a small amount of order y^2. When the temperature is increased K will become larger than $2/\pi$ and the large r limit will diverge. The renormalization procedure is able to extract the behaviour of K_R even when the integral in Eq. 53 diverges.

Our next task is to use Eq. 53 to establish a set of renormalization group equations for K_R and y from which we can extract the long distance behaviour.

4.4 The RG equations for the KT transition

The standard procedure of the renormalization group is to integrate over short wave length variations. One can think of this as if one is looking at how the system behaves at ever larger length scales. In the Wilson RG scheme one works directly with the Hamiltonian in Fourier space and "filter" out the modes corresponding to large k-vectors or short wave lengths. In the Kadanoff-Migdal real space approach one decimates the lattice by eliminating successively every other spin. In Kosterlitz's approach one study how the spin wave stiffness changes as one gradually increases the lower cut-off $a \to e^{dl}a$. Increasing a does correspond to eliminating the short distance fluctuations. For instance, since a is the minimum separation of two vortices, increasing a excludes configurations of small separation. One way to see the effect of increasing a is to break the integral in Eq. 53 up [8] in the following way

$$\int_a^L \mapsto \int_a^\infty = \int_a^{ae^{dl}} + \int_{ae^{dl}}^\infty . \tag{54}$$

The integral in Eq. 53 is broken up in this way and a new renormalized stiffness[3] $\tilde{K}$is introduced according to

$$\frac{1}{\tilde{K}} = \frac{1}{K} + 2\pi^3 y^2 \int_a^{ae^{dl}} \frac{dr}{a} \left(\frac{r}{a}\right)^{3-2\pi K} . \tag{55}$$

This leads to the equation

$$\frac{1}{K_R} = \frac{1}{\tilde{K}} + 2\pi^3 \int_{ae^{dl}}^\infty \frac{dr}{a} \left(\frac{r}{a}\right)^{3-2\pi K} . \tag{56}$$

Next step is to write this equation in a form that is invariant with respect to the change $a \mapsto ae^{dl}$. This is done by introducing a renormalized fugacity

$$\tilde{y} = ye^{2-\pi K}. \tag{57}$$

We are interested in the changes of $\tilde{K}$ and $\tilde{y}$ induced by an infinitesimal change in a, i.e. for small dl. Expanding in dl one arrives at the Kosterlitz renormalization equations (correct to order y^4 and y^3 respectively)

$$\frac{d\tilde{K}^{-1}}{dl} = 2\pi^3 \tilde{y}^2 \tag{58}$$

$$\frac{d\tilde{y}}{dl} = (2 - \pi\tilde{K})\tilde{y}. \tag{59}$$

This set of equations are integrated to obtain the trajectories as function of length scale l in the phase plane $(\tilde{K}^{-1}, y)$. The trajectories are sketched in the figure on the next page.

[3] This time renormalized due to change of the length scale, not as before when we introduced ρ_s^R where the renormalization was relative to the zero temperature value ρ_s.

Phase Plane Trajectories for the Kosterlitz RG Flow

- Note: $y = 0$ is a line of fixed points
- Note: $y(l)$ is a decreasing function of l for $\tilde{K}^{-1}(l) < \pi/2$ and increasing with l for $\tilde{K}^{-1}(l) > \pi/2$.
- Note: The straight line indicates the set of initial conditions for different temperatures. Increasing the temperature corresponds to moving up to the right along the line.
- Note: Most interesting is what happens in the limit $l \to \infty$. For initial conditions below the separatrix $\lim_{l\to\infty}(\tilde{K}^{-1}, \tilde{y}) = (\tilde{K}^{-1}(\infty), 0)$ where $\tilde{K}^{-1}(\infty) < \infty$. For initial conditions above the separatrix $\lim_{l\to\infty}(\tilde{K}^{-1}, \tilde{y}) = (\infty, \infty)$
- Note: The separatrix corresponds to $T = T_{KT}$, and $\lim_{l\to\infty} \tilde{K}^{-1}(T_{KT}^{-}) = \pi/2$. Whereas $\lim_{l\to\infty} \tilde{K}^{-1}(T_{KT}^{+}) = 0$, this is the universal jump mentioned in Eq. 37. We see from Eq. 29 that the universal jump determines the correlation function exponent at $T = T_{KT}$ to be $\eta(T_{KT}) = 1/4$.

5 The Vortex Unbinding Transition in Other Systems

We have a number of times alluded to the fact that not only the XY-model exhibit the Kosterlitz-Thouless vortex unbinding transition. Any two dimensional system that supports thermally induced "charges" or topological defects that interact logarithmically will undergo this transition. The $U(1)$ symmetry of the phase field $\theta(\mathbf{r})$ of the XY-model is also present in the Ginzburg-Landau free energy of superfluids and of superconductors. The topological excitations in the case of a superfluid consist of vortices in the flow of the superfluid. Vortices like the those observed when one empty a bath tub. For thin superfluid helium film experiments find that the destruction of the superfluid phase with increasing temperature occurs according to the scenario of the KT-transition.

The situation is slightly more complicated in superconductors. Because the superfluid in this case is charged (the superconducting pairs of electrons) screening effects play a role. However, for thin superconducting films of thickness δ the effective screening length is given by $\lambda_{eff} = \lambda^2/\delta$ which can easily become a macroscopic length. In this case the loss of superconductivity is caused by the unbinding of vortex pairs according to the KT-transition. The broken pairs can move freely when they respond to an applied electric current. As they move they cause phase slips in the superconducting order parameter. These phase slips induce a voltage drop according to the Josephson relation. The superconductor is now unable to support an electric current without a voltage drop, i.e. it is not a superconductor any longer.

Dislocations in two dimensional crystals interact through the strain field. Two edge dislocations of opposite sign correspond to an extra row of atoms inserted along the line connecting the location of the two dislocation cores. The extra line of atoms produces strain and leads to an increase in the energy which is logarithmic in the separation between the two dislocations. Thus the situation is very similar to the one encountered in the XY-model. When the dislocations unbind free dislocations are produced. A shear applied to the system can now be accommodated by the mobile dislocations without an increase in the (free) energy. I.e., the shear constant has dropped to zero and the system is melted. The 2d melting theory of Kosterlitz-Thouless-Halperin-Nelson-Young predicts that melting occur in two stages. At the first stages dislocations unbind and make the shear constant drop to zero. The dislocations are topological defects, their effect on the order of the lattice are, however, not very dramatic. Before the unbinding of dislocations the translational and the orientational order of the lattice are both described by correlation functions that depends algebraically on distance. When the dislocations unbind the translational correlation function becomes exponential but the orientational correlations remain algebraic. At a somewhat higher temperature topological defects called disclinations unbind with the effect that the orientational order becomes exponential. Details can be found in Chaikin and Lubensky [3].

There are many other cases where the logarithmic vortex interaction and the KT-transition plays a role. For instance, the shape of surfaces in three dimensions may undergo a transition from smooth to rough. This transition can be described in the limit of a discrete surface constructed by columns of different height[4] as a KT-transition.

[4]This is called the solid-on-solid model

6 Conclusion

We have describe a specific example where one can follow in great detail the step from one level of structure to the next in the hierarchical order of matter. Topological defects arise as coherent structures of the “atoms” of one level and can be considered as (composite) particles at the next level. Their interaction can be derived from the behaviour of the constituent “atoms”. Many different systems may support composite particles that interact in the same way. In the case we have discussed the specific physical properties and the subtleties of the Kosterlitz-Thouless transition are caused by the logarithmic interact between the topological defects. The one most important fact in determining the KT-transition is the fact that both energy and entropy depends logarithmically on length scale for the two dimensional topological charges.

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